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Research Article

Common Fixed Point Theorems for Mappings Satisfying ϕImplicit Relation in *G*-Metric Spaces

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Abstract: In this paper some common fixed point theorems are established for two and four mappings satisfying ϕ-implicit relation in G-metric spaces. Also, a common fixed point theorem for T-contraction mapping is proved. our results are improve the results of Popa and Patriciu [24]. An example is given to justify some of our results.

Keywords: G-metric space, common fixed point, weak-compatible maps, ϕ-implicit relation, T-contraction

1. INTRODUCTION

Jungck [5] proved a common fixed point theorem for commuting mappings as generalizing the Banach's fixed point theorem. The concept of the commutativity has generalized in several ways. For this Sessa [25] introduced the concept of weakly commuting mappings, Jungck [6] extend this concept to compatible maps. In 1998, Jungck and Rhoades [7] introduced the notion of weak compatibility and showed that compatible maps are weakly compatible but the converse need not to be true, for example see Pathak [21].

The notion of *G*-metric space was introduced by Mustafa and Sims [17, [18] as a generalization of the notion of metric spaces. Afterwards Mustafa, Sims and others authors introduced and developed several fixed point theorems for mappings satisfying different contractive conditions in *G*-metric spaces, also extend known theorems in metric spaces to *G*-metric spaces see [4, 9-20, 26] and many other papers.

Beiranvand, Moradi, Omid and Pazandeh [3] introduce the classes of T-contraction and T-contractive mappings, which are depending on another function. Moradi in [10] introduce the T-Kannan contractivemapping. Morales and Rojas [11, 12] have extended the concept of T-contraction mappings to cone metric space by proving fixed point theorems for T-Kannan, T-Chatterjea T-Zamfirescu, T-weakly contraction mappings. Sumitra, Rhymend Uthariaraj and Hemavathy [27] proved a fixed point theorem in the setting of cone metric space for T-Hardy-Rogers type contraction condition.

Karayian and Telici [8] and Shatanawi [26] proved some fixed point theorems for mappings satisfying ϕ- maps. Popa [22, 23] initiated the study of fixed points for mappings satisfying implicit relations. Altun and Turkoglu [2] introduced a new type of implicit relations satisfying ϕ-map. Popa and Patriciu [24] proved a fixed point theorem in a complete *G*-metric spaces for mappings satisfying ϕ-implicit relation.

The purpose of this paper is to study some common fixed point theorems for two and fourmappings satisfying ϕ-implicit relation in *G*-metric spaces. Also, a common fixed point theorem for T-contraction mapping is proved. our results are improve the results of Popa and Patriciu [24].

2. PRELIMINARIES

Definition 2.1. [18] Let X be a nonempty set, and let $G : X^3 \to [0, \infty)$, be a function satisfying:

(G_1) $G(x, y, z) = 0$ if $x = y = z$,

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(G_2) $0 < G(x,x,y)$, for all $x,y \in X$,with $x \neq y$,

(G_3) $G(x,x,y) \leq G(x,y,z), \forall x,y,z \in X$, with$z \neq y$,

$(G4)$$G(x,y,z) = G(x,z,y) = G(y,z,x) \ldots$, (symmetry in all three variables),

$(G5)$ $G(x,y,z) \leq G(x,a,a) + G(a,y,z), \forall x,y,z,a \in X$, (rectangle inequality).

Then the function G is called a generalized metric, or more specifically a G-metric on X, and the pair (X,G) is called a G-metric space.

Definition 2.2. [18] Let (X,G) be a G-metric space, a sequence (x_n) is said to be

(i) G-convergent if for every $\varepsilon > 0$, there exists an $x \in X$, and $k \in N$such that for all $m,n \geq k$, $G(x,x_n,x_m) < \varepsilon$.

(ii) G-Cauchy if for every $\varepsilon > 0$, there exists an $k \in \mathbf{N}$such that for all $m,n,p \geq k$, $G(x_m,x_n,x_p) < \varepsilon$, that is $G(x_m,x_n,x_p) \to 0$ as $m,n,p \to \infty$.

(iii) A space (X,G) is said to be G-complete if every G-Cauchy sequence in (X,G) is G-convergent.

Definition 2.3. [18] A G metric space X is symmetric if $G(x,y,y) = G(y,x,x)$ for all $x,y \in X$.

