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Research Article

Dynamics of Bosons in Two Wells of an External Trap

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Abstract: We study the dynamics of the atoms of Bose-Einstein condensate in a double well potential by deriving the two mode model for the well known Gross-Pitaevskii equation. The symmetric and anti-symmetric basis functions have been used for the development of the two mode model. The stability of these basis functions has been investigated. It is found that both solutions are stable. The time dependent Gross-Pitaevskii equation and the two mode approximations are solved numerically and then compare the results. It is shown that the solution obtained from two mode model demonstrates good agreement with the solution of the Gross-Pitaevskii equation.

Keywords: Bose-Einstein condensate, double well potential, Gross-Pitaevskii equation, Josephson tunneling.

1. INTRODUCTION

Multi-component Hamiltonian systems have gained a lot of attention in the past few years due to the development of theoretical and experimental results in coupled Bose-Einstein condensates (BECs) [1] and coupled nonlinear optical systems [2]. In BECs, mixtures of distinct spin states of rubidium [3,4] and sodium [5] were created experimentally. The two components BECs with different atomic species were also formed in laboratories, e.g. potassium-rubidium [6] and lithium-cesium [7]. Due to these experiments, many theoretical studies have been done to investigate the ground state solutions [8, 9] and the small amplitude excitations [8, 10, 11]. Several other nonlinear structures were also formed such as domain walls [12, 13, 14], dark-dark and dark-bright solitons [15, 16], vortex rings [17] and so on.

In 1962, Josephson presented the idea of electron tunneling between two superconductors which were separated by a thin insulator [18]. The effect of tunneling was named as Josephson tunneling. Since weak coupling is the only requirement for the effect of Josephson tunneling, it was thought that the weakly linked macroscopic quantum samples may admit such tunneling. In BECs, such tunneling was predicted by Smerzi and coresearchers [19, 20, 21]. The experimental realization of Josephson tunneling for a single [22, 23] and array of short Bose-Josephson junction [24] were made. Kaurov and Kuklov [25, 26] extended the idea of Bose-Josephson junction to long Bose-Josephson junction. This junction was analogues to long superconducting Josephson junction. They proposed that atomic vortices could be seen in weakly coupled BECs and that these vortices are similar to Josephson fluxons in superconducting long Josephson junction [27]. Further it was shown that due to the presence of a critical coupling, atomic Josephson vortices can be transformed to a dark soliton and vice versa. Josephson tunneling of dark solitons in a double-well potential was studied in [28].

The dynamics of Josephson tunneling in BECs was explained using a two-mode approximation in [29, 30, 31, 32, 33]. The coupled-mode equations were modified and improved in [34]. In this paper, we study the validity of the coupled-mode equations. The stability of the basis functions

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(which are used for the approximations) is also studied by investigating the eigenvalues structures.

2. MATHEMATICAL MODEL AND DESCRIPTION

Let us consider the atoms of BECs at very low temperature that is nearly at zero Kelvin. If U(x, t) is the wave function of the atoms of BECs which are interacting with each other, then the equation that describes the dynamics of atoms of BECs is the famous Gross-Pitaevskii (GP) equation. The GP equation in the dimensionless form is given as

$$i\frac{\partial U}{\partial t} = -\frac{1}{2}\frac{\partial^2 U}{\partial x^2} + \sigma|U|^2U + VU, \qquad (1)$$

where x and t are the space and time variables, σ is the nonlinearity coefficient and $i = \sqrt{-1}$. V is the external potential which in our case is a combination of a harmonic potential with Gaussian barrier and is given as

$$V = \frac{1}{2}\,\Omega^2 x^2 + Ae^{-(x/b)^2}, \qquad (2)$$

with Ω representing the frequency of oscillation and A and b are respectively the height and width of the Gaussian barrier.

