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Research Article

Some New Nonlinear Dynamical Integral Inequalities with Applications on Time Scales

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Abstract: We investigated some non-linear dynamical integral inequalities on time scales, providing explicit bounds on unknown functions. These types of inequalities, consolidate and develop some existing well-known inequalities, and can be utilized in qualitative theory of dynamical equations.

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1. INTRODUCTION

Substantial attention has been given in recent years to the theory of dynamic equations on time scales, which was introduced by Hilger's landmark [1].For instance see [2] and the references cited therein. Therefore, several researchers have discussed numerous aspects of dynamic equations on time scales. Dynamic inequalities play a significant role in the qualitative study of dynamic equations [3-7]. Various researchers have been studied integral inequalities of different types on the time scales [8].

The primary objective of our work is to analyze some non-linear dynamic integral inequalities on time scales which not only generalized few existing well known results. But this work also came handy to determine the explicit bounds of the solutions of particular dynamical equations on time scales. Along with we provide some continuous and discrete inequalities for different time scales. As a whole in this work, we have deeply studied time scales and time scales essentials.$\mathbb{T}$ is considered to be a time scales and $C_{rd}(\mathbb{T})$ denotes the set of all rd-continuous functions defined on $\mathbb{T}$. For convenience throughout the whole discussion we assume that $t_0 \in \mathbb{T}$. The work is structured as follows: Non-linear dynamic inequalities on time scales are given in section 2. In section 3 some applications to illustrate our main results are given.

2. MAIN RESULTS

In this paper, the following non-linear dynamic integral inequalities would be under consideration.

$$[\Theta(t)]^p \le \tilde{a}(t) + b(t) \int_{t_0}^{t} \{g(\tau)[\Theta(\tau)]^q + h(\tau)[\Theta(\tau)]^r + j(\tau)[\Theta(\tau)]^s\}\Delta\tau \quad (1)$$

$$[\Theta(t)]^p \le \tilde{a}(t) + b(t) \int_{t_0}^{t} \{g(\tau)[\Theta^\sigma(\tau)]^q + h(\tau)[\Theta^\sigma(\tau)]^r + j(\tau)[\Theta^\sigma(\tau)]^s\}\Delta\tau \quad (2)$$

$$[\Theta(t)]^p \le \tilde{a}(t) + b(t) \int_{t_0}^{t} K(t,\tau)\{g(\tau)[\Theta(\tau)]^q + h(\tau)[\Theta(\tau)]^r + j(\tau)[\Theta(\tau)]^s\}\Delta\tau \quad (3)$$

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$$[\Theta(\mathfrak{t})]^{\mathfrak{p}} \leq \tilde{a}(\mathfrak{t}) + \mathfrak{b}(\mathfrak{t}) \int_{\mathfrak{t}_0}^{\mathfrak{t}} K(\mathfrak{t},\mathfrak{v})\{g(\mathfrak{v})[\Theta^{\sigma}(\mathfrak{v})]^{\mathfrak{q}} + h(\mathfrak{v})[\Theta^{\sigma}(\mathfrak{v})]^{r} + j(\mathfrak{v})[\Theta^{\sigma}(\mathfrak{v})]^{s}\}\Delta\mathfrak{v} \quad (4)$$

$$[\Theta(\mathfrak{t})]^{\mathfrak{p}} \leq \rho + \int_{\mathfrak{t}_0}^{\mathfrak{t}} \{g(\mathfrak{v})[\Theta(\mathfrak{v})]^{\mathfrak{q}} + h(\mathfrak{v})[\Theta(\mathfrak{v})]^{r} + j(\mathfrak{v})[\Theta(\mathfrak{v})]^{s}\}\Delta\mathfrak{v} \quad (5)$$

provided that $\tilde{a}, \mathfrak{b}, g, h, j, \Theta\colon \mathbb{T}^K \to \boldsymbol{R}^+(= [0,\infty)$ are rd-continuous functions, and $K\colon \mathbb{T} \times \mathbb{T}^k \to \mathbf{R}^+(= [0,\infty))$ is a continuous function.

Lemma 1. [2,p.255] Let n, y $\in C_{rd}(\mathbb{T})$ and $m \in \boldsymbol{R}_+(\mathbb{T})$, then

$$y^{\Delta}(y) \leq m(\mathfrak{t})y(\mathfrak{t}) + n(\mathfrak{t}), \quad \mathfrak{t} \in \mathbb{T},$$

implies

$$y(\mathfrak{t}) \leq y(\mathfrak{t}_0)e_m(\mathfrak{t},\mathfrak{t}_0) + \int_{\mathfrak{t}_0}^{\mathfrak{t}} e_m(\mathfrak{t},\sigma(\mathfrak{v}))\Delta\mathfrak{v}, \quad \mathfrak{t} \in \mathbb{T}.$$