Lemma 2.1. [18] Let (X,G) be a G-metric space. Then the following are equivalent:

(i) (x_n) is convergent to x,

(ii) $G(x_n,x_n,x) \to 0$ as $n \to \infty$,

(iii) $G(x_n,x,x) \to 0$ as $n \to \infty$,

(iv) $G(x_n,x_m,x) \to 0$ as $n,m \to \infty$,

Lemma 2.2. [18] Let (X,G) be a G-metric space. Then the following are equivalent:

(i) The sequence (x_n) is G-Cauchy,

(ii) for every $\varepsilon > 0$, there exists $k \in \mathrm{N}$ such that $G(x_n,x_m,x_m) < \varepsilon$ for $m,n \geq k$.

Lemma 2.3. Mustafa and Sims [18] Let (X,G) be a G-metric space. Then the function $G(x,y,z)$ is jointly continuous in all three of its variables.

Definition 2.4. Let T and S be self maps of a nonempty set X. If $w = Tx = Sx$ for some $x \in X$, then x is called a coincidence point of T and S and w is called a point of coincidence of T and S.

Definition 2.5. Two self -mappings T and S are said to be weakly compatible if they commute at their coincidence points, that is, $Tx = Sx$ implies that $TSx = STx$.

3. IMPLICIT RELATIONS

Definition 3.1. Popa and Patriciu [24] A function $f: [0,\infty) \to [0,\infty)$is called a ϕ-function, $f \in \phi$, if f is a nondecreasing function such that $\sum_{n=1}^{\infty} f^n(t) < \infty$, for all $f(t) < t$ for $t > 0$ and $f(0) = 0$.

Definition 3.2. Let$\boldsymbol{F_\phi}$be the set of all continuous functions$F(t_1,\ldots,t_6): R_+^6 \to \mathrm{R}$ such that:

(F_1): $F$is nonincreasing in t_5,

(F_2):there exists a function $\phi_1, \phi_2 \in \phi$ such that for all $u,v \geq 0$, with

(F_a): $F(u,v,v,u,u+v,0) \leq 0$implies$u \leq \phi_1(v)$,

(F_b): $F(u,v,u,v,0,u+v) \leq 0$implies$u \leq \phi_2(v)$,

$(F3)$:there exists a function $\phi_3 \in \phi$ such that for all $t,t' > 0, F(t,t,0,0,t,t') \leq 0$ implies $t \leq \phi_3(t')$.

Example 3.3. $F(t_1,\ldots,t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6$,where$a > 0, b,c,d,e \geq 0$, $a + b + c + 2d + e < 1$.

(F_1):Obviously. $F(t_1,\ldots,t_6)$

(F_2): Let $u,v \geq 0$, and $F(u,v,v,u,u+v,0) = u - av - bv - cu - d(u+v) \leq 0$ which implies $u \leq \frac{a+b+d}{1-c-d} v$. F_a is satisfied for $\phi_1(t) = \frac{a+b+d}{1-c-d}\mathrm{t}$. Similarly $F(u,v,u,v,0,u+v) = u - av - bu - cv - e(u+v) \leq 0$ whichimplies $u \leq \frac{a+c+e}{1-b-e} v$. F_b is satisfied for $\phi_2(t) = \frac{a+c+e}{1-b-e} t$.

$(F3)$: Let $t,t' > 0$ be and $F(t,t,0,0,t,t') = t - at - bt - et' \leq 0$ which implies

$t \leq \frac{e}{1-(a+d)} t'andF_3$ is satisfied for $\phi_3(t) = \frac{e}{1-(a+d)} t$.

Example 3.4 $F(t_1,\ldots,t_6) = t_1 - kmax\{t_2,t_3,t_4,t_5,t_6\}$, where $k \in (0,\frac{1}{2})$.

(F_1): Obviously.

(F_2): Let $u, v \geq 0$, and $F(u, v, v, u, u+v, 0) = F(u, v, u, v, 0, u+v) = u - kmax\{u, v, u+v\} \leq 0$. Hence $u \leq \frac{k}{1-k}\, v$ and F_a is satisfied for $\phi_1(t) = \frac{k}{1-k}\, t$. Similarly $F(u, v, u, v, 0, u+v) = u - kmax\{u, v, u+v\} \leq 0$. Then $u \leq \frac{k}{1-k} v$ and F_b is satisfied for $\phi_2(t) = \frac{k}{1-k} t$.

(F_3): Let $t, t' > 0$ be and $F(t, t, 0, 0, t, t') = t - kmax\{t, t'\} \leq 0$ if $t > t'$, then $t(1-k) \leq 0$, a contradiction. Hence $t \leq t'$ which implies $t \leq kt'$ and F_3 is satisfied for $\phi_3(t) = kt$.

Example 3.5 $F(t_1, \ldots, t_6) = t_1 - kmax\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\}$, where $k \in (0,1)$.

(F_1):Obviously.

