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Research Article

Properties of Univalent Solution for Complex Fractional Differential Equation

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Abstract: In this note, we discuss some properties of univalent solutions for fractional differential equation in the unit disk in the sense of Srivastava-Owa operators. By employing the differential subordination concept, we will study the upper bound of these solutions. Furthermore, by applying the Rogosinski theorem and Goluzin theorem, we illustrate some inequalities involving the coefficients and integral representation of these solutions. Moreover, the uniqueness is studied by using Rouche's theorem.

Keywords: Fractional calculus; fractional differential equation; unit disk; univalent function; analytic function; subordination; superordination

AMS Mathematics Subject Classification: 30C45.

1. INTRODUCTION

In [1], Srivastava and Owa, gave definitions for fractional operators (derivative and integral) in the complex z-plane $\mathbb{C}$ as follows:

Definition 1.1 The fractional derivative of order α is defined, for a function $f(z)$ by

$$D_z^\alpha f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\alpha} d\zeta;\ 0 \le \alpha < 1,$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 1.2 The fractional integral of order α is defined, for a function $f(z)$, by

$$I_z^\alpha f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} d\zeta;\ \alpha > 0,$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane $(\mathbb{C})$ containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Let $\mathcal{A}$ denote the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U. \tag{1}$$

Also, let $\mathcal{S}$ and $\mathcal{C}$ denote the subclasses of $\mathcal{A}$ consisting of functions which are, respectively, univalent and convex in U. It is well known that; if the function $f(z)$ given by (3) is in the class $\mathcal{S}$, then $|a_n| \le n, \quad n \in \mathbb{N} \setminus \{1\}$. Moreover, if the

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function $f(z)$ given by (1) is in the class $\mathbb{C}$, then $|a_n| \le 1, \quad n \in \mathbb{N}$.

Definition 1.3 (Subordination Principal). For two functions f and g analytic in U, we say that the function $f(z)$ is subordinate to $g(z)$ in U and we write as $f(z) \prec g(z) (z \in U)$, if there exists a Schwarz function $w(z)$ analytic in U with $w(0) = 0$, and $|w(z)| < 1$, such that $f(z) = g(w(z)), z \in U$. In particular, if the function $g(z)$ is univalent in U, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

Definition 1.4 (Differential subordination) Let $\phi: \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the differential subordination $\phi(p(z)), zp'(z)) \prec h(z)$ then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, if $p \prec q$. If p and $\phi(p(z)), zp'(z))$ are univalent in U and satisfy the differential superordination $h(z) \prec \phi(p(z)), zp'(z))$, then p is called a solution of the differential superordination. An analytic function q is called subordinant of the solution of the differential superordination if $q \prec p$.

In [2] the authors imposed a linear fractional differential operator based on $D_z^\alpha f(z)$ as follows:

$$
\begin{aligned}
D^0 f(z) &= f(z) \\
&= z + \sum_{n=2}^{\infty} a_n z^n, \\
D^{1,\alpha}_{\beta,\lambda} f(z) &= (\beta - \lambda)\Phi^\alpha f(z) + \lambda z(\Phi^\alpha f(z))' + (1-\beta)z \\
&= z + \sum_{n=2}^{\infty} [\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}][\lambda(n-1)+\beta] a_n z^n \\
&= D^\alpha_{\beta,\lambda}(f(z)), \\
D^{2,\alpha}_{\beta,\lambda} f(z) &= D^\alpha_{\beta,\lambda}(D^{1,\alpha}_{\beta,\lambda} f(z)) \\
&= z + \sum_{n=2}^{\infty} \{[\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}][\lambda(n-1)+\beta]\}^2 a_n z^n, \\
&\vdots \\
D^{k,\alpha}_{\beta,\lambda} f(z) &= D^\alpha_{\beta,\lambda}(D^{k-1,\alpha}_{\beta,\lambda} f(z)) \\
&= z + \sum_{n=2}^{\infty} \{[\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}][\lambda(n-1)+\beta]\}^k a_n z^n \\
&:= z + \sum_{n=2}^{\infty} \Psi_{n,k}(\alpha,\beta,\lambda) a_n z^n,
\end{aligned} \tag{2}
$$

for $0 \le \alpha < 1, \beta \ge 1, \lambda \ge 0$ and $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ with $D^{k,\alpha}_{\beta,\lambda} f(0) = 0$.