Lemma 2. [4] Let us consider $0 \leq \tilde{a}$, $\mathfrak{p} \geq \mathfrak{q} \geq 0$, and $0 \neq \mathfrak{p}$ and if we take $0 \leq k$ then

$$\tilde{a}^{\frac{\mathfrak{p}}{\mathfrak{q}}} \leq \frac{\mathfrak{q}}{\mathfrak{p}} k^{\frac{\mathfrak{q}-\mathfrak{p}}{\mathfrak{p}}} \tilde{a} + \frac{\mathfrak{p}-\mathfrak{q}}{\mathfrak{p}} k^{\frac{\mathfrak{q}}{\mathfrak{p}}}.$$

Lemma 3. [2,p.46] Suppose $K\colon \mathbb{T} \times \mathbb{T}^k \to \mathbf{R}$ is continuous at $(\mathfrak{t},\mathfrak{t})$, then for any $\epsilon > 0, \exists\, \mathfrak{U}$ neighbourhood of $\mathfrak{t} \in \mathbb{T}^k$, which does not depend on $\mathfrak{v} \in [\mathfrak{t}_0, \sigma(\mathfrak{t})]$, s.t

$$\left|K(\sigma(\mathfrak{t}),\mathfrak{v}) - K(s,\mathfrak{v}) - K_1^{\Delta}(\mathfrak{t},\mathfrak{v})(\sigma(\mathfrak{t}) - s)\right| \leq \epsilon|\sigma(\mathfrak{t}) - s|, \qquad s \in \mathfrak{U}$$

provided that $K_1^{\Delta}(\mathfrak{t},.)$ (the derivative of K w.r.t the first variable) is rd-continuous on $[\mathfrak{t}_0, \sigma(\mathfrak{t})], \mathfrak{t} > \mathfrak{t}_0$, then

$$\mathfrak{d}(\mathfrak{t}) := \int_{\mathfrak{t}_0}^{\mathfrak{t}} K(\mathfrak{t},\mathfrak{v})\Delta\mathfrak{v} \Longrightarrow \mathfrak{d}^{\Delta}(\mathfrak{t}) = \int_{\mathfrak{t}_0}^{\mathfrak{t}} K_1^{\Delta}(\mathfrak{t},\mathfrak{v})\Delta\mathfrak{v} + K(\sigma(\mathfrak{t}),\mathfrak{t})$$

Before stating main results, some symbolic representation for the sake of brevity and compact understanding are given as:

$$\mathfrak{H}(\mathfrak{t}) := g(\mathfrak{t})\left\{\frac{\mathfrak{q}}{\mathfrak{p}} k^{\frac{\mathfrak{q}-\mathfrak{p}}{\mathfrak{p}}} \tilde{a}(\mathfrak{t}) + \frac{\mathfrak{p}-\mathfrak{q}}{\mathfrak{p}} k^{\frac{\mathfrak{q}}{\mathfrak{p}}}\right\} + h(\mathfrak{t})\left\{\frac{r}{\mathfrak{p}} k^{\frac{r-\mathfrak{p}}{\mathfrak{p}}} \tilde{a}(\mathfrak{t}) + \frac{\mathfrak{p}-r}{\mathfrak{p}} k^{\frac{r}{\mathfrak{p}}}\right\} + j(\mathfrak{t})\left\{\frac{s}{\mathfrak{p}} k^{\frac{s-\mathfrak{p}}{\mathfrak{p}}} \tilde{a}(\mathfrak{t}) + \frac{\mathfrak{p}-s}{\mathfrak{p}} k^{\frac{s}{\mathfrak{p}}}\right\} \quad (6)$$

$$\tilde{\mathfrak{H}}(\mathfrak{t}) := g(\mathfrak{t})\left\{\frac{\mathfrak{q}}{\mathfrak{p}} k^{\frac{\mathfrak{q}-\mathfrak{p}}{\mathfrak{p}}} \tilde{a}^{\sigma}(\mathfrak{t}) + \frac{\mathfrak{p}-\mathfrak{q}}{\mathfrak{p}} k^{\frac{\mathfrak{q}}{\mathfrak{p}}}\right\} + h(\mathfrak{t})\left\{\frac{r}{\mathfrak{p}} k^{\frac{r-\mathfrak{p}}{\mathfrak{p}}} \tilde{a}^{\sigma}(\mathfrak{t}) + \frac{\mathfrak{p}-r}{\mathfrak{p}} k^{\frac{r}{\mathfrak{p}}}\right\} + j(\mathfrak{t})\left\{\frac{s}{\mathfrak{p}} k^{\frac{s-\mathfrak{p}}{\mathfrak{p}}} \tilde{a}^{\sigma}(\mathfrak{t}) + \frac{\mathfrak{p}-s}{\mathfrak{p}} k^{\frac{s}{\mathfrak{p}}}\right\} \quad (7)$$