To obtain the two mode approximations, we use a pair of real symmetric and real antisymmetric functions which are denoted by φ_e and φ_o respectively. It is easy to see that if we substitute $U(x,t) = \sqrt{N}e^{-it\gamma_{e,o}}\varphi_{e,o}(x)$ into eq.(1), (where γ_e and γ_o are constants which represent the chemical potential in each well) the basis functions φ_e and φ_o will satisfy the following steady state equations

$$\gamma_e\varphi_e = -\frac{1}{2}\frac{d^2\varphi_e}{dx^2} + V\varphi_e + \sigma_1{\varphi_e}^3, \qquad (3)$$

$$\gamma_o\varphi_o = -\frac{1}{2}\frac{d^2\varphi_o}{dx^2} + V\varphi_o + \sigma_1{\varphi_o}^3, \qquad (4)$$

where $\sigma_1 = \sigma N$.

To seek the solution φ_e numerically, we discretize eq. (3) and approximate the second order derivative by the central difference approximation so that a system of nonlinear algebraic equations is obtained. The system can be solved using Newton's method with the Neumann boundary conditions to obtain the solution φ_e which is shown in Fig. 1. Similarly, from eq. (4) we get the solution φ_o and is depicted in Fig. 2. The stability of these basis functions will be discussed later.

One can now express the wave function $U(x,t)$ as [30]

$$U(x,t) = \sqrt{N}[\varphi_1(x)U_1(t) + \varphi_2(x)U_2(t)]. \qquad (5)$$

Here, $\varphi_1(x) = \left(\frac{\varphi_e+\varphi_o}{\sqrt{2}}\right)$ and $\varphi_2(x) = \left(\frac{\varphi_e-\varphi_o}{\sqrt{2}}\right)$ with $\int_{-\infty}^{\infty}\varphi_i\varphi_j dx = \delta_{ij}, i,j = e,o,$ and δ denotes the Kronecker delta function. N represents the number of boson atoms such that $\int_{-\infty}^{\infty}|U(x,t)|^2 = N$. So, we have

$$U = \sqrt{N}\left[U_1\left(\frac{\varphi_e+\varphi_o}{\sqrt{2}}\right) + U_2\left(\frac{\varphi_e-\varphi_o}{\sqrt{2}}\right)\right].$$

Substituting this value in (1), we get

$$\sqrt{N}\left[\dot{U}_1\left(\frac{\varphi_e+\varphi_o}{\sqrt{2}}\right) + \dot{U}_2\left(\frac{\varphi_e-\varphi_o}{\sqrt{2}}\right)\right]$$

$$= -\frac{\sqrt{N}}{2}\left[U_1\left(\frac{\ddot{\varphi}_e+\ddot{\varphi}_o}{\sqrt{2}}\right) + U_2\left(\frac{\ddot{\varphi}_e-\ddot{\varphi}_o}{\sqrt{2}}\right)\right]$$

$$+ V\sqrt{N}\left[U_1\left(\frac{\varphi_e+\varphi_o}{\sqrt{2}}\right) + U_2\left(\frac{\varphi_e-\varphi_o}{\sqrt{2}}\right)\right]$$

$$+ \sigma N\sqrt{N}\left[U_1^2\left(\frac{\varphi_e+\varphi_o}{\sqrt{2}}\right)^2 + U_2^2\left(\frac{\varphi_e-\varphi_o}{\sqrt{2}}\right)^2\right.$$

$$\left.+ 2U_1U_2\left(\frac{\varphi_e+\varphi_o}{\sqrt{2}}\right)\left(\frac{\varphi_e-\varphi_o}{\sqrt{2}}\right)\right]\left[\bar{U}_1\left(\frac{\varphi_e+\varphi_o}{\sqrt{2}}\right)\right.$$