(F_2):Let $u, v \geq 0$, and $F(u, v, v, u, u+v, 0) = F(u, v, u, v, 0, u+v) = u - kmax\{u, v, \frac{u+v}{2}\} \leq 0$. If $u > v$, then $u(1-k) \leq 0$, a contradiction. Hence $u \leq v$ which implies $u \leq kv$ and F_a is satisfied for $\phi_1(t) = kt$. Similarly $F(u, v, u, v, 0, u+v) = u - kmax\left\{u, v, \frac{u+v}{2}\right\} \leq 0$. Then $u \leq kv$ and F_b is satisfied for $\phi_2(t) = kt$.

(F_3): Let $t, t' > 0$ be and $F(t, t, 0, 0, t, t') = t - kmax\{t, t'\} \leq 0$ if $t > t'$, then $t(1-k) \leq 0$, a contradiction. Hence $t \leq t'$ which implies $t \leq kt'$ and F_3 is satisfied for $\phi_3(t) = kt$.

For more examples see [24], where those examples also satisfying (F_b).

4. MAIN RESULTS

Lemma 4.1. Let (X, G) be a G-metric space and $T, f: (X, G) \to (X, G)$ two mappings such that T is one to one and

$$F(G(Tfx, Tfy, Tfy), G(Tx, Ty, Ty),$$
$$G(Tx, Tfx, Tfx), G(Ty, Tfy, Tfy),$$
$$G(Tx, Tfy, Tfy), G(Ty, Tfx, Tfx)) \leq 0, \qquad (1)$$

for all $x, y \in X$ and F satisfying property (F_3)Then, $f$has at most a fixed point.

Proof. Suppose that u = f u and v = f v. Then by(1) we have successively

$$F(G(Tfu, Tfv, Tfv), G(Tu, Tv, Tv),$$
$$G(Tu, Tfu, Tfu), G(Tv, Tfv, Tfv),$$
$$G(Tu, Tfv, Tfv), G(Tv, Tfu, Tfu)) \leq 0,$$

by (F_3) we obtain that

$$G(Tu, Tv, Tv) \leq \phi_3(G(Tv, Tu, Tu)).$$

Similarly, we obtain that

$$G(Tv, Tu, Tu) \leq \phi_3(G(Tu, Tv, Tv)).$$

Hence

$$G(Tu, Tv, Tv) \leq \phi_3\big(G(T v, Tu, Tu)\big) \leq \phi_3^2\big(G(Tu, Tv, Tv)\big) < G(Tu, Tv, Tv),$$

which is a contradiction. Hence $Tu = Tv$, since T is one to one then $u = v$.

Theorem 4.1. Let(X, G) be a G-metric space. Assume that T and f are two self mappings of (X, G). Assume that $T(X)$ is a G-complete subspace of X and T is one to one mapping. If T and f satisfying inequality (1) for all $x, y \in X$, where $F \in F_\phi$, then f has a unique fixed point in X. Moreover, if T and f arecommuting at the fixed points of f , then T and f have a unique common fixed point.

Proof. Let x_0 be an arbitrary point of X. Define a sequence (x_n) in X such that $x_{n+1} = fx_n$ for each $n = 0,1,\cdots$. Then by (1) we have successively

$$F(G(Tfx_{n-1}, Tfx_n, Tfx_n), G(Tx_{n-1}, Tx_n, Tx_n),$$
$$G(Tx_{n-1}, Tfx_{n-1}, Tfx_{n-1}), G(Tx_n, Tfx_n, Tfx_n),$$
$$G(Tx_{n-1}, Tfx_n, Tfx_n), G(Tx_n, Tfx_{n-1}, Tfx_{n-1})) \leq 0,$$

$$F(G(Tx_n, Tx_{n+1}, Tx_{n+1}), G(Tx_{n-1}, Tx_n, Tx_n),$$
$$G(Tx_{n-1}, Tx_n, Tx_n), G(Tx_n, Tx_{n+1}, Tx_{n+1}),$$
$$G(Tx_{n-1}, Tx_{n+1}, Tx_{n+1}),$$
$$G(Tx_n, Tx_n, Tx_n)) \leq 0.$$

By (F_1) and (G_5) we obtain

$$F(G(Tx_n, Tx_{n+1}, Tx_{n+1}), G(Tx_{n-1}, Tx_n, Tx_n),$$
$$G(Tx_{n-1}, Tx_n, Tx_n), G(Tx_n, Tx_{n+1}, Tx_{n+1}),$$
$$G(Tx_{n-1}, Tx_n, Tx_n) +$$
$$G(Tx_n, Tx_{n+1}, Tx_{n+1}), 0) \leq 0.$$