Corresponding to the differential operator $D^{k,\alpha}_{\beta,\lambda}$, the fractional integral operator is given as follows

$$I^{k,\alpha}_{\beta,\lambda} f(z) = z + \sum_{n=2}^{\infty} \frac{a_n z^n}{\{[\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}][\lambda(n-1)+\beta]\}^k}$$

and hence we obtain

$$I^{k,\alpha}_{\beta,\lambda} D^{k,\alpha}_{\beta,\lambda} f(z) = f(z).$$

Remark 1.1. The operator (2) generalizes various types of operators such as Sălăgean's differential operator [3], Al-Oboudi's differential operator [4], Srivastava-owa fractional differential operator, the linear multiplier fractional differential operator which introduced in [5]. Finally, when $\beta = 1$, we obtain the linear fractional operator as in [6].

Here, we consider the following class of fractional differential equation:

$$\frac{z(D^{k,\alpha}_{\beta,\lambda} f(z))'}{D^{k,\alpha}_{\beta,\lambda} f(z)} = p(z) \tag{3}$$

such that $p(z) \prec q(z)$ and $q(0) = p(0) = 1$. We denote this class by $S^{k,\alpha}_{\beta,\lambda}(q)$.

We need the following preliminaries in the sequel.

The Libera-Pascu integral operator $L_a : \mathbb{A} \to \mathbb{A}$ defined by

$$F(z) := L_a f(z) = \frac{1+a}{z^a} \int_0^z f(t) t^{a-1} dt, a \in \mathbb{C}, \Re(a) \ge 0.$$

For $a = 1$ we obtain the Libera integral operator, for $a = 0$ we obtain the Alexander integral operator and in the case $a = 1,2,3,...$ we obtain the Bernardi integral operator.

Lemma 1.1 [7] Let h be convex in U and θ, ϕ be analytic in domain D. Let p be analytic in U, with $h(0) = \theta(p(0))$ and $p(U) \subset D$. If the differential equation

$$\theta[q(z)] + zq'(z)\phi[q(z)] = h(z)$$

has a univalent solution in U that satisfies $q(0)=p(0)$ and $\theta[q(z)]\prec h(z)$ then the differential subordination

$$\theta[p(z)]+zp'(z)\phi[p(z)]\prec h(z)$$

implies that $p(z)\prec q(z)$. The function q is the best dominant.

Lemma 1.2 (Rogosinski Theorem) [8] Let $f(z)=\sum_{n=1}^{\infty}a_n z^n$ and $g(z)=\sum_{n=1}^{\infty}b_n z^n$ be analytic in U and suppose $g\prec f$. Then

$$\sum_{n=1}^{k}|b_n|^2\le\sum_{n=1}^{k}|a_n|^2 .$$

Lemma 1.3 (Goluzin Theorem) [8] If $g\prec f$ and $\lambda_1\ge\lambda_2\ge\ldots\ge 0,$ then

$$\sum_{n=1}^{\infty}\lambda_n|b_n|^2\le\sum_{n=1}^{\infty}\lambda_n|a_n|^2 .$$

2. MAIN RESULTS

Our main results are provided in this section.