$$\left.+ \bar{U}_2\left(\frac{\varphi_e-\varphi_o}{\sqrt{2}}\right)\right],$$

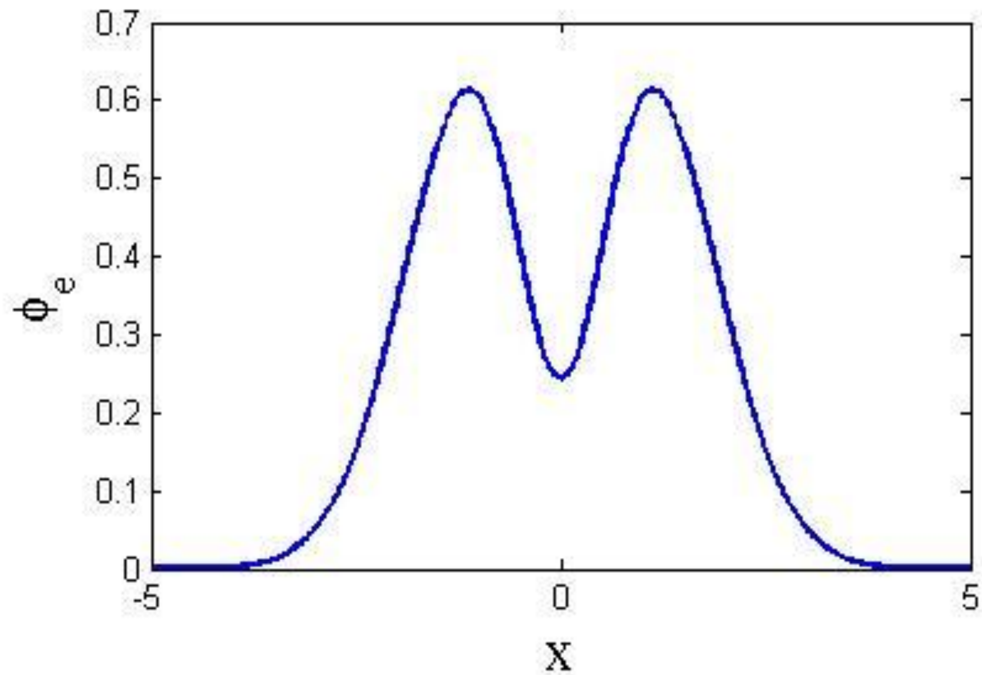

Fig. 1. Numerically obtained solution $\boldsymbol{\varphi_e}$ for the parameter values $\boldsymbol{\sigma_1}$ = 1, $\boldsymbol{\gamma_e}$ = 1.

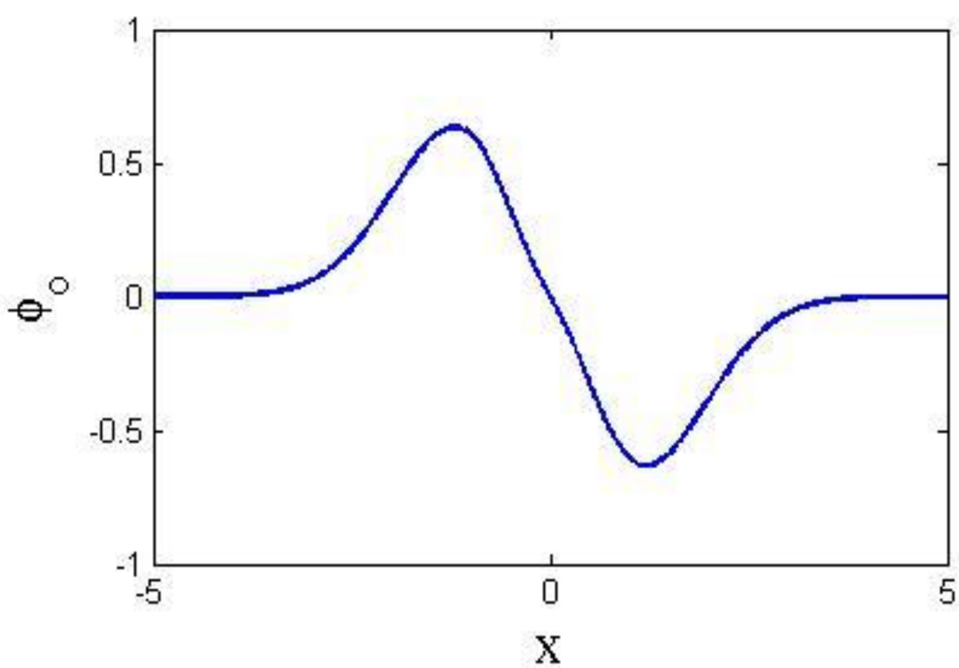

Fig. 2. Numerically obtained solution $\boldsymbol{\varphi_o}$ for the parameter values $\boldsymbol{\sigma_1}$ = 1, $\boldsymbol{\gamma_e}$ = 1.

where bar represents the complex conjugate, primes are used for the second order derivative with respect to x and dot for derivative with respect to time.