By (F_a) we obtained

$$G(Tx_n, Tx_{n+1}, Tx_{n+1}) \leq \phi_1(G(Tx_{n-1}, T x_n, Tx_n)) \leq \phi_1^2 (G(Tx_{n-2}, Tx_{n-1}, Tx_{n-1})) \\ \vdots \\ \leq \phi_1^n (G(Tx_0, Tx_1, Tx_1)).$$

Therefore

$$G(Tx_n, Tx_m, Tx_m) \leq G(Tx_n, T x_{n+1}, Tx_{n+1}) + \ldots + G(Tx_{m-1}, Tx_m, Tx_m) \leq \phi_1^n(G(Tx_0, Tx_1, Tx_1)) + \ldots + \phi_1^{m-1}(G(Tx_0, Tx_1, Tx_1)) = \sum_{k=n}^{m-1} \phi_1^k (G(Tx_0, Tx_1, Tx_1)).$$

Since, $\sum_{k=n}^{\infty} \phi_1^k (G(Tx_0, Tx_1, Tx_1)) < \infty$ then for any $\varepsilon > 0$, there exists $k \in N$ such that for $m > n \geq k, \sum_{k=n}^{m-1} \phi_1^k (G(Tx_0, Tx_1, Tx_1)) < \varepsilon$. Hence by Lemma (2.2)$(Tx_n)$ is a G-Cauchy sequence. Since $T(X)$ is a G-complete metric subspace of X, there exists a point q in $T(X)$ such that $\lim_{n\to\infty} Tx_n = q$.Also, we can find a point $u \in X$ such that $Tu = q$. Now, we prove $Tu = Tfu$.By(1) we have

$$F(G(Tfx_n, Tfu, Tfu), G(Tx_n, Tu, Tu), G(Tx_n, Tfx_n, Tfxn), G(Tu, Tfu, Tfu), G(Tx_n, Tfu, Tfu), G(Tu, Tfx_n, Tfx_n)) \leq 0,$$

$$F(G(Tx_{n+1}, Tfu, Tfu), G(Tx_n, Tu, Tu), G(Tx_n, Tx_{n+1}, Tx_{n+1}), G(Tu, Tfu, Tfu), G(Tx_n, Tfu, Tfu), G(Tu, Tx_{n+1}, Tx_{n+1})) \leq 0.$$

Letting n tend to infinity, we obtain

$$F(G(Tu, Tfu, Tfu), 0, 0, G(Tu, Tfu, Tfu), G(Tu, Tfu, Tfu), 0) \leq 0$$

By (F_a) we have $G(Tu, Tf u, Tf u) \leq \phi_1(0) = 0$, hence $G(Tu, Tfu, Tfu) = 0$, then $Tu = Tfu$. Since T is one to one, $fu = u$. By Lemma (4.1), u is the unique fixed point of f. Moreover, if T and f are commuting at the fixed points of f, then $fTu = Tfu = u$ this implies that Tu is another fixed point of f . By uniqueness of fixed point of f , we have $Tu = u$. Hence $Tu = fu = u$ is a unique common fixed point of f and T.If we put $T = I$, where I is the identity mapping, we have the following Corollary.

Corollary 4.1. (Theorem 4.2 [24]) Let (X, G) be a complete G-metric space. Assume that T satisfying the condition

$$F(G(Tx, Ty, Ty), G(x, y, y), G(x, Tx, Tx), G(y, Ty, Ty), G(x, Ty, Ty), G(y, Tx, Tx)) \leq 0,$$

for all $x, y \in X$, where $F \in \boldsymbol{F}_\phi$, then T has a uniquefixed point.

The following Lemmas are fundamental in the sequel.

Lemma 4.2. Abbas and Rhoades [1] Let T and S be weakly compatible self- mappings of nonempty set X. If T and S have a unique point of coincidence $w = Tx = Sx$, then w is the unique common fixed point of T and S.

Lemma 4.3. Let (X, G) be a G-metric space and $T, S : (X, G) \to (X, G)$ two mappings such that

$$F(G(Tx, Ty, Ty), G(Sx, Sy, Sy), G(Sx, Tx, Tx), G(Sy, Ty, Ty), G(Sx, Ty, Ty), G(Sy, Tx, Tx)) \leq 0, \tag{2}$$

for all $x, y \in X$ and F satisfying property (F_3). Then, T and S have at most a point of coincidence.

