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Research Article

New Integral Inequalities of the Hermite-Hadamard Type through Invexity

Shahid Qaisar[1], Muhammad Muddassar[2]* and M. Iqbal[3]

[1]College of Mathematics and Statistics, Chongqing University, Chongqing, 401331, P. R. China
[2] Department of Mathematics, University of Engineering and Technology, Taxila, Pakistan
[3] Department of Mathematics, University of Engineering and Technology, Lahore, Pakistan

Abstract: In this paper, we establish some new inequalities of Hermite-Hadamard type based on preinvexity for differentiable mapping that are linked with the illustrious Hermite-Hadamard type inequality for mappings whose derivatives are preinvexity. Also a parallel development is made base on concavity. Applications to some special means of real numbers are found. Also applications to numerical integration are provided. This contributes to new better estimates than presented already.

Keywords: Hermite-Hadamard inequality; preinvex function;Hölder Integral Inequality; power mean inequality.

AMS Subject Classification(2000): 26D15; 26D10

1. INTRODUCTION

One of the cornerstones of analysis is the Hadamard inequality, if $[a,b]$ with $a<b$ is a real interval and $f:[a,b]\to\mathbb{R}$a convex function, then

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{2} \qquad (1.1)$$

Over the last decade this has been extended in a number of ways. An important question is the estimating the difference between the middle and rightmost term in the (1.1). In recent years, this classical inequality has been improved and generalized in a number of ways and a large number of research papers have been written on this inequality [see 5-8, 10-13, 15-16, 20] and the references therein. The following identity is a useful building block.

In recent years, lots of efforts have been made by many mathematicians to generalize the classical convexity. For example, Ben-Israel and Mond [4] discuss the role of invexity in optimization; some more researchers extended the idea of this generalization [1, 3, 14, 15, 17-19, 21].These studies include among others the work of Hanson [8], Weir and Mond [18]. Noor [13, 14] has studied basic properties of the preinvex functions and their role in optimization, variational inequalities and equilibrium problems. Hanson in [9], introduced invex functions as a significant generalization of convex functions. Ben-Israel and Mond [4] gave the concept of preinvex function which is special case of invexity. Let us first recall the definition of pre-invexity and some related results.

Let K be a closed set $\mathbb{R}^n$and let $f:K\to\mathbb{R}$and $\eta:K\times K\to\mathbb{R}$be continuous functions. Let $x\in K$, then the set K is said to be invex at $x$with respect to $\eta(.,.)$.

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*Corresponding author. E-mail address: malik.muddassar@gmail.com.

If $x + t\eta(y,x) \in K.\ \forall\, x, y \in K, t \in [0,1]$

$K$is said to be invex set with respect to η if $K$is invex at each $x \in K$. The invex set K is also called a η-connected set.

Definition 1[16]**.** The function f on the invex set $K$is said to be preinvex with respect to η, if

$$f\big(x + t\eta(y,x)\big) \le (1-t)f(x) + t\, f(y), \forall\, x, y \in K, t \in [0,1]$$

The function f is said to be preconcave if and only if $-f$ is preinvex.

It is to be noted that every convex function is preinvex with respect to the map $\eta(\mathrm{x},\mathrm{y}) = \mathrm{x} - \mathrm{y}$ but the converse is not true [17-18] and [21].

Dragomir and Agrawal [7] established the following result connected with the Hadamard inequality, as well as to apply them for some elementary inequalities for real numbers and numerical integration.

Lemma 1. Let $f: I^o \subseteq \mathbb{R} \to E$ be differentiable function on I^o , $a, b \in I^o$, with $a < b$. If $f' \in L^1[a,b],\ then$

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx$$

$$= \frac{(b-a)}{2}\int_0^1\int_0^1 \begin{pmatrix} f'(ta + (1-t)b) - \\ f'(ua + (1-u)b) \end{pmatrix}(u-t)dtdu.$$

Definition 2 [10]**.** A function $f: [0,\infty) \to \mathbb{R}$ is said to be s-convex or $f$belongs to the class K_s^iif

$$f(\mu x + \nu y) \le \mu^s\, f(x) + \nu^s\, f(y)$$

holds for all $x, y \in [0.,\infty)$ and $u, v \in [0,1]$, for some fixed$s \in (0,1]$.

Note that, if $\mu^\alpha + \nu^\alpha = 1$, the above class of convex functions is called s-convex functions in first sense and represented by K_s^1 and if $\mu + \nu = 1$ the above class is called s-convex in second sense and represented by K_s^2

It may be noted that every 1-convex function is convex.

This paper is in the continuations of [12], which provides more general and refine results aspresented in [12].This paper is organized as follows: after Introduction, we discuss some new s-preinvex-Hermite Hadamard type inequalities for differentiable function in Section 2, and in Section 3 we give some applications for some special means of real numbers of the results formulated in Section 2. In Section 4 we gave some applications to quadrature formulae. Finally conclude our results and applications in Section 5.

2. MAIN RESULTS

Before proceeding towards our main theorem we need the following equality which is the generalization of Lemma 1 for invex sets. We begin with the following Lemma.