Theorem 2.1 Let $f\in S_{\beta,\lambda}^{k,\alpha}(q),$

$$P(z):=\frac{z(D_{\beta,\lambda}^{k,\alpha}F(z))'}{D_{\beta,\lambda}^{k,\alpha}F(z)},$$

where F is the Libera-Pascu integral operator, be analytic in U such that $P(U)\subset D$ (a domain) and

$$h(z)=(\frac{a}{1+a}q(z)+\frac{q^2(z)}{1+a}+\frac{zq'(z)}{1+a})\varphi(z),\quad \varphi(0)=1$$

has a univalent solution in the unit disk U such that $P(0)=q(0).$ Then the subordination

$$(\frac{a}{1+a}P(z)+\frac{P^2(z)}{1+a}+\frac{zP'(z)}{1+a})\phi(z)\prec$$
$$(\frac{a}{1+a}q(z)+\frac{q^2(z)}{1+a}+\frac{zq'(z)}{1+a})\phi(z)$$

implies $P(z)\prec q(z)$ for some analytic function φ and q is the best dominant.

Proof. Since $f\in S_{\beta,\lambda}^{k,\alpha}(q),$ then

$$\frac{z(D_{\beta,\lambda}^{k,\alpha}f(z))'}{D_{\beta,\lambda}^{k,\alpha}f(z)}\prec q(z).$$

From the definition of the Libera-Pascu integral operator we have

$$(1+a)f(z)=aF(z)+zF'(z),$$

and by using the linear operator $D_{\beta,\lambda}^{k,\alpha},$ we have

$$(1+a)D_{\beta,\lambda}^{k,\alpha}f(z)=aD_{\beta,\lambda}^{k,\alpha}F(z)+$$
$$D_{\beta,\lambda}^{k,\alpha}(zF'(z)),\quad \Re(a)\ge 0.$$

By making use the first derivative of the last assertion, we obtain

$$z(D_{\beta,\lambda}^{k,\alpha}f(z))'=\frac{a}{(1+a)}z(D_{\beta,\lambda}^{k,\alpha}F(z))'$$
$$+\frac{1}{(1+a)}z[D_{\beta,\lambda}^{k,\alpha}(zF'(z))]'$$

and using the fact that

$$D_{\beta,\lambda}^{k,\alpha}(zF'(z))=z(D_{\beta,\lambda}^{k,\alpha}F(z))',$$

yields

$$\begin{aligned} z(D_{\beta,\lambda}^{k,\alpha}f(z))' &= \frac{a}{(1+a)}z(D_{\beta,\lambda}^{k,\alpha}F(z))'+\frac{1}{(1+a)}z[z(D_{\beta,\lambda}^{k,\alpha}(F(z))']' \\ &= \{\frac{\frac{a}{(1+a)}z(D_{\beta,\lambda}^{k,\alpha}F(z))'}{D_{\beta,\lambda}^{k,\alpha}F(z)}+\frac{\frac{1}{(1+a)}z[z(D_{\beta,\lambda}^{k,\alpha}(F(z))']'}{D_{\beta,\lambda}^{k,\alpha}F(z)}\}D_{\beta,\lambda}^{k,\alpha}F(z) \\ &= (\frac{a}{1+a}P(z)+\frac{P^2(z)}{1+a}+\frac{zP'(z)}{1+a})D_{\beta,\lambda}^{k,\alpha}F(z) \end{aligned}$$

A computation implies that

$$\frac{z(D_{\beta,\lambda}^{k,\alpha}f(z))'}{D_{\beta,\lambda}^{k,\alpha}f(z)}=\theta[P(z)]+zP'(z)\phi(P(z))$$

where

$$\theta[P(z)]:=(\frac{a}{1+a}P(z)+\frac{P^2(z)}{1+a})\varphi(z),$$

$$\phi(P(z)):=\frac{\varphi(z)}{1+a}$$

and

$$\varphi(z) := \frac{D_{\beta,\lambda}^{k,\alpha} F(z)}{D_{\beta,\lambda}^{k,\alpha} f(z)}$$

are analytic in U. It is clear that $\theta[q(z)] \prec h(z)$. Hence in view of Lemma 1.1, we have $P(z) \prec q(z)$ and q is the best dominant.

Corollary 2.1 Let the assumptions of Theorem 2.1 hold. Then the Libera-Pascu integral operator $F(z) \in S_{\beta,\lambda}^{k,\alpha}(q)$.