Proof. Suppose that $u = Tp = Sp$ and $v = Tq = Sq$.Then by (2) we have successively

$$F(G(Tp, Tq, Tq), G(Sp, Sq, Sq), G(Sp, Tp, Tp), G(Sq, Tq, Tq), G(Sp, Tq, Tq), G(Sq, Tp, Tp)) \leq 0$$

$$F(G(u, v, v), G(u, v, v), 0, 0, G(u, v, v), G(v, u, u)) \leq 0,$$

by(F_3) we obtain that

$$G(u, v, v) \leq \phi_3(G(v, u, u)).$$

Similarly, we obtain that

$$G(v, u, u) \leq \phi_3(G(u, v, v)).$$

Hence

$$G(u, v, v) \leq \phi_3(G(v, u, u)) \leq \phi_3^2 (G(u, v, v)) < G(u, v, v),$$

which is a contradiction. Hence $u = v$.

Lemma 4.4. Let (X, G) be a G-metric space and $A, B, S, T : (X, G) \to (X, G)$ such that

$$F(G(Tx, By, By), G(Sx, Ay, Ay), G(Sx, Tx, Tx), G(Ay, By, By), G(Sx, By, By), G(Ay, Tx, Tx)) \leq 0, \quad (3)$$

for all $x, y \in X$ and F satisfying property (F_3). Then, A, B, S and T have at most a common fixed point.

Proof. Suppose that $p = Tp = Sp = Ap = Bp$ and $q = Tq = Sq = Aq = Bq, p \neq q$.Then by (3) we have successively

$$F(G(Tp, Bq, Bq), G(Sp, Aq, Aq), G(Sp, Tp, Tp), G(Aq, Bq, Bq), G(Sp, Bq, Bq), G(Aq, Tp, Tp)) \leq 0,$$

$$F(G(p, q, q), G(p, q, q), 0, 0, G(p, q, q), G(q, p, p)) \leq 0,$$

by(F_3)we obtain that

$$G(p, q, q) \leq \phi_3(G(q, p, p)).$$

Similarly, we obtain that

$$G(q, p, p) \leq \phi_3(G(p, q, q)).$$

Hence

$$\begin{aligned} G(p, q, q) &\leq \phi_3\big(G(q, p, p)\big) \\ &\leq \phi_3^2(G(p, q, q)) \\ &< G(p, q, q), \end{aligned}$$

which is a contradiction. Hence $p = q$.

Theorem 4.2. Let (X, G)be a G-metric space and$T, S: (X, G) \to (X, G)$satisfying inequality(2) forall $x, y \in X$, where $F \in \boldsymbol{F}_\phi$. If $T(X) \subseteq S(X)$ and$S(X)$ is a G-complete metric subspace of (X, G),then T and S have a unique point of coincidence. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point.

Proof. Let x_0 be an arbitrary point of X since $T(X) \subseteq S(X)$ we can choose $x_1 \in X$ such that$Tx_0 = Sx_1$. Continuing this process, having chosenx_n $in\, X$, we obtain x_{n+1} such that $Tx_n = Sx_{n+1}$. Then, by (2) we have successively

$$F(G(Tx_{n-1}, Tx_n, Tx_n), G(Sx_{n-1}, Sx_n, Sx_n), G(Sx_{n-1}, Tx_{n-1}, Tx_{n-1}), G(Sx_n, Tx_n, Tx_n), G(Sx_{n-1}, Tx_n, Tx_n), G(Sx_n, Tx_{n-1}, Tx_{n-1})) \leq 0,$$

$$F(G(Sx_n, Sx_{n+1}, Sx_{n+1}), G(Sx_{n-1}, Sx_n, Sx_n), G(Sx_{n-1}, Sx_n, Sx_n), G(Sx_n, Sx_{n+1}, Sx_{n+1}), G(Sx_{n-1}, Sx_{n+1}, Sx_{n+1}), 0) \leq 0.$$

By $(F_1)$and (G_5) we obtain

$$F(G(Sx_n, Sx_{n+1}, Sx_{n+1}), G(Sx_{n-1}, Sx_n, Sx_n), G(Sx_{n-1}, Sx_n, Sx_n), G(Sx_n, Sx_{n+1}, Sx_{n+1}), G(Sx_{n-1}, Sx_n, Sx_n) + G(Sx_n, Sx_{n+1}, Sx_{n+1}), 0) \leq 0.$$

By (F_a) we obtain

$$\begin{aligned} &G(S\,x_n, Sx_{n+1}, Sx_{n+1}) \\ &\leq \phi_1(G(Sx_{n-1}, S\,x_n, Sx_n)) \\ &\leq \phi_1^2\,(G(Sx_{n-2}, Sx_{n-1}, Sx_{n-1})) \\ &\vdots \\ &\leq \phi_1^n\,(G(Sx_0, Sx_1, Sx_1)). \end{aligned}$$