Lemma 2[3]**.** Let$K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta: K \times K \to \mathbb{R}$ and suppose $f: K \to \mathbb{R}$ be differentiable function. If $|\boldsymbol{f}'|$ is integrable on the η-path $P_{ac}, c = a + \eta(b,a)$,then following inequality holds:

$$\left|\frac{f(a) + f\big(a+\eta(b,a)\big)}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\le \frac{\eta(b,a)}{2}\int_0^1 |1-2t|\, \boldsymbol{f}'\big(a + t\eta(b,a)\big)dt$$

Theorem 2. Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta: K \times K \to \mathbb{R}$suppose that $f: K \to \mathbb{R}$be differentiable function. If$|\boldsymbol{f}'|^{\mathbf{q}}$ is preinvex on$K$, then the following inequality holds:

$$\left|\frac{f(a) + f\big(a+\eta(b,a)\big)}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\le \frac{\eta(b,a)}{2^s}\left[\frac{s.2^s}{(s+1)(s+2)}\right][|\boldsymbol{f}'(a)| + |\boldsymbol{f}'(b)|] \quad (2.6)$$

For $\mathrm{q} \ge 1$and every$\mathrm{a}, \mathrm{b} \in \mathrm{K}$with $\eta(b,a) \ne 0$.

Proof. By using Lemma 2, we get

$$\left|\frac{f(a) + f\big(a+\eta(b,a)\big)}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\le \frac{\eta(b,a)}{2}\int_0^1 |1-2t|\, \boldsymbol{f}'\big(a + t\eta(b,a)\big)dt$$

since$|\boldsymbol{f}'|$ is s-preinvex on K for every$\mathrm{a}, \mathrm{b} \in \mathrm{K}$and $\mathrm{t} \in [0,1]$.we have

$$\boldsymbol{f}'\big(a+t\eta(b,a)\big) \le (1-t)^s|\boldsymbol{f}'(a)| + (t)^s|\boldsymbol{f}'(b)|.$$

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\le \frac{\eta(b,a)}{2}\int_0^1 |1-2t|\,[(1-t)^s|\boldsymbol{f}'(a)| + (t)^s|\boldsymbol{f}'(b)|]dt(2.7)$$

Where we use the fact

$$\int_0^1 |1-2t|\,(1-t)^s dt = \int_0^1 |1-2t|\,t^s dt = \left[\frac{s.2^s}{(s+1)(s+2)}\right] (2.8)$$

By (2.8), and (2.7),we get (2.6).

Theorem 3. Let the assumption of Theorem 2 are satisfied with $\mathrm{p} > 1$,such that $\mathrm{q} = \frac{\mathrm{p}}{\mathrm{p}-1}$ if the mapping $|\boldsymbol{f}'|^{\mathbf{q}}$is concave on $[a,b]$.then,

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\le \frac{\eta(b,a)}{2}(p+1)^{\frac{1}{\mathrm{p}}}\left|\boldsymbol{f}'\frac{a+b}{2}\right|. \quad (2.9)$$

Proof. By using Lemma 2, we obtain

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \le \frac{\eta(b,a)}{2}\int_0^1 |1-2t|\,\boldsymbol{f}'\big(a+t\eta(b,a)\big)dt \quad (2.10)$$

By applying Hӧlder'sinequality on the right side of (2.10). we have**,**

$$\int_0^1 |1-2t|\,\boldsymbol{f}'\big(a+t\eta(b,a)\big)dt \le$$

$$\left(\int_0^1 |1-2t|^p dt\right)^{\frac{1}{\mathrm{p}}}\left(\int_0^1 \left|\boldsymbol{f}'\big(+t\eta(b,a)\big)\right|^q dt\right)^{\frac{1}{\mathrm{q}}} \quad (2.11)$$

Here

$$\int_0^1 |1-2t|^p dt = \int_0^{\frac{1}{2}} (1-2t)^p\,dt = \int_{\frac{1}{2}}^1 (2t-1)^p\,dt = \frac{1}{p+1}, (2.12)$$

Since$|\boldsymbol{f}'|^{\mathbf{q}}$isconcave, by applying Jensen's integral inequality on the second integral of R.H.S of (2.11.) we have

$$\int_0^1 \left|\boldsymbol{f}'\big(a+t\eta(b,a)\big)\right|^q dt \le \left(\int_0^1 t^0 dt\right)$$

$$\left|\frac{\int_0^1 \left|\boldsymbol{f}'\big(a+t\eta(b,a)\big)\right|^q dt}{\int_0^1 t^0 dt}\right| = \left|\boldsymbol{f}'\frac{a+b}{2}\right|^q \quad (2.13)$$

By (2.10), and (2.12),and (2.13)we get (2.9).