$$\mathfrak{X}(\mathfrak{t}) := \mathfrak{b}(\mathfrak{t})\left\{\frac{\mathfrak{q}}{\mathfrak{p}} g(\mathfrak{t}) k^{\frac{\mathfrak{q}-\mathfrak{p}}{\mathfrak{p}}} + \frac{r}{\mathfrak{p}} h(\mathfrak{t}) k^{\frac{r-\mathfrak{p}}{\mathfrak{p}}} + \frac{s}{\mathfrak{p}} j(\mathfrak{t}) k^{\frac{s-\mathfrak{p}}{\mathfrak{p}}}\right\} \quad (8)$$

$$\tilde{\mathfrak{X}}(\mathfrak{t}) := \mathfrak{b}^{\sigma}(\mathfrak{t})\left\{\frac{\mathfrak{q}}{\mathfrak{p}} g(\mathfrak{t}) k^{\frac{\mathfrak{q}-\mathfrak{p}}{\mathfrak{p}}} + \frac{r}{\mathfrak{p}} h(\mathfrak{t}) k^{\frac{r-\mathfrak{p}}{\mathfrak{p}}} + \frac{s}{\mathfrak{p}} j(\mathfrak{t}) k^{\frac{s-\mathfrak{p}}{\mathfrak{p}}}\right\} \quad (9)$$

$$\widetilde{\widetilde{\mathfrak{H}}}(\mathfrak{t}) := \mathfrak{H}(\mathfrak{t})K(\sigma(\mathfrak{t}),\mathfrak{t}) + \int_{\mathfrak{t}_0}^{\mathfrak{t}} K^{\Delta}(\mathfrak{t},\tau)\{\mathfrak{H}(\tau) + \beta(\tau)\mathfrak{X}(\tau)\}\Delta\tau. \tag{10}$$

$$\widetilde{\mathfrak{E}}(\mathfrak{t}) := \frac{\mathfrak{q}.\, g(\mathfrak{t})k^{\frac{\mathfrak{q}}{\mathfrak{p}}} + r.\, h(\mathfrak{t})k^{\frac{r}{\mathfrak{p}}} + s.\, j(\mathfrak{t})k^{\frac{s}{\mathfrak{p}}}}{k\mathfrak{p}} \tag{11}$$

$$\widetilde{\mathfrak{D}}(\mathfrak{t}) := \frac{\rho - k}{k\mathfrak{p}}\left[\mathfrak{q}.\, g(\mathfrak{t})k^{\frac{\mathfrak{q}}{\mathfrak{p}}} + r.\, h(\mathfrak{t})k^{\frac{r}{\mathfrak{p}}} + s.\, j(\mathfrak{t})k^{\frac{s}{\mathfrak{p}}}\right] + g(\mathfrak{t})k^{\frac{\mathfrak{q}}{\mathfrak{p}}} + h(\mathfrak{t})k^{\frac{r}{\mathfrak{p}}} + j(\mathfrak{t})k^{\frac{s}{\mathfrak{p}}} \tag{12}$$

$$\widehat{\mathfrak{Y}}(\mathfrak{t}) := \widetilde{\mathfrak{H}}(\mathfrak{t})K(\sigma(\mathfrak{t}),\mathfrak{t}) + \int_{\mathfrak{t}_0}^{\mathfrak{t}} K^{\Delta}(\mathfrak{t},\tau)\widetilde{\mathfrak{H}}(\tau)\Delta\tau. \tag{13}$$

$$\mathfrak{z}(\mathfrak{t}) := y^{\sigma}(\mathfrak{t})K(\sigma(\mathfrak{t}),\mathfrak{t})\widetilde{\mathfrak{X}}(\mathfrak{t}) + \int_{\mathfrak{t}_0}^{\mathfrak{t}} K^{\Delta}(\mathfrak{t},\tau)y^{\sigma}(\tau)\widetilde{\mathfrak{X}}(\tau)\Delta\tau. \tag{14}$$

$$\mathfrak{X}_1(\mathfrak{t}) := \frac{\widetilde{\mathfrak{X}}(\mathfrak{t})\omega^{\sigma}(\mathfrak{t})}{1 - \mu(\mathfrak{t})\widetilde{\mathfrak{X}}(\mathfrak{t})\omega^{\sigma}(\mathfrak{t})} \tag{15}$$

$$\widetilde{\mathfrak{Y}}(\mathfrak{t}) := \frac{\widehat{\mathfrak{Y}}(\mathfrak{t})}{1 - \mu(\mathfrak{t})\widehat{\mathfrak{Y}}(\mathfrak{t})} \tag{16}$$

$$\widetilde{\widetilde{\mathfrak{X}}}(\mathfrak{t}) := \mathfrak{X}(\mathfrak{t})K(\sigma(\mathfrak{t}),\mathfrak{t}) \tag{17}$$

The main result of this paper is as follow.