Then for $m > n$, and by G_5

$$\begin{aligned} G(Sx_n, Sx_m, Sx_m) &\leq G(Sx_n, Sx_{n+1}, Sx_{n+1}) + \ldots \\ &+ G(Sx_{m-1}, Sx_m, Sx_m) \\ &\leq \phi_1^n\big(G(Sx_0, Sx_1, Sx_1)\big) + \ldots \\ &+ \phi_1^{m-1}(G(Sx_0, Sx_1, Sx_1)) \\ &= \sum_{k=n}^{m-1} \phi_1^k\,(G(Sx_0, Sx_1, Sx_1)). \end{aligned}$$

Since$\sum_{k=n}^{\infty} \phi_1^k\big(G(Sx_0, Sx_1, Sx_1)\big) < \infty$, then for any $\varepsilon > 0$, there exists $k \in N$ such that for $m > n \geq k, \sum_{k=n}^{m-1} \phi_1^k\big(G(Sx_0, Sx_1, Sx_1)\big) < \varepsilon$.Hence by Lemma (2.2)$(Sx_n)$ is aG-Cauchy sequence. Since $S(X)$ is a G-complete metric subspace of X, there exists a point q in $S(X)$ such that $\lim_{n\to\infty} Sx_n = q$.

Also, we can find a point $p \in X$ such that $Sp = q$.We provethat $Tp = Sp$. By (2) we have

$$F(G(T\,x_{n-1}, Tp, Tp), G(Sx_{n-1}, Sp, Sp), G(Sx_{n-1}, Tx_{n-1}, Tx_{n-1}), G(Sp, Tp, Tp), G(Sx_{n-1}, Tp, Tp), G(Sp, Tx_{n-1}, Tx_{n-1})) \leq 0,$$

$$F(G(Sx_n, Tp, Tp), G(Sx_{n-1}, Sp, Sp),$$

$G(Sx_{n-1}, Sx_n, Sx_n), G(Sp, Tp, Tp),$

$G(Sx_{n-1}, Tp, Tp), G(Sp, Sx_n, Sx_n)) \leq 0.$

Letting n tend to infinity, we obtain

$F(G(Sp, Tp, Tp), 0, 0, G(Sp, Tp, Tp),$

$G(Sp, Tp, Tp), 0) \leq 0,$

By (F_a) it follows that there exists a function $\phi_1 \in \phi$ such that $G(Sp, Tp, Tp) \leq \phi_1(0) = 0$, a contradiction. Hence $Tp = Sp$. Then $Tp = Sp = q$ is apoint of coincidence of T and S. By Lemma (4.3), q is the unique point of coincidence. Moreover, if T and S are weakly compatible, by Lemma (4.2), q is the unique common fixed point of T and S.

Remark 4.3. If we put $S = I$, where I is the identitymapping, we have the Corollary (4.1).

Now, we extend the above theorem for four mappings in G-metric space.

Theorem 4.4. Let (X, G) be a symmetric G-metricspace and $A, B, S, T: (X, G) \to (X, G)$ satisfyingcondition (3) for all $x, y \in X$, where $F \in \boldsymbol{F}_\phi$, suchthat

$(a) T(X) \subseteq A(X)$ and $B(X) \subseteq S(X)$,

(b) the pairs (A, B) and (T, S) are weakly compatible,

(c) one of $A(X), B(x), S(X)$, or $T(X)$ is a G-completesubspace of X, Then A, B, S, and T have a uniquecommonfixed point.

Proof. Let x_0 be an arbitrary point of X From (a) we can choose $x_1, x_2 \in X$ such that $y_0 = Tx_0 = Ax_1$ and $y_1 = Bx_1 = Sx_2$. Continuing this process, we obtain

$y_{2n} = Tx_{2n} = Ax_{2n+1}$, $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$, For $n = 0, 1, 2, \cdots$. Then, by (3) we have successively