Theorem 4. Let the assumption of Theorem 2 are satisfied with $\mathrm{p} > 1$,such that $\mathrm{q} = \frac{\mathrm{p}}{\mathrm{p}-1}$ if the mapping $|\boldsymbol{f}'|^{\mathbf{q}}$is s-preinvex on $[a,b]$.then,

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\le \frac{\eta(b,a)}{2(p+1)^{\frac{1}{\mathrm{p}}}}\left(\frac{|\boldsymbol{f}'(a)|^q + |\boldsymbol{f}'(b)|^q}{s+1}\right)^{\frac{1}{\mathrm{q}}}. (2.14)$$

Proof. By using Lemma 2, we obtain

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\le \frac{\eta(b,a)}{2}\int_0^1 |1-2t|\,\boldsymbol{f}'\big(a+t\eta(b,a)\big)dt \quad (2.15)$$

By applying Hӧlder's inequality on the right side of (2.15). we have**,**

$$\int_0^1 |1-2t|\,\boldsymbol{f}'\big(a+t\eta(b,a)\big)dt \le$$

$$\left(\int_0^1 |1-2t|^p dt\right)^{\frac{1}{\mathrm{p}}}\left(\int_0^1 \left|\boldsymbol{f}'\big(a+t\eta(b,a)\big)\right|^q dt\right)^{\frac{1}{\mathrm{q}}} (2.16)$$

since $|\boldsymbol{f}'|$ is s-preinvex on K for every$\mathrm{a},\mathrm{b} \in$ K$\mathrm{and}\ \mathrm{t} \in [0,1]$. we have

$$\left|\boldsymbol{f}'\big(a+t\eta(b,a)\big)\right|^q$$

$$\le (1-t)^s|\boldsymbol{f}'(a)|^q + (t)^s|\boldsymbol{f}'(b)|^q.$$

And

$$\int_0^1 |1-2t|^p\,dt = \int_0^{\frac{1}{2}} (1-2t)^p\,dt$$

$$= \int_{\frac{1}{2}}^1 (2t-1)^p\,dt = \frac{1}{p+1},$$

$$\int_0^1 \left|\boldsymbol{f}'\big(a+t\eta(b,a)\big)\right|^q dt =$$

$$\frac{|\boldsymbol{f}'(a)|^q + |\boldsymbol{f}'(b)|^q}{s+1} \qquad (2.17)$$

By (2.16), and (2.17), we get (2.14).

Corollary 5. From theorem 4 the assumption of Theorem 2 are satisfied with $p > 1$,such that $q = \frac{p}{p-1}$ if the mapping $|\boldsymbol{f}'|^q$is s-preinvex on $[a, b]$.then,

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\leq \frac{\eta(b,a)}{2(p+1)^{\frac{1}{p}}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}} |\boldsymbol{f}'(a)| + |\boldsymbol{f}'(b)|.$$

Proof. The above inequality is obtained by using the fact $\sum_{i=1}^n(\alpha_i+\beta_i)^k \leq \sum_{i=1}^n(\alpha_i)^k + \sum_{i=1}^n(\beta_i)^k$for $k \in (0,1)$ with $0 \leq \frac{p}{p-1}$, for $p > 1$.

Theorem 6. Let the assumption of Theorem 2 are satisfied with $p > 1$,such that $q = \frac{p}{p-1}$ if the mapping $|\boldsymbol{f}'|^q$is s-preconcave on $[a, b]$.then,

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \left|\boldsymbol{f}'\frac{a+b}{2}\right|^q \leq$$

$$\frac{\eta(b,a)}{2}(p+1)^{\frac{1}{p}}.2^{\frac{s-1}{q}}\left|\boldsymbol{f}'\frac{a+b}{2}\right|^q \quad (2.18)$$

Proof. We proceed similarly as in Theorem 4. By s-preconcavity of $|\boldsymbol{f}'|^q$we obtain

$$\int_0^1 \left|\boldsymbol{f}'\big(a+t\eta(b,a)\big)\right|^q dt = 2^{\frac{s-1}{q}}\left|\boldsymbol{f}'\frac{a+b}{2}\right|^q. \quad (2.19)$$

Now (2.18) immediately follows from Theorem 1.

Theorem 7. Let the assumption of Theorem 4 are satisfied, we have another result:

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\leq \frac{\eta(b,a)}{2^{\frac{p+1}{p}}}\left[\frac{s.2^s+1}{2^s}\right]\left(\frac{|\boldsymbol{f}'(a)|^q + |\boldsymbol{f}'(b)|^q}{(s+1)(s+2)}\right)^{\frac{1}{q}}. \ (2.20)$$

Proof. By using Lemma 2, , we get

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\leq \frac{\eta(b,a)}{2}\int_0^1 |1-2t|\,\boldsymbol{f}'\big(a+t\eta(b,a)\big)dt$$

$$= \frac{\eta(b,a)}{2}\int_0^1 |1-2t|^{\frac{1}{p}}$$

$$|1-2t|^{\frac{1}{p}}\left|\boldsymbol{f}'\big(a+t\eta(b,a)\big)\right|dt \qquad (2.21)$$

