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Research Article

On Stability for a Class of Fractional Differential Equations in Complex Banach Spaces

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Abstract: In this paper, we consider the Hyers-Ulam stability for the following fractional differential equations in sense of Caputo fractional derivative defined in the unit disk: ${}^cD_z^{\beta} f(z) = G(f(z), z\,{}^cD_z^{\alpha} f(z); z)$, $0 < \alpha < 1 < \beta < 2$, in a complex Banach space. Furthermore, a generalization of the admissible functions in complex Banach spaces is imposed and applications are illustrated.

Keywords: Analytic function; unit disk; Hyers-Ulam stability, admissible functions; Fractional calculus: fractional differential equation, Caputo fractional derivative

AMS Mathematics Subject Classification: 30C45, 39B72, 39B82

1. INTRODUCTION

A classical problem in the theory of functional equations is that: If a function f approximately satisfies functional equation E when does there exists an exact solution of E which f approximates. In 1940, S. M. Ulam [1] imposed the question of the stability of Cauchy equation and in 1941, D. H. Hyers [2] solved it. In 1978, Th. M. Rassias [3] provided a generalization of Hyers theorem by proving the existence of unique linear mappings near approximate additive mappings. The problem has been considered for many different types of spaces (see [4-6]). Recently, Li and Hua [7] discussed and proved the Hyers-Ulam stability of spacial type of finite polynomial equation, and Bidkham, Mezerji and Gordji [8] introduced the Hyers-Ulam stability of generalized finite polynomial equation. Finally, M.J. Rassias [9] imposed a Cauchy type additive functional equation and investigated the generalised Hyers-Ulam "product-sum" stability of this equation. Recently, the author studied stabilities of different kinds of fractional problems [10-15].

The main advantage of Caputo fractional derivative is that the fractional differential equations with Caputo fractional derivative use the initial conditions (including the mixed boundary conditions) on the same character as for the integer-order differential equations. In the present work, we will show another advantage of Caputo fractional derivative based on admissible functions in complex Banach spaces.

2. MATERIALS AND METHODS

Let $U := \{z \in \mathsf{C} : |z| < 1\}$ be the open unit disk in the complex plane C and H denote the space of all analytic functions on U. Here we suppose that H as a topological vector space endowed with the topology of uniform convergence over compact subsets of U. Also for $a \in \mathsf{C}$ and $m \in \mathsf{N}$, let $\mathsf{H}[a, m]$ be the subspace of H consisting of functions of the form

$$f(z) = a + a_m z^m + a_{m+1} z^{m+1} + \ldots, \quad z \in U.$$

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Srivastava and Owa [16], posed definitions for fractional operators (derivative and integral) in the complex z-plane $\mathbb{C}$ as follows:

Definition 2.1 The fractional derivative of order α is defined, for a function $f(z)$ by

$$D_z^\alpha f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\alpha} d\zeta,$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 2.2 The fractional integral of order $\alpha > 0$ is defined, for a function $f(z)$, by

$$I_z^\alpha f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} d\zeta; \alpha > 0,$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane ($\mathbb{C}$) containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Remark 2.1

$$D_z^\alpha z^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} z^{\mu-\alpha}, \mu > -1$$

and

$$I_z^\alpha z^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} z^{\mu+\alpha}, \mu > -1.$$

In [16], it was shown the relation

$$I_z^\alpha D_z^\alpha f(z) = D_z^\alpha I_z^\alpha f(z) = f(z).$$

Definition 2.3. The Caputo fractional derivative of order $\mu > 0$ is defined, for a function $f(z)$ by

$$^c D_z^\mu f(z) := \frac{1}{\Gamma(n-\mu)} \int_0^z \frac{f^{(n)}(\zeta)}{(z-\zeta)^{\mu-n+1}} d\zeta,$$

where $n = [\mu] + 1$, (the notation $[\mu]$ stands for the largest integer not greater than μ , for example, when μ =3/2, then [μ]=1 and hence n=2), the function $f(z)$ is analytic in simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z-\zeta)^{n-\mu-1}$ is removed by requiring $log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Remark 2.2

Let $\mu, \nu > 0$ and $n = [\mu] + 1$, then the following relations hold:

(i) $^c D_z^\mu z^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma(\nu-\mu)} z^{\nu-1}, \quad \nu > n;$

(ii) $^c D_z^\mu z^m = 0, \quad m = 0, ..., n-1;$

(iii) $I_z^\mu \, {}^c D_z^\mu f(z) = f(z), \quad z \in U$

(iv) $^c D_z^\mu I_z^\mu f(z) = f(z) - \sum_{j=0}^{n-1} c_j z^j; \quad z \in U,$

where c_j are some constants and

(v) $^c D_z^\mu I_z^\alpha f(z) = I_z^{\alpha-\mu} f(z), \quad \alpha - \mu > 0.$

More details on fractional derivatives and their properties and applications can be found in [17,18].