$F(G(T x_{2n}, B x_{2n+1}, B x_{2n+1}),$

$G(S x_{2n}, Ax_{2n+1}, Ax_{2n+1}), G(S x_{2n},$

$T x_{2n}, T x_{2n}),$

$G(A x_{2n+1}, B x_{2n+1}, B x_{2n+1}), G(S x_{2n},$

$B x_{2n+1}, B x_{2n+1}), G(A x_{2n+1},$

$T x_{2n}, T x_{2n})) \leq 0,$

$F(G(y_{2n}, y_{2n+1}, y_{2n+1}), G(y_{2n-1}, y_{2n}, y_{2n}),$

$G(y_{2n-1}, y_{2n}, y_{2n}), G(y_{2n}, y_{2n+1}, y_{2n+1}),$

$G(y_{2n-1}, y_{2n+1}, y_{2n+1}), 0) \leq 0.$

By (F_1) and (G_5) we obtain

$F(G(y_{2n}, y_{2n+1}, y_{2n+1}), G(y_{2n-1}, y_{2n}, y_{2n}),$

$G(y_{2n-1}, y_{2n}, y_{2n}), G(y_{2n}, y_{2n+1}, y_{2n+1}),$

$G(y_{2n-1}, y_{2n}, y_{2n}) +$

$G(y_{2n}, y_{2n+1}, y_{2n+1}), 0) \leq 0.$

By (F_a) we obtain

$(y_{2n}, y_{2n+1}, y_{2n+1}) \leq \phi_1(G(y_{2n-1}, y_{2n}, y_{2n}))$

$\leq \phi_1^2(G(y_{2n-2}, y_{2n-1}, y_{2n-1}))$

$\leq \phi_1^n(G(y_0, y_1, y_1)).$

Hence, for all n even or odd we have

$G(y_n, y_{n+1}, y_{n+1}), \leq \phi_1^n(G(y_0, y_1, y_1)).$

Then for $m > n$ and by G_5 we obtain

$$\begin{aligned} &G(y_n, y_m, y_m) \\ &\leq G(y_n, y_{n+1}, y_{n+1}) + \ldots + G(y_{m-1}, y_m, y_m) \\ &\leq \phi_1^n(G(y_0, y_1, y_1)) + \ldots + \phi_1^{m-1}(G(y_0, y_1, y_1)) \\ &= \sum_{k=n}^{m-1} \phi_1^k(G(y_0, y_1, y_1)). \end{aligned}$$

Since $\sum_{k=n}^{\infty} \phi_1^k(G(y_0, y_1, y_1)) < \infty$, then for any $\varepsilon > 0$, there exists $k \in N$ such that for $m > n \geq k$, $\sum_{k=n}^{m-1} \phi_1^k(G(y_0, y_1, y_1)) < \varepsilon$. Hence by Lemma (2.2) (y_n) is a G-Cauchy sequence. Let $A(X)$ is G-completesubspace of X, there exists a point q in $A(X)$, such that

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Ax_{2n+1} = q.$$

Also, we can find a point $p \in X$ such that $Ap = q$. Since $\lim_{n\to\infty} y_{2n} = q$, then also

$\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = q$. We provethat $Bp = Ap$. By (3) we have

$F(G(Tx_{2n}, Bp, Bp), G(Sx_{2n}, Ap, Ap), G(Sx_{2n},$

$Tx_{2n}, Tx_{2n}), G(Ap, Bp, Bp),$

$G(Sx_{2n}, Bp, Bp), G(Ap, Tx_{2n}, Tx_{2n})) \leq 0,$

Letting n tend to infinity, we obtain

$F(G(q, Bp, Bp), 0, 0, G(q, Bp, Bp), G(q, Bp, Bp), 0) \leq 0.$

By (F_a) it follows that there exists a function $\phi_1 \in \phi$ such that $G(q, Bp, Bp) \leq \phi_1(0) = 0$, a contradiction. Hence $Bp = q$. Then $Bp = Ap =$

$q$is apoint of coincidence of $A$and B. Since (A, B) is weakly compatible then$Aq = ABp = BAp = Bq$. Since $B(X) \subseteq S(X)$, $Bp = q$ then $q \in S(X)$, so there is $r \in X$such that $Sr = q$. We prove $Tr = q$. By (3)we obtain successively

$F(G(Tr, Bp, Bp), G(Sr, Ap, Ap),$

$G(Sr, Tr, Tr), G(Ap, Bp, Bp), G(Sr, Bp, Bp),$

$G(Ap, Tr, Tr)) \leq 0,$

$F(G(Tr, q, q), 0, G(q, Tr, Tr), 0, 0,$

$G(q, Tr, Tr)) \leq 0.$

Since (X, G) is symmetric and by (F_b) there exists a function $\phi_2 \in \phi$such that $G(Tr, q, q) \leq \phi_2(0) = 0$. So $G(Tr, q, q) = 0$, hence $Tr = q$. Therefore $Tr = Sr = q$. Since (S, T) is weakly compatible then $Sq = STr = TSr = Tq$. By (3) replacing $x = q$ and $y = q$we obtain