$$i\left[\dot{U}_1(\varphi_e+\varphi_o)+\dot{U}_2(\varphi_e-\varphi_o)\right]$$
$$=-\frac{1}{2}\left[U_1\left(\varphi''_e+\varphi''_o\right)+U_2\left(\varphi''_e-\varphi''_o\right)\right]+V\left[U_1\left(\varphi''_e+\varphi''_o\right)+U_2\left(\varphi''_e-\varphi''_o\right)\right]$$
$$+\sigma_1\left[U_1^2\left(\frac{\varphi_e^2+\varphi_o^2+2\varphi_e\varphi_o}{2}\right)+U_2^2\left(\frac{\varphi_e^2+\varphi_o^2-2\varphi_e\varphi_o}{2}\right)\right.$$
$$\left.+2U_1U_2\left(\frac{\varphi_e^2-\varphi_o^2}{2}\right)\right]\left[\bar{U}_1\left(\frac{\varphi_e+\varphi_o}{\sqrt{2}}\right)+\bar{U}_2\left(\frac{\varphi_e-\varphi_o}{\sqrt{2}}\right)\right].$$

$$i\left[\dot{U}_1(\varphi_e+\varphi_o)+\dot{U}_2(\varphi_e-\varphi_o)\right]$$
$$=(U_1+U_2)\left(-\frac{1}{2}\varphi''_e+V\varphi_e\right)+(U_1-U_2)\left(-\frac{1}{2}\varphi''_o+V\varphi_o\right)$$
$$+\frac{\sigma_1}{2}[U_1^2(\varphi_e^2+\varphi_o^2+2\varphi_e\varphi_o)+U_2^2(\varphi_e^2+\varphi_o^2-2\varphi_e\varphi_o)$$
$$+2U_1U_2(\varphi_e^2-\varphi_o^2)][\bar{U}_1(\varphi_e+\varphi_o)+\bar{U}_2(\varphi_e-\varphi_o)].$$

Using eq.(3) and eq.(4), we get

$$i\left[\dot{U}_1(\varphi_e+\varphi_o)+\dot{U}_2(\varphi_e-\varphi_o)\right]$$
$$=(U_1+U_2)(\gamma_e\varphi_e-\sigma_1{\varphi_e}^3)+(U_1-U_2)(\gamma_o\varphi_o-\sigma_1{\varphi_o}^3)$$
$$+\frac{\sigma_1}{2}[|U_1|^2U_1(\varphi_e^3+3\varphi_e\varphi_o^2+3\varphi_e^2\varphi_o+\varphi_o^3)+U_1^2\bar{U}_2(\varphi_e^3-\varphi_e\varphi_o^2+\varphi_e^2\varphi_o-\varphi_o^3)$$
$$+\bar{U}_1U_2^2(\varphi_e^3-\varphi_e\varphi_o^2-\varphi_e^2\varphi_o+\varphi_o^3)+|U_2|^2U_2(\varphi_e^3+3\varphi_e\varphi_o^2-3\varphi_e^2\varphi_o-\varphi_o^3)$$
$$+2|U_1|^2U_2(\varphi_e^3-\varphi_e\varphi_o^2+\varphi_e^2\varphi_o-\varphi_o^3)+2U_1|U_2|^2(\varphi_e^3-\varphi_e\varphi_o^2-\varphi_e^2\varphi_o+\varphi_o^3)] \quad (6)$$

Multiplying both sides by $(\varphi_e+\varphi_o)$ and integrating with respect to x from $-\infty$ to ∞ and using $\int_{-\infty}^{\infty}\varphi_i\varphi_j dx=\delta_{ij}, i,j=e,o$, we obtain

$$2i\dot{U}_1=(U_1+U_2)[\gamma_e-\mu_{ee}]+(U_1-U_2)[\gamma_o-\mu_{oo}]$$
$$+\frac{1}{2}[\mu_{ee}(|U_1|^2U_1+U_1^2\bar{U}_2+\bar{U}_1U_2^2+|U_2|^2U_2+2|U_1|^2U_2+2U_1|U_2|^2)$$
$$+\mu_{00}(|U_1|^2U_1-U_1^2\bar{U}_2+\bar{U}_1U_2^2-|U_2|^2U_2-2|U_1|^2U_2+2U_1|U_2|^2)$$
$$+\mu_{eo}(6|U_1|^2U_1-2\bar{U}_1U_2^2-4U_1|U_2|^2)],$$