In the present paper, we study the generalized Hyers-Ulam stability for holomorphic solutions of the fractional differential equation in complex Banach spaces X and Y

$$^c D_z^\beta f(z) = G(f(z), z \, {}^c D_z^\alpha f(z); z),$$
$$0 < \alpha < 1 < \beta < 2, \tag{1}$$

where $G: X^2 \times U \to Y$ and $f: U \to X$ are holomorphic functions such that $f(0) = \Theta$ (Θ is the zero vector in X).

3. RESULTS

In this section we present extensions of the generalized Hyers-Ulam stability to holomorphic vector-valued functions. Let X, Y represent complex Banach space. The class of admissible functions $\mathcal{G}(X,Y)$, consists of those functions

$g: X^2 \times U \to Y$ that satisfy the admissibility conditions:

$$\| g(r, ks; z) \| \ge 1, when \| r \| = 1, \| s \| = 1, \quad (z \in U, k \ge 1). \tag{2}$$

We next introduce the generalized Hyers-Ulam stability depending on the properties of the fractional operators.

Definition 3.1 Let p be a real number. We say that

$$\sum_{n=0}^{\infty} a_n z^{n+\alpha} = f(z) \tag{3}$$

has the generalized Hyers-Ulam stability if there exists a constant $K > 0$ with the following property:

for every $\varepsilon > 0, w \in \overline{U} = U \bigcup \partial U,$ if

$$\left| \sum_{n=0}^{\infty} a_n w^{n+\alpha} \right| \le \varepsilon \left(\sum_{n=0}^{\infty} \frac{|a_n|^p}{2^n} \right)$$

then there exists some $z \in \overline{U}$ that satisfies equation (3) such that

$$| z^i - w^i | \le \varepsilon K,$$

$$(z, w \in \overline{U}, \quad i \in \mathbb{N}).$$

We need the following results:

Lemma 3.1 [19] If $f: D \to X$ is holomorphic, then $\| f \|$ is a subharmonic of $z \in D \subset \mathbb{C}$. It follows that $\| f \|$ can have no maximum in D unless $\| f \|$ is of constant value throughout D.

Lemma 3.2 [20] Let $f: U \to X$ be the holomorphic vector-valued function defined in the unit disk U with $f(0) = \Theta$ (the zero element of X). If there exists a $z_0 \in U$ such that

$$\| f(z_0) \| = \max_{|z| = |z_0|} \| f \|,$$

then

$$\| z_0 f'(z_0) \| = \kappa \| f(z_0) \|, \quad \kappa \ge 1.$$

Theorem 3.1 Let $G \in \mathrm{G}(X, Y)$. If $f: U \to X$ is a holomorphic vector-valued function defined in the unit disk U, with $f(0) = \Theta$, then then

$$\| G(f(z), z\, {}^c D_z^\alpha f(z); z) \| < 1 \Rightarrow \| f(z) \| < 1. \tag{4}$$

Proof. From Definition 2.3, we observe that

$$\begin{aligned} \| z\, {}^c D_z^\alpha f(z) \| &= \left\| \frac{z}{\Gamma(1-\alpha)} \int_0^z \frac{f'(\zeta)}{(z-\zeta)^\alpha} d\zeta \right\| \\ &\le \frac{\| z f'(z) \|}{\Gamma(2-\alpha)} |z|^{1-\alpha} \\ &\le \frac{\| z f'(z) \|}{\Gamma(2-\alpha)}, \quad z \in U. \end{aligned}$$

Assume that $\| f(z) \| \ge 1$ for $z \in U$. Thus, there exists a point $z_0 \in U$ for which $\| f(z_0) \| = 1.$ According to Lemma 3.1, we have

$$\| f(z) \| < 1, \quad z \in U_{r_0} = \{ z : |z| < |z_0| = r_0 \},$$

and

$$\max_{|z| \le |z_0|} \| f(z) \| = \| f(z_0) \| = 1.$$

In view of Lemma 3.2, at the point z_0 there is a constant $k \ge 1$ such that

$$\| z_0 f'(z_0) \| = \kappa \| f(z_0) \| = \kappa.$$

Therefore,

$$\begin{aligned} \| z_0\, {}^c D_{z_0}^\alpha f(z_0) \| &= \frac{\| z_0 f'(z_0) \|}{\Gamma(2-\alpha)} \\ &= \frac{\kappa \| f(z_0) \|}{\Gamma(2-\alpha)} = \frac{\kappa}{\Gamma(2-\alpha)}, \end{aligned}$$

consequently, we obtain

$$\| f(z_0) \| = \frac{\Gamma(2-\alpha)}{\kappa} \| z_0\, {}^c D_{z_0}^\alpha f(z_0) \| = 1, \quad \kappa \ge 1.$$