By applying Hölder's inequality on (2.21), for $q > 1$,we have,

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\leq \frac{\eta(b,a)}{2}\left(\int_0^1 |1-2t|\,dt\right)^{\frac{1}{p}}$$

$$\left(\int_0^1 |1-2t|\left|\boldsymbol{f}'\big(a+t\eta(b,a)\big)\right|^q dt\right)^{\frac{1}{q}} (2.22)$$

since $|\boldsymbol{f}'|$ is s-preinvex on K for every$a, b \in K$and $t \in [0,1]$.(2.22) can be written as:

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\leq \frac{\eta(b,a)}{2}\left(\frac{1}{2}\right)^{\frac{1}{p}}\int_0^1 |1 - 2t|\,[(1-t)^s|\boldsymbol{f}'(a)| + (t)^s|\boldsymbol{f}'(b)|]^{\frac{1}{q}}dt$$

$$\leq \frac{\eta(b,a)}{2^{\frac{p+1}{p}}}\int_0^1 |-2t|\,[(1-t)^s|\boldsymbol{f}'(a)| + (t)^s|\boldsymbol{f}'(b)|]^{\frac{1}{q}}dt \qquad (2.23)$$

Where we use the fact from Muddassar et al [12],

$$\int_0^1 |1-2t|\,(1-t)^s dt = \int_0^1 |1-2t|\,t^s dt = \left[\frac{s.2^s}{(s+1)(s+2)}\right]$$

By(2.23), and (2.24), in(2.21),we get (2.20).

Corollary 8. From Theorem 7 Let the assumption of Theorem 4 be satisfied with $p > 1$,such that $q = \frac{p}{p-1}$ if the mapping $|\boldsymbol{f'}|$ is s-preinvex on $[a,b]$.then,

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\leq \frac{\eta(b,a)}{2^{\frac{p+1}{p}}}\left[\frac{s.2^s+1}{2^s}\right]^{\frac{1}{q}} |\boldsymbol{f'}(a)| + |\boldsymbol{f'}(b)|$$

Proof. Theproof is similar to that of Corollary 5.

Theorem 9. Let the assumption of Theorem 2 are satisfied with $p > 1$,such that $q = \frac{p}{p-1}$ if the mapping $|\boldsymbol{f'}|$ is s-preconcave on $[a,b]$.then

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\leq \frac{\eta(b,a)}{2^{\frac{p+1}{p}}}\left[\frac{s.2^s+1}{2^s}\right]^{\frac{1}{q}} \left|\boldsymbol{f'}\frac{a+b}{2}\right|^q . \quad (2.25)$$

Proof. We proceed similarly as in theorem 6. By s-concavity of $|\boldsymbol{f'}|^{\mathbf{q}}$we obtain

$$\int_0^1 |1-2t|\left|\boldsymbol{f'}\left(a+t\eta(b,a)\right)\right|^q dt =$$

$$\left[\frac{s.2^s+1}{2^s}\right]^{\frac{1}{q}} \left|\boldsymbol{f'}\frac{a+b}{2}\right|^q . \quad (2.26)$$

Now (2.25) immediately follows from Theorem 1.

Theorem 10. Let the assumption of Theorem 2 are satisfied, then

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\leq \frac{\eta(b,a)}{2}\left(\frac{s^2+3s+4}{(s+1)(s+2)(s+3)}\right)^{\frac{1}{q}} |\boldsymbol{f'}(a)| + |\boldsymbol{f'}(b)|. \quad (2.27)$$

Proof. From Lemma 2

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\leq \frac{\eta(b,a)}{2}\int_0^1\int_0^1 \begin{pmatrix} \boldsymbol{f'}\left(a+t\eta(b,a)\right) \\ -\boldsymbol{f'}\left(a+u\eta(b,a)\right)\end{pmatrix} |u-t|dtdu$$

$$\leq \frac{\eta(b,a)}{2}\int_0^1\int_0^1 \left(\boldsymbol{f'}\left(a+t\eta(b,a)\right)\right)|u-t|dtdu$$

$$+\frac{\eta(b,a)}{2}\int_0^1\int_0^1 \left(\boldsymbol{f'}\left(a+u\eta(b,a)\right)\right)|u-t|dtdu$$

$$= \eta(b,a)\int_0^1\int_0^1 \left(\boldsymbol{f'}\left(a+t\eta(b,a)\right)\right)|u-t|dtdu. \quad (2.28)$$

since $|\boldsymbol{f'}|$ is s-preinvex on K for every$a, b \in K$and $t \in [0,1]$.we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\leq \frac{\eta(b,a)}{2}\int_0^1\int_0^1 [(1-t)^s|\boldsymbol{f'}(a)| + (t)^s|\boldsymbol{f'}(b)|]|u-t|dtdu \quad (2.29)$$

But

$$\int_0^1\int_0^1 t^s|u-t|dtdu =$$

$$\int_0^1\int_0^1 (1-t)^s|u-t|dtdu = \frac{s^2+3s+4}{(s+1)(s+2)(s+3)} \quad (2.30)$$

By (2.29) and (2.30),we get (2.27).