Theorem 1. Let $\mathfrak{a}, \mathfrak{b}, g, h, j, \Theta: \mathbb{T}^{K} \to \mathbf{R}$ be non-negative rd-continuous functions and $(0 \neq)\mathfrak{p}, \mathfrak{q}, r, s$ are constant with $\mathfrak{p} \geq \mathfrak{q}$; $\mathfrak{p} \geq r$ and $\mathfrak{p} \geq s$ such that (1) holds, then

$$\Theta(\mathfrak{t}) \leq \sqrt[\mathfrak{p}]{\mathfrak{a}(\mathfrak{t}) + \mathfrak{b}(\mathfrak{t})\int_{\mathfrak{t}_0}^{\mathfrak{t}} e_{\mathfrak{X}}(\mathfrak{t},\sigma(\tau))\mathfrak{H}(\tau)\Delta\tau}, \tag{18}$$

where $\mathfrak{H}$ and $\mathfrak{X}$ are defined above in equation (6) and (8), respectively.

Proof. Define a function

$$z(\mathfrak{t}) = \int_{\mathfrak{t}_0}^{\mathfrak{t}} \{g(\tau)[\Theta(\tau)]^{\mathfrak{q}} + h(\tau)[\Theta(\tau)]^{r} + j(\tau)[\Theta(\tau)]^{s}\}\Delta\tau,$$

so that $z(\mathfrak{t}_0) = 0$ and z is non decreasing

$$z^{\Delta}(\mathfrak{t}) = g(\mathfrak{t})[\Theta(\mathfrak{t})]^{\mathfrak{q}} + h(\mathfrak{t})[\Theta(\mathfrak{t})]^{r} + j(\mathfrak{t})[\Theta(\mathfrak{t})]^{s} \tag{19}$$

$$\Longrightarrow \Theta(\mathfrak{t}) \leq \sqrt[\mathfrak{p}]{\mathfrak{a}(\mathfrak{t}) + \mathfrak{b}(\mathfrak{t})z(\mathfrak{t})} \tag{20}$$

Direct application of Lemma 2 and inequality (20) in (19) yield*:*

$$z^{\Delta}(\mathfrak{t}) \leq g(\mathfrak{t})[\mathfrak{a}(\mathfrak{t}) + \mathfrak{b}(\mathfrak{t})z(\mathfrak{t})]^{\frac{\mathfrak{q}}{\mathfrak{p}}} + h(\mathfrak{t})[\mathfrak{a}(\mathfrak{t}) + \mathfrak{b}(\mathfrak{t})z(\mathfrak{t})]^{\frac{r}{\mathfrak{p}}} + j(\mathfrak{t})[\mathfrak{a}(\mathfrak{t}) + \mathfrak{b}(\mathfrak{t})z(\mathfrak{t})]^{\frac{s}{\mathfrak{p}}}$$

$$\leq g(\mathfrak{t})\left[\frac{\mathfrak{q}}{\mathfrak{p}}k^{\frac{\mathfrak{q}-\mathfrak{p}}{\mathfrak{p}}}\big(\mathfrak{a}(\mathfrak{t}) + \mathfrak{b}(\mathfrak{t})z(\mathfrak{t})\big) + \frac{\mathfrak{p}-\mathfrak{q}}{\mathfrak{p}}k^{\frac{\mathfrak{q}}{\mathfrak{p}}}\right] + h(\mathfrak{t})\left[\frac{r}{\mathfrak{p}}k^{\frac{r-\mathfrak{p}}{\mathfrak{p}}}\big(\mathfrak{a}(\mathfrak{t}) + \mathfrak{b}(\mathfrak{t})z(\mathfrak{t})\big) + \frac{\mathfrak{p}-r}{\mathfrak{p}}k^{\frac{r}{\mathfrak{p}}}\right]$$

$$+j(ŧ)\left[\frac{s}{p}k^{\frac{s-p}{p}}\left(ã(ŧ)+ƀ(ŧ)z(ŧ)\right)+\frac{p-s}{p}k^{\frac{s}{p}}\right]\leq \mathfrak{H}(ŧ)+z(ŧ)\mathfrak{X}(ŧ). \tag{21}$$

Application of Lemma 1 yields:

$$z(ŧ)\leq\int_{ŧ_0}^{ŧ}e_{\mathfrak{X}}\left(ŧ,\sigma(ט)\right)\mathfrak{H}(ט)\Delta ט.$$

Hence, the result.

Theorem 2. Let $ã, ƀ, g, h, j, \Theta: \mathbb{T}^{K} \to \mathbf{R}$ be non-negative rd-continuous functions and $(0\neq)p, q, r, s$ are constants with $p \geq q;\ p \geq r$ and $p \geq s$. If $1 > \mu(ŧ)\tilde{\mathfrak{X}}(ŧ)\omega^{\sigma}(ŧ)$ s.t (2) satisfied,then

$$\Theta(ŧ)\leq\sqrt[p]{ã(ŧ)+ƀ(ŧ)\int_{ŧ_0}^{ŧ}\left(\tilde{\mathfrak{H}}\oplus\mathfrak{X}_1\right)(ט)\,\Delta ט}, \tag{22}$$

provided that ω is defined in the proof.Functions $\tilde{\mathfrak{H}}$ and $\mathfrak{X}_1$ are defined by equation (6) and (15), respectively.