where $\mu_{ij}=\sigma_1\int\varphi_i^2(x)\varphi_j^2(x)dx,\ i,j=e,o$, and the integrals with odd powers of φ_e and φ_o will be zero. Since $\int_{-\infty}^{\infty}|U(x,t)|^2=N\Rightarrow|U_1|+|U_2|=1$. Using this equation, the above equation can be written as

$$2i\dot{U}_1 = (U_1+U_2)[\gamma_e-\mu_{ee}] + (U_1-U_2)[\gamma_o-\mu_{oo}]$$
$$+\frac{1}{2}[\mu_{ee}\{2(U_1+U_2)-|U_1|^2U_1-|U_2|^2U_2+U_1^2\bar{U}_2+\bar{U}_1U_2^2\}$$
$$+\mu_{oo}\{2(U_1-U_2)-|U_1|^2U_1+|U_2|^2U_2-U_1^2\bar{U}_2+\bar{U}_1U_2^2\}$$
$$+\mu_{eo}(-4U_1+10|U_1|^2U_1-2\bar{U}_1U_2^2)].$$

$$i\dot{U}_1 = \left[\frac{\gamma_e+\gamma_o}{2}-\left(\frac{\mu_{ee}+\mu_{oo}}{2}\right)+\frac{1}{4}(2\mu_{ee}+2\mu_{oo}-4\mu_{eo})\right]U_1+\frac{1}{4}(-\mu_{ee}-\mu_{oo}+10\mu_{eo})|U_1|^2U_1$$
$$+\frac{1}{4}(\mu_{ee}-\mu_{oo})U_1^2\bar{U}_2+\left[\frac{\gamma_e-\gamma_o}{2}-\left(\frac{\mu_{ee}-\mu_{oo}}{2}\right)+\frac{1}{4}(2\mu_{ee}-2\mu_{oo})\right]U_2$$
$$+\frac{1}{4}(-\mu_{ee}+\mu_{oo})|U_2|^2U_2+\frac{1}{4}(\mu_{ee}+\mu_{oo}-2\mu_{eo})\bar{U}_1U_2^2$$

$$i\dot{U}_1 = \left[\frac{\gamma_e+\gamma_o}{2}-\mu_{eo}\right]U_1+\left(\frac{10\mu_{eo}-\mu_{ee}-\mu_{oo}}{4}\right)|U_1|^2U_1+\left(\frac{\mu_{ee}-\mu_{oo}}{4}\right)U_1^2\bar{U}_2+\left(\frac{\gamma_e-\gamma_o}{2}\right)U_2$$
$$-\left(\frac{\mu_{ee}-\mu_{oo}}{4}\right)|U_2|^2U_2+\left(\frac{\mu_{ee}+\mu_{oo}-2\mu_{eo}}{4}\right)\bar{U}_1U_2^2$$

$$i\dot{U}_1 = \left(B+C|U_1|^2+\frac{\Delta\mu}{4}U_1\bar{U}_2\right)U_1+\left(\frac{\Delta\gamma}{2}-\frac{\Delta\mu}{4}|U_2|^2+D\bar{U}_1U_2\right)U_2. \quad (7)$$

Similarly, multiplying both sides of eq.(6) by $(\varphi_e-\varphi_o)$ and integrating with respect to x from -∞ to ∞ and following the same procedure as before, we obtain

$$i\dot{U}_2 = \left(B+C|U_2|^2+\frac{\Delta\mu}{4}U_2\bar{U}_1\right)U_2+\left(\frac{\Delta\gamma}{2}-\frac{\Delta\mu}{4}|U_1|^2+D\bar{U}_2U_1\right)U_1, \quad (8)$$

where

$$B=\frac{\gamma_e+\gamma_o}{2}-\mu_{eo},$$
$$C=\frac{10\mu_{eo}-\mu_{ee}-\mu_{oo}}{4},$$
$$D=\frac{\mu_{ee}+\mu_{oo}-2\mu_{eo}}{4},$$
$$\Delta\mu=\mu_{ee}-\mu_{oo},$$
$$\Delta\gamma=\gamma_e-\gamma_o.$$