Theorem 11. Let the assumption of Theorem 2 are satisfied. Furthermore, if the mapping $|\boldsymbol{f'}|^{\mathbf{q}}$is s-concave on $[a,b]$. For $q > 1$,then,

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leq$$

$$\eta(b,a)\left(\frac{2}{(p+1)(p+2)}\right)^{\frac{1}{p}} \left|\boldsymbol{f'}\frac{a+b}{2}\right|. \quad (2.31)$$

Proof. From Lemma 2

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\leq \frac{\eta(b,a)}{2}\int_0^1\int_0^1 \begin{pmatrix} \boldsymbol{f}'\big(a+t\eta(b,a)\big) \\ -\boldsymbol{f}'\big(a+u\eta(b,a)\big)\end{pmatrix}|u-t|dtdu$$

$$\leq \frac{\eta(b,a)}{2}\int_0^1\int_0^1 \Big(\boldsymbol{f}'\big(a+t\eta(b,a)\big)\Big)|u-t|dtdu$$

$$+\frac{\eta(b,a)}{2}\int_0^1\int_0^1 \Big(\boldsymbol{f}'\big(a+u\eta(b,a)\big)\Big)|u-t|dtdu$$

$$= \eta(b,a)\int_0^1\int_0^1 \Big(\boldsymbol{f}'\big(a+t\eta(b,a)\big)\Big)|u-t|dtdu. \quad (2.32)$$

By applying Hölder's inequality in (2.32),we have**,**

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\leq \frac{\eta(b,a)}{2}\left(\left|\boldsymbol{f}'\big(a+t\eta(b,a)\big)\right|^q\right)^{\frac{1}{q}} dtdu\left(\int_0^1\int_0^1 |u-t|^p\, dtdu\right)^{\frac{1}{\mathrm{p}}}$$

But

$$\int_0^1\int_0^1 |u-t|^p\, dtdu =$$

$$\int_0^1\left\{\int_0^u (u-t)^p dt+\int_u^1 (t-u)^p dt\right\} du =$$

$$\frac{2}{(p+1)(p+2)} \quad (2.34)$$

Since$|\boldsymbol{f}'|^{\mathbf{q}}$isconcave, by applying Jensen's integral inequality on the first integral in R.H.S we have

$$\int_0^1\int_0^1 \left|\boldsymbol{f}'\big(a+t\eta(b,a)\big)\right|^q dtdu \leq$$

$$\int_0^1\left[\left(\int_0^1 t^0 dt\right)\left|\frac{\int_0^1 \left|\boldsymbol{f}'\big(a+t\eta(b,a)\big)\right|^q dt}{\int_0^1 t^0 dt}\right|\right] du$$

$$= \int_0^1 \left|\boldsymbol{f}'\frac{a+b}{2}\right|^q du = \left|\boldsymbol{f}'\frac{a+b}{2}\right|^q \quad (2.35)$$

Hence (2.33), (2.34), and (2.35), together imply (2.31).

Theorem 12. Let the assumption of Theorem 2 are satisfied. Futhermore, if the mapping $|\boldsymbol{f}'|^{\mathbf{q}}$is s-preinvex on $[a,b]$. For $q>1$,then,

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\leq \eta(b,a)\left[\frac{2}{(p+1)(p+2)}\right]^{\frac{1}{p}}\left(\frac{|\boldsymbol{f}'(a)|^q+|\boldsymbol{f}'(b)|^q}{(s+1)}\right) \quad (2.36)$$

Proof. From Lemma 2

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\leq \frac{\eta(b,a)}{2}\int_0^1\int_0^1 \begin{pmatrix} \boldsymbol{f}'\big(a+t\eta(b,a)\big) \\ -\boldsymbol{f}'\big(a+u\eta(b,a)\big)\end{pmatrix}|u-t|dtdu \leq$$

$$\frac{\eta(b,a)}{2}\int_0^1\int_0^1 \Big(\boldsymbol{f}'\big(a+t\eta(b,a)\big)\Big)|u-t|dtdu$$

$$+\frac{\eta(b,a)}{2}\int_0^1\int_0^1 \Big(\boldsymbol{f}'\big(a+u\eta(b,a)\big)\Big)|u-t|dtdu$$

$$= \eta(b,a)\int_0^1\int_0^1 \Big(\boldsymbol{f}'\big(a+t\eta(b,a)\big)\Big)|u-t|dtdu. \quad (2.37)$$

By applying Hölder's inequality in (2.37), we have**,**

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\leq \eta(b,a)\left(\int_0^1\int_0^1 \left|\boldsymbol{f}'\big(a+t\eta(b,a)\big)\right|^q\right)^{\frac{1}{q}} dtdu\left(\int_0^1\int_0^1 |u-t|^p\, dtdu\right)^{\frac{1}{\mathrm{p}}} \quad (2.38)$$

Since $|\boldsymbol{f}'|$ is s-preinvex on K for every$a,\mathrm{b}\in \mathrm{K}$and $\mathrm{t}\in[0,1]$.we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right|$$
$$\leq \eta(b,a)\int_0^1\int_0^1([(1-t)^s|f'(a)|^q+(t)^s|f'(b)|^q]dtdu)^{\frac{1}{q}} \quad (2.39)$$

But

$$\int_0^1\int_0^1|u-t|^p\,dtdu=\int_0^1\left\{\int_0^u (u-t)^p dt+\int_u^1(t-u)^p dt\right\}du=\frac{2}{(p+1)(p+2)} \quad (2.40)$$

And

$$\int_0^1\int_0^1 t^s dtdu=\int_0^1\int_0^1(1-t)^s dtdu=\frac{1}{(s+1)}. \quad (2.41)$$

By (2.39), (2.40), (2.40), and (2.41),(2.41), we have(2.36).