Proof. Define a function

$$\omega(ŧ)\ =\int_{ŧ_0}^{ŧ}\{g(ט)[\Theta^{\sigma}(ט)]^{q}+h(ט)[\Theta^{\sigma}(ט)]^{r}+j(ט)[\Theta^{\sigma}(ט)]^{s}\}\Delta ט,$$

so that $\omega\ (ŧ_0) = 0$ and ω is non decreasing

$$\omega^{\Delta}(ŧ)=g(ŧ)[\Theta^{\sigma}(ט)]^{q}+h(ŧ)[\Theta^{\sigma}(ט)]^{r}+j(ŧ)[\Theta^{\sigma}(ט)]^{s} \tag{23}$$

$$\Rightarrow \Theta(ŧ)\leq\sqrt[p]{ã(ŧ)+ƀ(ŧ)\omega(ŧ)} \tag{24}$$

Direct application of Lemma 2 and inequality (24) in (23) yield:

$$\omega^{\Delta}(ŧ)\leq g(ŧ)[ã^{\sigma}(ŧ)+ƀ^{\sigma}(ŧ)\omega^{\sigma}(ŧ)]^{\frac{q}{p}}+h(ŧ)[ã^{\sigma}(ŧ)+ƀ^{\sigma}(ŧ)\omega^{\sigma}(ŧ)]^{\frac{r}{p}}+j(ŧ)[ã^{\sigma}(ŧ)+ƀ^{\sigma}(ŧ)\omega^{\sigma}(ŧ)]^{\frac{s}{p}}$$

$$\leq g(ŧ)\left[\frac{q}{p}k^{\frac{q-p}{p}}\left(ã^{\sigma}(ŧ)+ƀ^{\sigma}(ŧ)\omega^{\sigma}(ŧ)\right)+\frac{p-q}{p}k^{\frac{q}{p}}\right]+h(ŧ)\left[\frac{r}{p}k^{\frac{r-p}{p}}\left(ã^{\sigma}(ŧ)+ƀ^{\sigma}(ŧ)\omega^{\sigma}(ŧ)\right)+\ \frac{p-r}{p}k^{\frac{r}{p}}\right]$$

$$+j(ŧ)\left[\frac{s}{p}k^{\frac{s-p}{p}}\left(ã^{\sigma}(ŧ)+ƀ^{\sigma}(ŧ)\omega^{\sigma}(ŧ)\right)+\frac{p-s}{p}k^{\frac{s}{p}}\right]\leq\ \tilde{\mathfrak{H}}(ŧ)+\omega^{\sigma}(ŧ)\tilde{\mathfrak{X}}(ŧ). \tag{25}$$

It is observed that:

$$\tilde{\mathfrak{X}}(t)\omega^{\sigma}(t)=\frac{\mathfrak{X}_1(t)}{1+\mu(t)\mathfrak{X}_1(t)}$$

$\Rightarrow$ (25) becomes

$$\omega^{\Delta}(ŧ)\leq\tilde{\mathfrak{H}}(ŧ)+\frac{\mathfrak{X}_1(ŧ)}{1+\mu(ŧ)\mathfrak{X}_1(ŧ)}\left(=(\tilde{\mathfrak{H}}\oplus\mathfrak{X}_1)(ŧ)\right)$$

Hence, the result.

The following two theorems are the weighted variants of the last theorems respectively.

Theorem 3. Let $\tilde{a}, b, g, h, j, \Theta: \mathbb{T}^K \to \mathbf{R}$ be non-negative rd-continuous functions and $(0 \neq)p, q, r, s$ are constants with $p \geq q$; $p \geq r$ and $p \geq s$.Let $K(t,s)$ be a weight function as defined in lemma 3 s.t $K_1^{\Delta}(t,s) \geq 0$ for $t \geq s$ such that (3) holds, then

$$\Theta(t) \leq \sqrt[p]{\tilde{a}(t) + b(t)\int_{t_0}^{t} e_{\tilde{\tilde{\mathfrak{X}}}}(t,\sigma(\tau))\tilde{\tilde{\mathfrak{H}}}(\tau)\Delta\tau},$$

where $\tilde{\tilde{\mathfrak{H}}}$ and $\tilde{\tilde{\mathfrak{X}}}$ are defined above in equation (10) and (17), respectively.