We put $k:=\frac{\kappa}{\Gamma(2-\alpha)}\geq 1$; hence from equation (3), we deduce

$$\| \; G(f(z_0), z_0\, {}^cD^{\alpha}_{z_0} f(z_0); z_0) \; \|$$
$$= \| \; G(f(z_0), k[z_0\, {}^cD^{\alpha}_{z_0} f(z_0)/k]; z_0) \; \|$$
$$\geq 1,$$

which contradicts the hypothesis in (4), we must have $\| \; f(z) \; \| < 1.$

Corollary 3.1 Assume the problem (2). If $G \in \mathsf{G}(X,Y)$ is the holomorphic vector-valued function defined in the unit disk U then

$$\| \; G(f(z), z\, {}^cD^{\alpha}_z f(z); z) \; \| < 1$$
$$\Rightarrow \| \; I^{\alpha}_z G(f(z), z\, {}^cD^{\alpha}_z f(z); z) \; \| < 1. \tag{5}$$

Proof. By continuity of $G,$ the fractional differential equation (1) has at least one holomorphic solution f. According to Remark 1.2, the solution $f(z)$ of the problem (1) takes the form

$$f(z) = I^{\alpha}_z G(f(z), z\, {}^cD^{\alpha}_z f(z); z).$$

Therefore, in virtue of Theorem 3.1, we obtain the assertion (5).

Theorem 3.2 Let $G \in \mathsf{G}(X,Y)$ be holomorphic vector-valued functions defined in the unit disk U then the equation (1) has the generalized Hyers-Ulam stability for $z \to \partial U.$

Proof. Assume that

$$G(z) := \sum_{n=0}^{\infty} \varphi_n z^n, \quad z \in U$$

therefore, by Remark 1.2, we have

$$I^{\alpha}_z G(z) = \sum_{n=0}^{\infty} a_n z^{n+\alpha} = f(z).$$

Also, $z \to \partial U$ and thus $|z| \to 1.$ According to Theorem 3.1, we have

$$\| \; f(z) \; \| < 1 = |z|.$$

Let $\varepsilon > 0$ and $w \in \overline{U}$ be such that

$$|\sum_{n=1}^{\infty} a_n w^{n+\alpha}| \leq \varepsilon(\sum_{n=1}^{\infty} \frac{|a_n|^p}{2^n}).$$

We will show that there exists a constant K independent of ε such that

$$|w^i - u^i| \leq \varepsilon K, \quad w \in \overline{U}, u \in U$$

and satisfies (3). We put the function

$$f(w) = \frac{-1}{\lambda a_i} \sum_{n=1, n\neq i}^{\infty} a_n w^{n+\alpha}, \quad a_i \neq 0, 0 < \lambda < 1, \tag{6}$$

thus, for $w \in \partial U,$ we obtain

$$\begin{aligned} |w^i - u^i| \; &= |w^i - \lambda f(w) + \lambda f(w) - u^i| \\ &\leq |w^i - \lambda f(w)| + \lambda |f(w) - u^i| \\ &< |w^i - \lambda f(w)| + \lambda |w^i - u^i| \\ &= |w^i + \frac{1}{a_i} \sum_{n=1, n\neq i}^{\infty} a_n w^{n+\alpha}| + \lambda |w^i - u^i| \\ &= \frac{1}{|a_i|} |\sum_{n=1}^{\infty} a_n w^{n+\alpha}| + \lambda |w^i - u^i|. \end{aligned}$$

Without loss of generality, we consider $|a_i| = \max_{n\geq 1}(|a_n|)$ yielding

$$\begin{aligned} |w^i - u^i| \; &\leq \frac{1}{|a_i|(1-\lambda)} |\sum_{n=1}^{\infty} a_n w^{n+\alpha}| \\ &\leq \frac{\varepsilon}{|a_i|(1-\lambda)} (\sum_{n=0}^{\infty} \frac{|a_n|^p}{2^n}) \\ &\leq \frac{\varepsilon |a_i|^{p-1}}{(1-\lambda)} (\sum_{n=0}^{\infty} \frac{1}{2^n}) \\ &\leq \frac{2\varepsilon |a_i|^{p-1}}{(1-\lambda)} \\ &:= K\varepsilon. \end{aligned}$$

This completes the proof.