Corollary 13. From Theorem 12, Let the assumption of Theorem 2 are satisfied. Futhermore, if the mapping $|f'|^q$is s-preinvex on $[a,b]$.For $q>1$,then,

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right|$$
$$\leq \eta(b,a)\left[\frac{2}{(p+1)(p+2)}\right]^{\frac{1}{p}}\left(\frac{1}{(s+1)}\right)^{\frac{1}{q}}(|f'(a)|+|f'(b)|)$$

Proof. The proof is similar to that of Corollary 5.

Theorem 14. Let the assumption of Theorem 2 are satisfied. Futhermore, if the mapping $|f'|^q$is s-preconcave on $[a,b]$.For $q>1$,then,

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right|$$
$$\leq \eta(b,a)\left[\frac{2}{(p+1)(p+2)}\right]^{\frac{1}{p}}.2^{\frac{s-1}{q}}\left|f'\frac{a+b}{2}\right|. \quad (2.42)$$

Proof. We proceed similarly as in theorem 10. By s-preconcavity of $|f'|^q$we obtain

$$\int_0^1\int_0^1\left|f'(a+t\eta(b,a))\right|^q dtdu$$
$$\leq 2^{s-1}\left|f'\frac{a+b}{2}\right|^q \quad (2.43)$$

Now (2.42), immediately follows from Theorem 1.

Theorem 15. Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta: K\times K\to\mathbb{R}$suppose that$f: K\to\mathbb{R}$be differentiable function. If$|f'|^q$ is s-preinvex on K, then the following inequality holds:

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right|$$
$$\leq\frac{\eta(b,a)}{3^{\frac{1}{p}}}\left(\frac{s^2+3s+4}{2(s+1)(s+2)(s+3)}\right)^{\frac{1}{q}}$$
$$(|f'(a)|^q+|f'(b)|^q)^{\frac{1}{q}} \quad (2.44)$$

Proof. From Lemma 2

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right|$$
$$\leq\frac{\eta(b,a)}{2}\int_0^1\int_0^1\left(\begin{matrix}f'(a+t\eta(b,a))\\-f'(a+u\eta(b,a))\end{matrix}\right)|u-t|dtdu$$
$$\leq\frac{\eta(b,a)}{2}\int_0^1\int_0^1\left(f'(a+t\eta(b,a))\right)|u-t|dtdu$$
$$+\frac{\eta(b,a)}{2}\int_0^1\int_0^1\left(f'(a+u\eta(b,a))\right)|u-t|dtdu$$
$$=\eta(b,a)\int_0^1\int_0^1\left(f'(a+t\eta(b,a))\right)|u-t|dtdu. \quad (2.45)$$

By applying Holder's inequality in(2.45) we follows as,

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$
$$\leq \eta(b,a)\left(\int_0^1\int_0^1 |u-t||\boldsymbol{f}'(a+t\eta(b,a))|^q\right)^{\frac{1}{q}} dtdu\left(\int_0^1\int_0^1 |u-t|\,dtdu\right)^{\frac{1}{p}} \quad (2.46)$$

Here

$$\int_0^1\int_0^1 |u-t|\,dtdu=\frac{1}{3} \quad (2.47)$$

And

$$\int_0^1\int_0^1 |u-t||\boldsymbol{f}'(a+t\eta(b,a))|^q\,dtdu$$
$$\leq \eta(b,a)\int_0^1\int_0^1 (|u-t|(1-t)^s|\boldsymbol{f}'(a)|^q + |u-t|(t)^s|\boldsymbol{f}'(b)|^q dtdu) \quad (2.48)$$

Since $|\boldsymbol{f}'|$ is s-preinvex on K for every$a,b\in K$ and $t\in[0,1]$.we have

By solving (2.48), we have

$$\int_0^1\int_0^1 |u-t||\boldsymbol{f}'(a+t\eta(b,a))|^q\,dtdu$$
$$\leq\left(\frac{s^2+3s+4}{2(s+1)(s+2)(s+3)}\right)^{\frac{1}{q}}(|\boldsymbol{f}'(a)|^q+|\boldsymbol{f}'(b)|^q)^{\frac{1}{q}} \quad (2.49)$$

Relations (2.46), (2.47), and (2.49) together imply (2.44).