Proof. Define a function

$$\beta(t) = \int_{t_0}^{t} \alpha(t,\tau)\Delta\tau,$$

provided that

$$\alpha(t,\tau) = K(t,\tau)\{g(\tau)[\Theta(\tau)]^q + h(\tau)[\Theta(\tau)]^r + j(\tau)[\Theta(\tau)]^s\}$$

so that $\beta(t_0) = 0$ and $\beta(t)$ is non decreasing.

$$(3) \Rightarrow \Theta(t) \leq \sqrt[p]{\tilde{a}(t) + b(t)\beta(t)}. \quad (27)$$

Direct application of lemma 2 and inequality (19), (21) and (27) yield:

$$\beta^{\Delta}(t) = \alpha(\sigma(t),t) + \int_{t_0}^{t} \alpha_1^{\Delta}(t,\tau)\Delta\tau$$

$$= K(\sigma(t),t)\{g(t)[\Theta(t)]^q + h(t)[\Theta(t)]^r + j(t)[\Theta(t)]^s\}$$

$$+ \int_{t_0}^{t} K^{\Delta}(t,\tau)\{g(\tau)[\Theta(\tau)]^q + h(\tau)[\Theta(\tau)]^r + j(\tau)[\Theta(\tau)]^s\}\Delta\tau$$

$$= K(\sigma(t),t)z^{\Delta}(t) + \int_{t_0}^{t} K^{\Delta}(t,\tau)z^{\Delta}(\tau)\Delta\tau \leq \tilde{\tilde{\mathfrak{H}}}(t) + \beta(t)\tilde{\tilde{\mathfrak{X}}}(t). \quad (28)$$

application of lemma 1 yields the required result.

Theorem 4. Under the assumptions of theorem 3 for $1 > \mu(t)\,\hat{\mathfrak{Y}}(t)$ such that (4) holds, then

$$\Theta(t) \leq \sqrt[p]{\tilde{a}(t) + b(t)\int_{t_0}^{t} \left(\tilde{\mathfrak{z}} \oplus \tilde{\mathfrak{Y}}\right)(\tau)\,\Delta\tau}, \quad (29)$$

where $\hat{\mathfrak{Y}}, \tilde{\mathfrak{z}}$ and $\tilde{\mathfrak{Y}}$ are defined by equation (13), (14) and (16), respectively.

Proof. Consider

$$y(t) = \int_{t_0}^{t} K(t,\tau)\{g(\tau)[\Theta^{\sigma}(\tau)]^q + h(\tau)[\Theta^{\sigma}(\tau)]^r + j(\tau)[\Theta^{\sigma}(\tau)]^s\}\Delta\tau = \int_{t_0}^{t} z(t,\tau)\Delta\tau$$

where, $z(\mathfrak{t},\mathfrak{r}) = K(\mathfrak{t},\mathfrak{r})\{g(\mathfrak{r})[\Theta^{\sigma}(\mathfrak{r})]^{q} + h(\mathfrak{r})[\Theta^{\sigma}(\mathfrak{r})]^{r} + j(\mathfrak{r})[\Theta^{\sigma}(\mathfrak{r})]^{s}\}$,

so that $y(\mathfrak{t}_0) = 0$ and $y(\mathfrak{t})$ is non-decreasing.

$$(4) \Rightarrow \quad \Theta(\mathfrak{t}) \leq \sqrt[p]{\tilde{a}(\mathfrak{t}) + \mathfrak{b}(\mathfrak{t})y(\mathfrak{t})}. \tag{30}$$

Direct application of lemma 2 and (7),(9),(13),(19) yield:

$$y^{\Delta}(\mathfrak{t}) = z(\sigma(\mathfrak{t}),\mathfrak{t}) + \int_{\mathfrak{t}_0}^{\mathfrak{t}} z_1^{\Delta}(\mathfrak{t},\mathfrak{r})\Delta\mathfrak{r} \leq K(\sigma(\mathfrak{t}),\mathfrak{t})\big[\tilde{\mathfrak{H}}(\mathfrak{t}) + y^{\sigma}(\mathfrak{t})\tilde{\mathfrak{X}}(\mathfrak{t})\big] + \int_{\mathfrak{t}_0}^{\mathfrak{t}} K^{\Delta}(\mathfrak{t},\mathfrak{r})\big[\tilde{\mathfrak{H}}(\mathfrak{r}) + y^{\sigma}(\mathfrak{r})\tilde{\mathfrak{X}}(\mathfrak{r})\big]\Delta\mathfrak{r}$$

$$= \tilde{\mathfrak{Y}}(\mathfrak{t}) + K(\sigma(\mathfrak{t}),\mathfrak{t})y^{\sigma}(\mathfrak{t})\tilde{\mathfrak{X}}(\mathfrak{t}) + \int_{\mathfrak{t}_0}^{\mathfrak{t}} K^{\Delta}(\mathfrak{t},\mathfrak{r})y^{\sigma}(\mathfrak{r})\tilde{\mathfrak{X}}(\mathfrak{r})\Delta\mathfrak{r}. \tag{31}$$

For (14), the above inequality (31) has the form

$$y^{\Delta}(\mathfrak{t}) \leq \tilde{\mathfrak{z}}(\mathfrak{t}) + \frac{\tilde{\mathfrak{Y}}(\mathfrak{t})}{1 + \mu(\mathfrak{t})\tilde{\mathfrak{Y}}(\mathfrak{t})} = \big(\tilde{\mathfrak{z}} + \tilde{\mathfrak{Y}}\big)(\mathfrak{t}).$$

Hence, the result.