$F(G(Tq, Bq, Bq), G(Tq, Bq, Bq), 0, 0,$

$G(Tq, Bq, Bq), G(Bq, Tq, Tq)) \leq 0.$

Since (X, G) is symmetric and by (F_3), there existsa function $\phi_3 \in \phi$ such that

$$\begin{aligned} G(Tq, Bq, Bq) &\leq \phi_3(G(Bq, Tq, Tq)) \\ &= \phi_3((G(Tq, Bq, Bq)) \\ &< G(Tq, Bq, Bq), \end{aligned}$$

a contradiction. Hence $Tq = Bq$. Therefor $Tq = Sq = Aq = Bq$. By (3) we obtain

$F(G(Tq, Bp, Bp), G(Tq, Bp, Bp), 0, 0,$

$G(Tq, Bp, Bp), G(Bp, Tq, Tq)) \leq 0,$

$F(G(Tq, q, q), G(Tq, q, q), 0, 0,$

$G(Tq, q, q), G(q, Tq, Tq)) \leq 0.$

Since (X, G) is symmetric and by (F_3) we conclude $Tq = q$. Therefore $Tq = Sq = Aq = Bq = q$, so q is a common fixed point of A, B, T, and S. ByLemma (4.4), q is the unique common fixed pointof A, B, T, and S. In the cases for$B(X)$, $S(X)$ *or* $T(X)$ is aG-complete subspace of $X$the proof is similar.

Remark 4.5. If we put $B = T$ and $A = S$we haveTheorem (4.2).

By Example (3.5) and theorem (4.4) we get the followingcorollary.

Corollary 4.2. Let (X, G) be a symmetric G-metricspace and $A, B, S, T : (X, G) \to (X, G)$ satisfying the condition

$$\begin{aligned} &G(Tx, By, By) \\ &\leq kmax\{G(Sx, Ay, Ay), G(Sx, Tx, Tx), \\ &G(Ay, By, By), G(Sx, By, By), G(Ay, Tx, Tx)\}, \quad (7) \end{aligned}$$

for all $x, y \in X$, where $k \in [0, \frac{1}{2})$, such that:

(a) $T(X) \subseteq A(X)$and$B(X) \subseteq S(X)$,

(b) the pairs (A, B) and(T, S) are weakly compatible,

(c) one of $A(X), B(x), S(X)$, or $T(X)$ is a G-completesubspace of X.

Then A, B, S, and T have a unique common fixed point.

Remark 4.6. Also, by Examples in [24] and Examples(3.3), (3.4) we have a new results.

Example 4.7. Let $X = [0, \infty)$ with the symmetric G-metric space $G(x, y, z) = |x - y| + |y - z| + |z - x|$,

and A, B, S and T are self mappings of X defined by

$$Tx = \begin{cases} 0, & if\ x \in [0,1) \\ 1, & if\ x \in [1, \infty) \end{cases}, \qquad Sx = \begin{cases} 3, & if\ x \in [0,1) \\ \frac{1}{x}, & if\ x \in [1, \infty) \end{cases}$$

$$Ax = \begin{cases} 0, & if\ x \in [0,1) \\ \frac{1}{\sqrt{x}}, & if\ x \in [1, \infty) \end{cases}, \qquad Bx = 1, \ if\ x \in [0, \infty),$$

Clearly $T(X) \subseteq A(X)$ and $B(X) \subseteq S(X)$, $A(X)$is G-complete subspace of X, and the pairs (T, S) and (A, B) are weakly compatible. Takethe implicit relation defined as $F(t_1, \ldots, t_6) = t_1 - kmax\{t_2, t_3, t_4, t_5, t_6\}$,where $k \in \left(0, \frac{1}{2}\right)$. Then

$$\begin{aligned} t_1 &= 2|Tx - By|, & t_2 &= 2|Sx - Ay|, \\ t_3 &= 2|Sx - Ty|, & t_4 &= 2|Ax - By|, \\ t_5 &= 2|Sx - By|, & t_6 &= 2|Ay - Tx|. \end{aligned}$$

- If$x, y \in [0,1)$ we obtain that $t_2 = t_3 = max\{t_2, t_3, t_4, t_5, t_6\}$, and$t_1 \leq \frac{1}{3}\ max\{t_2, t_3, t_4, t_5, t_6\}$.
- If $x, y \in [1, \infty)$ we obtain that $t_1 = 0$ so we done and choose $k = \frac{1}{3}$.
- If $x \in [0,1)$and $y \in [1, \infty)$we obtain that$t_3 = max\{t_2, t_3, t_4, t_5, t_6\}$, and$t_1 \leq \frac{1}{3}max\{t_2, t_3, t_4, t_5, t_6\}$.

- If $x \in [1, \infty)$ and $y \in [0,1)$ we obtain that $t_1 = 0$ so we done and choose $k = \frac{1}{3}$,

the inequality (7) holds for all $x, y \in X$. The hypotheses of Corollary (4.2) satisfied, and 1 is the unique common fixed point of the mappings $A, B, S$and T.

5. CONCLUSIONS

In this paper, we introduced some common fixed point theorems for two and four mappings satisfying $\varphi -$ implicit relation in G-metric spaces and a common fixed point theorem for T-contraction is proved .The results improved the results of Popa and Patriciu [24].

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