Thus, eq. (7) and eq. (8) represent a system of two ordinary differential equations of first order. These equations describe the dynamics of atoms of BEC in each well of the external potential. We solve this system of differential equations using Runge-Kutta method of order 4 to get U_1 and U_2. Substituting these solutions U_1 and U_2 into eq. (5), we obtain the solution U which is shown in Fig. 3 by dotted line. We then solve eq. (1) numerically and the solution obtained is shown in Fig. 3 by solid line. Figure shows that the solution obtained through two mode model is very close to the numerical solution of the GP equation and hence justifies the validity of the two mode model.

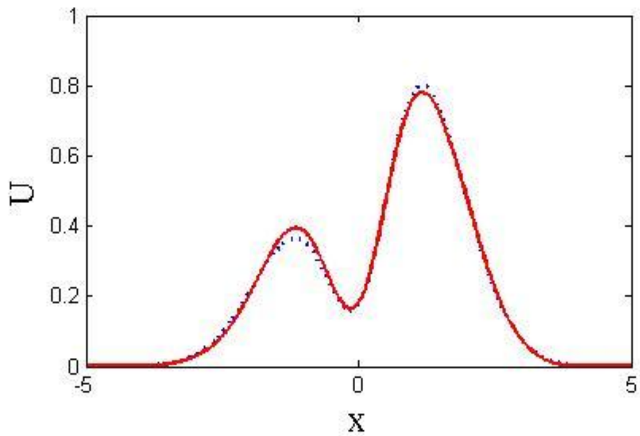

Fig. 3. Comparison of the two solutions. The solid curve represents the solution of eq. (1) while dotted curve is the solution obtained through two mode approximation. The graph shows very good agreement between the two solutions.

3. STABILITY OF BASIS FUNCTIONS

Let us discuss the stability of the solutions φ_e and φ_o. To do so, we first substitute

$$U(x,t) = e^{-it\gamma_e}\,\widetilde{U}(x,t)$$

into eq. (1) to obtain

$$i\frac{\partial \widetilde{U}}{\partial t} = -\frac{1}{2}\frac{\partial^2 \widetilde{U}}{\partial x^2} + \sigma|\widetilde{U}|^2\widetilde{U} + V\widetilde{U} - \gamma_e\widetilde{U}. \tag{9}$$

We now perturb the solution φ_e by adding a small perturbation $\eta(x,t)$ in it, i.e.

$$\widetilde{U}(x,t) = \varphi_e(x) + \eta(x,t), \tag{10}$$

where we assume that the perturbation η is so small that its squares and higher power terms can be neglected. Substituting the value of $\widetilde{U}(x,t)$ from eq. (10) into eq. (9) and using eq. (3), we get

$$i\frac{\partial \eta}{\partial t} = -\frac{1}{2}\frac{\partial^2 \eta}{\partial x^2} + (2\sigma\,{\varphi_e}^2 + V - \gamma_e)\eta + \sigma\,{\varphi_e}^2\bar{\eta}. \tag{11}$$

Taking complex conjugate of eq. (11) to obtain

$$-i\frac{\partial \bar{\eta}}{\partial t} = -\frac{1}{2}\frac{\partial^2 \bar{\eta}}{\partial x^2} + (2\sigma\,{\varphi_e}^2 + V - \gamma_e)\bar{\eta} + \sigma\,{\varphi_e}^2\eta. \tag{12}$$

For simplicity, we denote η by α and $\bar{\eta}$ by β so that eq. (11) and eq. (12) can be written as

$$i\frac{\partial \alpha}{\partial t} = -\frac{1}{2}\frac{\partial^2 \alpha}{\partial x^2} + (2\sigma\,{\varphi_e}^2 + V - \gamma_e)\alpha + \sigma\,{\varphi_e}^2\beta. \tag{13}$$

$$i\frac{\partial \beta}{\partial t} = \frac{1}{2}\frac{\partial^2 \beta}{\partial x^2} - (2\sigma\,{\varphi_e}^2 + V - \gamma_e)\beta - \sigma\,{\varphi_e}^2\alpha. \tag{14}$$