3. APPLICATIONS

Now we discuss here the few utilizations of the main results for the special cases $\mathbb{T} = \mathbf{R}$, $\mathbb{Z}$.

Corollary 1.(Continuous case)

Let $\mathbb{T} = \mathbf{R}$ and $\tilde{a}, \mathfrak{b}, g, h, j, \Theta : [\mathfrak{t}_0,\infty) \rightarrow \mathbf{R}_+$ be continuous functions; Let $(0 \neq)p, q, r, s$ are constants such that $p \geq q$; $p \geq r$ and $p \geq s$, then (18) implies

$$\Theta(\mathfrak{t}) \leq \sqrt[p]{\tilde{a}(\mathfrak{t}) + \mathfrak{b}(\mathfrak{t}) \int_{\mathfrak{t}_0}^{\mathfrak{t}} exp\left(\int_{\mathfrak{r}}^{\mathfrak{t}} \mathfrak{X}(\eta)d\eta\right)\mathfrak{H}(\mathfrak{r})d\mathfrak{r}}, \tag{32}$$

provided that $\mathfrak{X}$ and $\mathfrak{H}$ are defined as in theorem 1.

Corollary 2.(Discrete case)

Let $\mathbb{T} = \mathbb{Z}$ and $\tilde{a}, \mathfrak{b}, g, h, j, \Theta : \mathfrak{A} \rightarrow \mathbf{R}_+, \mathfrak{A} = \{\mathfrak{t}_0, \mathfrak{t}_0 + 1, ...\}$; let $(0 \neq)p, q, r, s$ are constants s.t $p \geq q$; $p \geq r$ and $p \geq s$, then (18) implies

$$\Theta(\mathfrak{t}) \leq \sqrt[p]{\tilde{a}(\mathfrak{t}) + \mathfrak{b}(\mathfrak{t}) \sum_{\mathfrak{r}=\mathfrak{t}_0}^{\mathfrak{t}-1} \prod_{\eta=\mathfrak{r}+1}^{\mathfrak{t}-1} \big(1 + \mathfrak{X}(\eta)\big)\mathfrak{H}(\mathfrak{r})} \tag{33}$$

provided that $\mathfrak{X}$ and $\mathfrak{H}$ are defined just like in Theorem 1.

Corollary 3. (Continuous case)

Let $\mathbb{T} = \mathbf{R}$ and $\tilde{a}, \mathfrak{b}, g, h, j, \Theta: [\mathfrak{t}_0,\infty) \rightarrow \mathbf{R}_+$ be continuous functions; let $(0 \neq)p, q, r, s$ are constants s.t $p \geq q$; $p \geq r$ and $p \geq s$. Let $K(\mathfrak{t}, s)$ be a weight function (defined in Lemma 3) s.t $\frac{\partial K(\mathfrak{t},s)}{\partial \mathfrak{t}} \geq 0$ for $\mathfrak{t} \geq s$ then (29) implies

$$\Theta(\mathfrak{t}) \leq \sqrt[p]{\tilde{a}(\mathfrak{t}) + \mathfrak{b}(\mathfrak{t}) \int_{\mathfrak{t}_0}^{\mathfrak{t}} (\mathfrak{z} \oplus \tilde{\mathfrak{Y}})(s)\, ds}, \tag{34}$$

where $\tilde{\mathfrak{Y}}$ and $\mathfrak{z}$ are as defined in Theorem 4.

Corollary 4. (Discrete case)

Assume $\mathbb{T} = \mathbb{Z}$ and $\tilde{a}, \mathfrak{b}, g, h, j, \Theta : \mathfrak{A} \to \mathbf{R}_+$; let $(0 \neq)p, q, r, s$ are constants s.t $p \geq q; p \geq r$ and $p \geq s$. Let $K(\mathfrak{t}, s)$ be a weight function as defined in lemma 3 s.t $K(\mathfrak{t}+1, s) - K(\mathfrak{t}, s) \geq 0$ for $\mathfrak{t} \geq s$ and $1 > \hat{\mathfrak{Y}}(\mathfrak{t})$, then (29)⇒

$$\Theta(\mathfrak{t}) \leq \sqrt[p]{\tilde{a}(\mathfrak{t}) + \mathfrak{b}(\mathfrak{t}) \sum_{s=\mathfrak{t}_0}^{\mathfrak{t}-1} (\tilde{\mathfrak{z}} \oplus \tilde{\mathfrak{Y}})(s)}, \tag{35}$$

provided

$$\hat{\mathfrak{Y}}(\mathfrak{t}) = K(\mathfrak{t}+1, \mathfrak{t})\tilde{\mathfrak{H}}(\mathfrak{t}) + \sum_{s=\mathfrak{t}_0}^{\mathfrak{t}-1} [K(\mathfrak{t}+1, s) - K(\mathfrak{t}, s)]\tilde{\mathfrak{H}}(s),$$

And $\tilde{\mathfrak{Y}}, \tilde{\mathfrak{H}}$ and $\mathfrak{z}$ are as defined in Theorem 4.