Corollary 16. From theorem 15,Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta: K\times K\to\mathbb{R}$ suppose that$f: K\to\mathbb{R}$be differentiable function. If$|\boldsymbol{f}'|^{\mathbf{q}}$ is s-preinvex on K, then the following inequality holds:

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$
$$\leq\frac{\eta(b,a)}{3^{\frac{1}{p}}}\left(\frac{s^2+3s+4}{2(s+1)(s+2)(s+3)}\right)^{\frac{1}{q}}(|\boldsymbol{f}'(a)|+|\boldsymbol{f}'(b)|)$$

Proof. Theproof is similar to that of Corollary 5.

Theorem 17. Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta: K\times K\to\mathbb{R}$suppose that$f: K\to\mathbb{R}$ be differentiable function. If$|\boldsymbol{f}'|^{\mathbf{q}}$ is s-preconcave on K, then the following inequality holds:

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$
$$\leq\frac{\eta(b,a)}{3^{\frac{1}{p}}}\left[\frac{s^2+3s+4}{(s+2)(s+3)}\right]^q.2^{\frac{s-2}{q}}\left|\boldsymbol{f}'\frac{a+b}{2}\right|. \quad (2.50)$$

Proof. We proceed similarly as in theorem 12. By s-preconcavity of $|\boldsymbol{f}'|^{\mathbf{q}}$we obtain

$$\int_0^1\int_0^1 |u-t||\boldsymbol{f}'(a+t\eta(b,a))|^q\,dtdu$$
$$\leq\left[\frac{s^2+3s+4}{(s+2)(s+3)}\right]^q\left|\boldsymbol{f}'\frac{a+b}{2}\right| \quad (2.51)$$

Now (2.50) immediately follows from Theorem 1.

3. APPLICATION TO SOME SPECIAL MEANS

We now consider the applications of our theorem to the special means.

(a) The arithmetic mean;

$$A=A(a,b):=\frac{a+b}{2}, \qquad a,b>0,$$

(b) The geometric mean;

$$G=G(a,b):=\sqrt{ab}, \qquad a,b>0,$$

(c) The Harmonic mean:

$$H=H(a,b):=\frac{2ab}{a+b}, \qquad a,b>0,$$

(d) The logarithmic mean:

$$L=L(a,b):=\begin{cases} a, & if a=b\\ \frac{b-a}{lnb-lna}, & if a\neq b\end{cases}, \quad a,b>0,$$

(e) The identric mean:

$$I=I(a,b):=\begin{cases} a, & if a=b\\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/b-a}, & if a\neq b\end{cases} \quad a,b>0,$$

(f) The p- logarithmic mean:

$$I = I(a,b) := \begin{cases} a, & if a = b \\ \sqrt[p]{\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}}, & if a \neq b \end{cases} \quad a,b > 0,$$

The following inequality is well known in the literature in [9]:

$\boldsymbol{H \leq G \leq L \leq I \leq A}$.

It is also known that L_P is monotonically increasing over $p \in \mathbb{R}$,denoting $L_0 = I$ and $L_{-1} = L$.

Proposition 1. Let$p > 1, 0 < a < b$ and $\frac{p}{p-1}$.then one has the inequality.

$$|A(a,b) - L(a,b)| \leq \frac{(lnb - lna)}{3} A(|a|,|b|). \quad (3.52)$$

Proof. By Theorem 10 applied for the mapping $f(x) = e^x$ for $s = 1$. .we have the above inequality (3.52)

Proposition 2. Let $p > 1$, , $0 < a < b$ and $q = \frac{p}{p-1}$.then one has the inequality.

$$\left|\frac{I(1-a, 1-b)}{G(1-a, 1-b)}\right| \leq exp\left(\left(\frac{b-a}{3}\right) H^{-1}(|1-a|, |1-b|)\right)$$

Proof. By Theorem 12 applied for the mapping $f(x) = -ln(1-x)$ for $s = 1$.

Another result which is connected with p-logarithmic mean$L_P(a,b)$ is following one :

Proposition 3. Let $p > 1, 0 < a < b$ and $q = \frac{p}{p-1}$.

$$\left|A[(1-a)^n, (1-b)^n] - L_n^p[(1-a)^n, (1-b)^n]\right| \leq |n|(b-a)\left[\frac{2}{(p+1)(p+2)}\right]^{\frac{1}{p}} \left[A\left(|1-a|^{\frac{q}{n-1}}, |1-b|^{\frac{q}{n-1}}\right)\right]^{\frac{1}{q}}.$$

Proof . Following by Theorem 15, setting$f(x) = (1-x)^n$, $|n| \geq 2$ and $n \in \mathbb{R}$ for $s = 1$.

4. APPLICATION TO QUADRATURE FORMULA

Let D be the division or the partition of the interval $[a,b]$,i.e., $d: a = x_0 < x_1 < \cdots < n - 1 < x_n = b$, and consider the quadrature formula

$$\int_a^b f(x)\,dx = S(f,D) + R(f,D) \quad (4.53)$$

Where

$S(f,D) = \sum_{k=0}^{n-1} \frac{f(x_k)+f(x_{k+1})}{2}(x_{k+1} - x_k)$For the trapezoidal version and $R(f,D)$ denotes the related approximation error.