Eq. (13) and eq. (14) can also be written as

$$-\frac{1}{2}\frac{\partial^2 \alpha}{\partial x^2} + (2\sigma\,{\varphi_e}^2 + V - \gamma_e)\alpha + \sigma\,{\varphi_e}^2\beta = \lambda\alpha, \tag{15}$$

$$\frac{1}{2}\frac{\partial^2 \beta}{\partial x^2} - (2\sigma\,{\varphi_e}^2 + V - \gamma_e)\beta - \sigma\,{\varphi_e}^2\alpha = \lambda\beta. \tag{16}$$

Discretizing eq. (15) and eq. (16) with step size h and using the Neumann boundary conditions yield an eigenvalue problem $AX = \lambda X$ with eigenvalues λ and

$$A = \begin{bmatrix} A_1 & -D_1 \\ D_1 & -A_1 \end{bmatrix},$$

where

$$A_1 = \begin{bmatrix} \frac{-1}{h^2} - \left(2\sigma\, {\varphi_{e,1}}^2 + V - \gamma_e\right) & \frac{1}{2h^2} & 0 & 0 & \dots & \frac{1}{2h^2} \\ \frac{1}{2h^2} & \frac{-1}{h^2} - \left(2\sigma\, {\varphi_{e,2}}^2 + V - \gamma_e\right) & \frac{1}{2h^2} & 0 & \dots & 0 \\ 0 & \frac{1}{2h^2} & \frac{-1}{h^2} - \left(2\sigma\, {\varphi_{e,3}}^2 + V - \gamma_e\right) & \frac{1}{2h^2} & \dots & 0 \\ & \ddots & & \ddots & & \ddots \\ & \ddots & & \ddots & & \ddots \\ \frac{1}{2h^2} & 0 & \dots & \frac{1}{2h^2} & & \frac{-1}{h^2} - \left(2\sigma\, {\varphi_{e,N}}^2 + V - \gamma_e\right) \end{bmatrix},$$

$$D_1 = \begin{bmatrix} \sigma\, {\varphi_{e,1}}^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma\, {\varphi_{e,2}}^2 & 0 & \dots & 0 \\ 0 & 0 & \sigma\, {\varphi_{e,3}}^2 & \dots & 0 \\ & & \ddots & & \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & \sigma\, {\varphi_{e,N}}^2 \end{bmatrix}.$$

The solution will be stable if all eigenvalues are real. We find the eigenvalues of above matrix A for the solution φ_e shown in Fig. (1). It is found that the imaginary parts of all eigenvalues are zero, i.e. all eigenvalues are real as they all are lying on the horizontal axis as shown in Fig. (4). This shows that solution φ_e is stable.

Following the same procedure as above, we found the eigen values structure for the solution φ_o depicting that the solution φ_o is also stable.

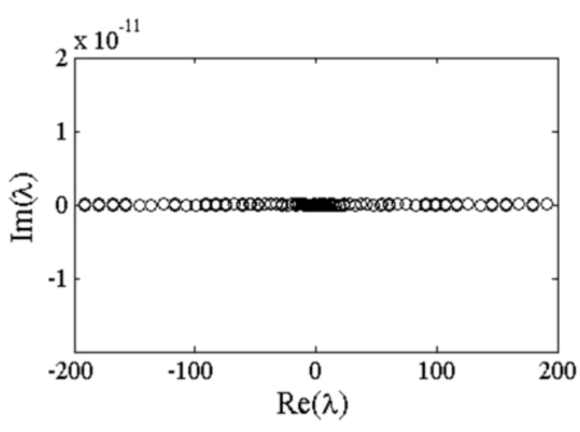

Fig. 4 . The eigenvalues structure for the solution $\boldsymbol{\varphi_e}$. All eigenvalues are lying on the horizontal axis showing the stability of the solution.

4. CONCLUSIONS

In this paper, we have presented the derivation of a two mode model using a symmetric and an anti-symmetric basis functions. It was found that the solutions obtained through two mode model and that from the time-dependent GP equation are very close to each other and validated the two mode model. The two mode model can be used to describe the dynamics of bosons in each well of the external potential. We also studied the stability of the basis functions by perturbing the solutions. Both solutions were found to be stable.

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