4. DISCUSSION

In this section, we introduce some applications of functions to achieve the generalized Hyers-Ulam stability.

Example 4.1 Consider the function $G: X^2 \times U \to \mathbb{R}$ by

$$G(r,s;z) = a(\| r \| + \| s \|)^n + b|z|^2, \quad n \in N_+$$

with $a \geq 0.5$, $b \geq 0$ and $G(\Theta,\Theta,0) = 0$. Our aim is to apply Theorem 3.1. this follows since

$$\| G(r,ks;z) \| = a(\| r \| + k\| s \|)^n + b|z|^2$$
$$= a(1+k)^n + b|z|^2 \geq 1,$$

when $\| r \| = \| s \| = 1$, $z \in U$. Hence by Theorem 3.1, we have : If $a \geq 0.5$, $b \geq 0$ and $f: U \to X$ is a holomorphic vector-valued function defined in U, with $f(0) = \Theta$, then

$$a(\| f(z) \| + \| z\, {}^cD_z^\alpha f(z) \|)^n + b|z|^2 < 1$$
$$\Rightarrow \| f(z) \| < 1.$$

Consequently, $\| I_z^\alpha G(f(z), z\, {}^cD_z^\alpha f(z); z) \| < 1$, thus in view of Theorem 3.2, f has the generalized Hyers-Ulam stability.

Example 4.2 Assume the function $G: X^2 \to X$ by

$$G(r,s;z) = G(r,s) = re^{\| s \|^m - 1}, \quad m \geq 1$$

with $G(\Theta,\Theta) = \Theta$. By applying Theorem 3.1, we need to show that $G \in \mathrm{G}(X,X)$. Since

$$\| G(r,ks) \| = \| re^{\| ks \|^m - 1} \|$$
$$= e^{k^m - 1} \geq 1,$$

when $\| r \| = \| s \| = 1, k \geq 1$. Hence, by Theorem 3.1, we have : For $f: U \to X$ is a holomorphic vector-valued function defined in U, with $f(0) = \Theta$, then,

$$\| f(z)e^{\| z\, {}^cD_z^\alpha f(z) \|^m - 1} \| < 1$$
$$\Rightarrow \| f(z) \| < 1.$$

Consequently, $\| I_z^\alpha G(f(z), z\, {}^cD_z^\alpha f(z); z) \| < 1$; thus in view of Theorem 3.2, f has the generalized Hyers-Ulam stability.

Example 4.3 Let $a, b: U \to \mathbb{C}$ satisfy

$$|a(z) + \mu b(z)| \geq 1,$$

for every $\mu \geq 1, \nu > 1$ and $z \in U$. Consider the function $G: X^2 \to Y$ by

$$G(r,s;z) = a(z)r + \mu b(z)s,$$

with $G(\Theta,\Theta) = \Theta$. Now for $\| r \| = \| s \| = 1$, we have

$$\| G(r,\mu s;z) \| = |a(z) + \mu b(z)| \geq 1$$

and thus $G \in \mathrm{G}(X,Y)$. If $f: U \to X$ is a holomorphic vector-valued function defined in U, with $f(0) = \Theta$, then

$$\| a(z)f(z) + b(z)z\, {}^cD_z^\alpha f(z) \| < 1$$
$$\Rightarrow \| f(z) \| < 1.$$

Hence according to Theorem 3.2, f has the generalized Hyers-Ulam stability.

Example 4.4 Let $\lambda: U \to \mathbb{C}$ be a function such that

$$\Re\left(\frac{1}{\lambda(z)}\right) > 0,$$

for every $z \in U$. Consider the function $G: X^2 \to Y$ by

$$G(r,s;z) = r + \frac{s}{\lambda(z)},$$

with $G(\Theta,\Theta) = \Theta$. Now for $\| r \| = \| s \| = 1$, we have

$$\| G(r,ks;z) \| = \left|1 + \frac{k}{\lambda(z)}\right| \geq 1, \quad k \geq 1$$

and thus $G \in \mathrm{G}(X,Y)$. If $f: U \to X$ is a holomorphic vector-valued function defined in U, with $f(0) = \Theta$, then

$$\| \, f(z) + \frac{z\, {}^{c}D_{z}^{\alpha} f(z)}{\lambda(z)} \, \| < 1$$
$$\Rightarrow \| \, f(z) \, \| < 1.$$

Hence according to Theorem 3.2, f has the generalized Hyers-Ulam stability.

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