Next, using Lemma 1.2 and Lemma 1.3, we introduce the coefficient inequalities as follows.

Theorem 2.2 Let $f \in S_{\beta,\lambda}^{k,\alpha}(q)$. Then

$$\sum_{n=1}^{k} |b_n|^2 + 1 \le \sum_{n=1}^{k} |q_n|^2 + 1, \tag{4}$$

where $q(z) = \sum_{n=0}^{\infty} q_n z^n, q(0) = 1$ and $b_1 := \Psi_{2,k}(\alpha,\beta,\lambda)a_2, b_2 := 2\Psi_{3,k}(\alpha,\beta,\lambda)a_3 - [\Psi_{2,k}(\alpha,\beta,\lambda)a_2]^2, \ldots$

Proof. Since $f \in S_{\beta,\lambda}^{k,\alpha}(q)$, then we have

$$\frac{z(D_{\beta,\lambda}^{k,\alpha} f(z))'}{D_{\beta,\lambda}^{k,\alpha} f(z)} \prec q(z).$$

A computation implies that

$$\begin{aligned}\frac{z(D_{\beta,\lambda}^{k,\alpha} f(z))'}{D_{\beta,\lambda}^{k,\alpha} f(z)} &= 1 + \Psi_{2,k}(\alpha,\beta,\lambda)a_2 z \\ &\quad + \{2\Psi_{3,k}(\alpha,\beta,\lambda)a_3 \\ &\quad - [\Psi_{2,k}(\alpha,\beta,\lambda)a_2]^2\}z^2 + \ldots \\ &\prec 1 + q_1 z + q_2 z^2 + \ldots.\end{aligned}$$

which is equivalent to

$$1 + \Psi_{2,k}(\alpha,\beta,\lambda)a_2 z + \{2\Psi_{3,k}(\alpha,\beta,\lambda)a_3 - [\Psi_{2,k}(\alpha,\beta,\lambda)a_2]^2\}z^2 + \ldots \prec 1 + q_1 z + q_2 z^2 + \ldots.$$

or

$$\Psi_{2,k}(\alpha,\beta,\lambda)a_2 z + \{2\Psi_{3,k}(\alpha,\beta,\lambda)a_3 - [\Psi_{2,k}(\alpha,\beta,\lambda)a_2]^2\}z^2 + \ldots \prec q_1 z + q_2 z^2 + \ldots.$$

thus in view of Lemma 1.2

$$\sum_{n=1}^{k} |b_n|^2 \le \sum_{n=1}^{k} |q_n|^2,$$

and consequently we obtain (4).

As applications of Lemma 1.3, we have the following theorems:

Theorem 2.3 Let $f \in S_{\beta,\lambda}^{k,\alpha}(q)$. Then for $m\lambda + \beta > 0, \quad m = 1,2,3,\ldots,$

$$\sum_{n=1}^{\infty} \lambda_n |b_n|^2 + 1 \le \sum_{n=1}^{\infty} |q_n|^2 \lambda_n + 1, \tag{5}$$

where

$$\begin{aligned}\lambda_1 &= \left(\frac{2(\lambda+\beta)}{2-\alpha}\right)^k \\ \lambda_2 &= \left(\frac{6(2\lambda+\beta)}{(2-\alpha)(3-\alpha)}\right)^k \\ \lambda_3 &= \left(\frac{24(3\lambda+\beta)}{(2-\alpha)(3-\alpha)(2-\alpha)}\right)^k \\ &\vdots,\end{aligned}$$

such that $\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge \ldots \ge 0.$

Theorem 2.4 Let $f \in \mathcal{A}$ such that $|a_1| \le |a_2| \le \ldots$. Assume that the solution of the fractional differential equation

$$D_{\beta,\lambda}^{k,\alpha} f(z) = g_\alpha(z) \tag{6}$$

satisfying the subordination $f(z) \prec G(z),\ G(0) = 0$. Then the solution is bounded.