Note 1. Let $\tilde{a}, g, \Theta: \mathbb{T}^K \to \mathbf{R}$ be non-negative rd-continuous functions and $p = 1 = q$ s.t $\mathfrak{b}(\mathfrak{t}) = 1, h \equiv 0 \equiv k$.Then, (18) implies

$$\Theta(\mathfrak{t}) \leq \tilde{a}(\mathfrak{t}) + \int_{\mathfrak{t}_0}^{\mathfrak{t}} e_g(\mathfrak{t}, \sigma(\mathfrak{v}))g(\mathfrak{v})\tilde{a}(\mathfrak{v})\Delta\mathfrak{v}\,, \tag{36}$$

which is nothing except that [2, Theorem 6.4].

Corollary 5. Assume that g, h, j, Θ are non-negative rd-continuous functions and $(0 \neq)p, q, r, s$ are constant with $p \geq q$; $p \geq r$ and $p \geq s$. If $\rho \geq 0$ is a real constant, then (5) implies

$$\Theta(\mathfrak{t}) \leq \sqrt[p]{\rho + \int_{\mathfrak{t}_0}^{\mathfrak{t}} e_{\tilde{\mathfrak{F}}}(\mathfrak{t}, \sigma(\mathfrak{v}))\tilde{\mathfrak{D}}(\mathfrak{v})\Delta\mathfrak{v}}. \tag{37}$$

Proof. By using Theorem 1, (37) follows from (5).

Finally, to illustrate our main results we give an application to initial value dynamical equation. Let us examine the following IVP on time scales

$$[\Theta^p(\mathfrak{t})]^\Delta = \mathfrak{G}\left(\mathfrak{t}, \emptyset_q(\Theta(\mathfrak{t})), \emptyset_r(\Theta(\mathfrak{t})), \emptyset_s(\Theta(\mathfrak{t}))\right), \Theta^p(\mathfrak{t}_0) = \Theta_0, \tag{38}$$

where, $\emptyset_q(\zeta) = |\zeta|^q.\operatorname{sgn}\zeta$ and $\mathfrak{G}: \mathbb{T} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is a continuous functions.t

$$|\mathfrak{G}(\mathfrak{t}, \Theta^q(\mathfrak{t}), \Theta^r(\mathfrak{t}), \Theta^s(\mathfrak{t}))| \leq g(\mathfrak{t}).|\Theta(\mathfrak{t})|^q + h(\mathfrak{t}).|\Theta(\mathfrak{t})|^r + j(\mathfrak{t}).|\Theta(\mathfrak{t})|^s,$$

provided that g, h, j are non-negative rd-continuous functions on $\mathbb{T}^K$, then

$$\Theta^p(\mathfrak{t}) = C + \int_{\mathfrak{t}_0}^{\mathfrak{t}} \mathfrak{G}\left(\mathfrak{v}, \emptyset_q(\Theta(\mathfrak{v})), \emptyset_r(\Theta(\mathfrak{v})), \emptyset_s(\Theta(\mathfrak{v}))\right)\Delta\boldsymbol{\tau},$$

$$\Rightarrow |\Theta^p(\mathfrak{t})| \leq |C| + \int_{\mathfrak{t}_0}^{\mathfrak{t}} \left|\mathfrak{G}\left(\mathfrak{v}, \emptyset_q(\Theta(\mathfrak{v})), \emptyset_r(\Theta(\mathfrak{v})), \emptyset_s(\Theta(\mathfrak{v}))\right)\right| \Delta\mathfrak{v}$$

$$\leq |C| + \int_{\mathfrak{t}_0}^{\mathfrak{t}} \{g(\mathfrak{v}).|\Theta(\mathfrak{v})|^q + h(\mathfrak{v}).|\Theta(\mathfrak{v})|^r + j(\mathfrak{v}).|\Theta(\mathfrak{v})|^s\}\Delta\mathfrak{v}.$$

In fact, $\mathrm{C} = \Theta^{\mathrm{p}}(\mathfrak{t}_0)$, then Corollary 5 yields:

$$|\Theta(\mathfrak{t})| \leq \sqrt[\mathrm{p}]{\Theta^{\mathrm{p}}(\mathfrak{t}_0) + \int_{\mathfrak{t}_0}^{\mathfrak{t}} e_{\widetilde{\mathfrak{E}}}(\mathfrak{t}, \sigma(\mathfrak{r}))\widetilde{\mathfrak{D}}(\mathfrak{r})\Delta\mathfrak{r}},$$

provided that $\Theta(\mathfrak{t})$ is a solution of (38).

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