Proposition 4. . Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta: K \times K \to \mathbb{R}$suppose that $f: K \to \mathbb{R}$be differentiable function. If$|\boldsymbol{f}'|^{\mathbf{p}/(\mathbf{p-1})}$ is s-preinvex on $K, p > 1$,for every division D of $[a,b]$, then the following inequality holds:

$$|R(f,D)| \leq \frac{1}{2^{\frac{p+1}{p}}}\left(\frac{s.2^s + 1}{(s+1)(s+2)}\right)^{\frac{1}{q}} \sum_{k=0}^{n-1} (x_{k+1} - x_k)^2 \left[|\boldsymbol{f}'(x_k)| + |\boldsymbol{f}'(x_{k+1})|\right] (4.54)$$

Proof. Applying Corollary 8 on the subintervals $[x_k, x_{k+1}]$, $(k = 0,1, \dots\dots . n-1)$ of the division D and using the fact :

$\sum_{m-1}^{n-1}(\emptyset_m + \mu_m)^r \leq \sum_{m-1}^{n-1}(\emptyset_m)^r + \sum_{m-1}^{n-1}(\mu_m)^r$ for $(0 < r < 1)$ and for each m both $\emptyset_m, \mu_m \geq 0$,we have

$$\left|\frac{1}{x_{k+1} - x_k}\int_{x_k}^{x_{k+1}} \boldsymbol{f}(x) - \boldsymbol{f}\left(\frac{x_{k+1} - x_k}{2}\right)\right| \leq \frac{x_{k+1} - x_k}{2^{\frac{p+1}{p}}}\left(\frac{s.2^s + 1}{(s+1)(s+2)}\right)^{\frac{1}{q}} \left[|\boldsymbol{f}'(x_k)| + |\boldsymbol{f}'(x_{k+1})|\right] (4.55),$$

Summing over k from 0 to $n-1$ and taking into account that $|\boldsymbol{f'}|$ is s-preinvex ,

By triangle inequality ,we have

$$\left|\int_a^b f(x)\,dx - S(f,D)\right| =$$

$$\left|\sum_{k=0}^{n-1}\left\{\int_{x_k}^{x_{k+1}} \boldsymbol{f}(x) - (x_{k+1}-x_k)\boldsymbol{f}\left(\frac{x_{k+1}+x_k}{2}\right)\right\}\right| \le$$

$$\sum_{k=0}^{n-1}\left|\left\{\int_{x_k}^{x_{k+1}} \boldsymbol{f}(x) - (x_{k+1}-x_k)\boldsymbol{f}\left(\frac{x_{k+1}-x_k}{2}\right)\right\}\right|$$

$$|R(f,D)| \le \sum_{k=0}^{n-1}(x_{k+1}-x_k)\left|f\left(\frac{x_{k+1}+x_k}{2}\right) - \frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}} \boldsymbol{f}(x)\right| \quad (4.56),$$

By combining(4.55), (4.56),we get (4.54).

Proposition 5. Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta: K \times K \to \mathbb{R}$suppose that $f: K \to \mathbb{R}$be differentiable function. If$|f'|^{\mathbf{p}/(\mathbf{p-1})}$ is s-preinvex on $K, p > 1$,for every division$D$ of $[a,b]$, then the following inequality holds:

$$|R(f,D)| \le \frac{1}{3^{\frac{1}{p}}}\left(\frac{s^2+3s+4}{2(s+1)(s+2)(s+3)}\right)^{\frac{1}{q}}$$

$$\sum_{k=0}^{n-1}(x_{k+1}-x_k)^2\,[|\boldsymbol{f'}(x_k)| + |\boldsymbol{f'}(x_{k+1})|]$$

Proof. The proof is similar to that of Proposition 4 and using Corollary 16.

5. CONCLUSIONS

Convexity has been playing a key role in mathematical programming, engineering, and optimization theory. The generalization of convexity is one of the most important aspects in mathematical programming and optimization theory. There have been many attempts to weaken the convexity assumptions in the literature. A significant generalization of convex functions is that of invex functions introduced by Hanson [9]. Ben-Israel and Mond [4] introduced the concept of preinvex functions, which is a special case of invexity. Pini [17] introduced the concept of prequasiinvex functions as a generalization of invex functions. Noor [13, 14] has established some Hermite-Hadamardtype inequalities for preinvex and log-preinvex functions. In this paper we developed more results on hermite-Hadamard's type inequalities by weaken the condition of convexity and found some new results which are related with some special mean; we also applied these results on quadrature rules that gave better estimates than previously presented. We can further find some new relations in the same way as above associating with some special means by taking some other convex functions. For example, choosing different convex functions like $f(x) = -ln\,x, f(x) = \frac{1}{x}$ and $f(x) = -ln(1-x)$ for different values of $s$in s-invexity (concavity), we get new relations relating to some special means.

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