Proof. The solution of the equation (6) has the form

$$f(z) = I_{\beta,\lambda}^{k,\alpha} g_\alpha(z).$$

Since $f(z) \prec G(z)$, then Lemma 1.3 implies

$$\begin{aligned}|f(z)| = \left|\sum_{n=1}^{\infty} a_n z^n\right| &\le \sum_{n=1}^{\infty} |a_n|, \quad (a_1 = 1) \\ &= \sum_{n=1}^{\infty} \frac{|a_n|^2}{|a_n|} \le \sum_{n=1}^{\infty} \frac{|g_n|^2}{|a_n|}.\end{aligned}$$

But $\frac{1}{|a_1|} \ge \frac{1}{|a_2|} \ge \ldots$ hence the solution $f(z)$ is bounded.

Theorem 2.5 If

$$\sum_{n=1,n\neq k}^{\infty} |a_n| \leq |a_k|, \quad a_k \neq 0, k \geq 2.$$

Then Eq. (3) has a unique solution in the unit disk.

Proof. By setting

$$\phi(z) = \frac{-1}{a_k} \sum_{n=1,n\neq k}^{\infty} a_n z^n, \quad k \geq 2$$

for $|z| \leq 1$, implies

$$\begin{aligned} |\phi(z)| &= |\frac{-1}{a_k} \sum_{n=1,n\neq k}^{\infty} a_n z^n | \\ &= |\frac{1}{a_k}|| \sum_{n=1,n\neq k}^{\infty} a_n z^n | \\ &\leq |\frac{1}{a_k}| \sum_{n=1,n\neq k}^{\infty} |a_n| \\ &< 1. \end{aligned}$$

Since $|\phi(z)| < 1$ for $|z| = 1$, and by Rouche's theorem, we observe that (3) has exactly one zero in U.

Theorem 2.6 Let the assumption of Theorem 2.4 hold. If for positive integer m

$$\sum_{n=1}^{\infty} |g_n|^2 < |a_m| \tag{7}$$

then the fractional differential equation (6) has a unique solution in U.

Proof. From the Proof of Theorem 2.4, we have

$$|f(z)| \leq \sum_{n=1}^{\infty} \frac{|g_n|^2}{|a_n|}.$$

By the assumption of the theorem, there exists a coefficient a_m such that $1/|a_n| \leq 1/|a_m|$ for all n. Therefore by (7), we obtain

$$\begin{aligned} |f(z)| &\leq |\frac{1}{a_m}| \sum_{n=1}^{\infty} |g_n|^2 \\ &< 1; \end{aligned}$$

thus by Rouche's theorem, Eq.(6) has a unique solution.

Theorem 2.7 Let the assumption of Theorem 2.6 hold. If

$$\sum_{n=1}^{\infty} |g_n|^2 < 1, \tag{8}$$

then the fractional differential equation (6) has a unique solution in U.

Proof. Since

$$|f(z)| \leq \sum_{n=1}^{\infty} \frac{|g_n|^2}{|a_n|}.$$

Then we impose

$$\begin{aligned} |f(z)| &\leq \sum_{n=1}^{\infty} \frac{|g_n|^2}{|a_n|} \\ &\leq \sum_{n=1}^{\infty} \frac{1}{|a_1|} |g_n|^2 \\ &= \sum_{n=1}^{\infty} |g_n|^2 \\ &< 1; \end{aligned}$$

hence by Rouche's theorem, Eq.(6) has a unique solution.

3. CONCLUSIONS

By employing a linear fractional differential operator in the unit disk in the sense of the Srivastava-Owa differential operator, fractional differential equations (3) and (6) are introduced. In Theorem 2.1, we considered that the Eq. (3) has a univalent solution. The important properties of this solution are described by making use of the Libera-Pascu integral operator, subordination concept, differential subordination and coefficients bound. Theorems 2.5-2.7 are proposing the unique solution of Eq.(3) and (6) in the unit disk by applying the Rouche's theorem. Theorem 2.6 showed the connection between the existence of unique solution and the coefficients bound. Similarly for Theorem 2.7.

4. ACKNOWLEDGEMENT

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