Proceedings of the Pakistan Academy of Sciences 50 (1):77–83 (2013)

ISSN: 0377 - 2969 (print), 2306 - 1448 (online)

Pakistan Academy of Sciences

Research Article

A New Criterion for Meromorphic Multivalent Starlike Functions of Order γ defined by Dziok and Srivastava Operator

Rabha M. El-Ashwah[1], Mohamed K. Aouf[2], Ali Shamandy[2] and Sheza M. El-Deeb[1*]

[1]Mathematics Department, Faculty of Science, Mansoura University, New Damietta 34517, Egypt
[2]Mathematics Department, Faculty of Science, Mansoura University, Mansoura 33516, Egypt

Abstract: In this paper we introduce a subclass $M_{p,q,s}(\alpha_1;\gamma)$ of meromorphic multivalent starlike functions of order γ defined by Dziok and Srivastava operator. The main object of this paper is to investigate various important properties and characteristics for this class. Further, a property preserving integrals is considered.

Keywords and pharses: Meromorphic functions, Hadamard product, generalized hypergeometric function.

2000 Mathematics Subject Classification. 30C45.

1. INTRODUCTION

Let $\sum_p$ be the class of functions of the form:

$$f(z) \quad \frac{1}{z^p}+\sum_{k\quad 0}^{\infty} a_k z^k \ (p\in \mathrm{N} \quad \{1,2,....\}), \qquad (1.1)$$

which are analytic and p-valent in the punctured unit disc $U^* = \{z : z\in \mathbb{C}$ and $0<|z|<1\} = U\setminus\{0\}$. For functions $f(z)\in\sum_p$ given by (1.1) and $g(z)\in\sum_p$ defined by

$$g(z) \quad \frac{1}{z^p}+\sum_{k\quad 0}^{\infty} b_k z^k \quad (p\in \mathrm{N}), \qquad (1.2)$$

then the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f*g)(z) \quad \frac{1}{z^p}+\sum_{k\quad 0}^{\infty} a_k b_k z^k \quad (g*f)(z). \qquad (1.3)$$

For real numbers $\alpha_1,...,\alpha_q$ and $\beta_1,...,\beta_s$ $(\beta_j \notin \mathrm{Z}_0^- \quad \{0,-1,-2,...\};\ j \quad 1,2,...,s)$, we now define the generalized hypergeometric function ${}_qF_s(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)$ by (see, for example, [15, p.19])

$$ {}_qF_s(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z) \quad \sum_{k\quad 0}^{\infty}\frac{(\alpha_1)_k...(\alpha_q)_k}{(\beta_1)_k...(\beta_s)_k}.\frac{z^k}{k!} \qquad (1.4)$$

$$(q\le s+1;\ q,s\in \mathrm{N}_0 \quad \mathrm{N}\cup\{0\};\ z\in U),$$

where $(\theta)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(\theta)_\nu=\frac{\Gamma(\theta+\nu)}{\Gamma(\theta)}=\begin{cases}1 & (\nu \quad 0;\ \theta\in \mathrm{C}^*=\mathrm{C}\setminus\{0\}) \\ \theta(\theta+1)...(\theta+\nu-1) & (\nu\in \mathrm{N};\ \theta\in \mathrm{C}).\end{cases} \qquad (1.5)$$

Corresponding to the function

Received, May 2012; Accepted, January 2013
*Corresponding author, Sheza M. El-Deeb: E-mail: shezaeldeeb@yahoo.com

$h_p(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)$, defined by

$$h_p(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z) = z^{-p}\,{}_qF_s(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z), \tag{1.6}$$

we consider a linear operator

$$H_p(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s):\ \sum_p \to \sum_p,$$

which is defined by the following Hadamard product:

$$H_p(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s)f(z) = h_p(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z) * f(z). \tag{1.7}$$

We observe that, for a function $f(z)$ of the form (1.1), we have

$$H_p(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s)f(z) = z^{-p} + \sum_{k=0}^{\infty}\Gamma_{k+p}(\alpha_1)a_k z^k. \tag{1.8}$$

where, for convenience

$$\Gamma_{k+p}(\alpha_1) = \frac{(\alpha_1)_{k+p}...(\alpha_q)_{k+p}}{(\beta_1)_{k+p}...(\beta_s)_{k+p}}.\frac{1}{(k+p)!}. \tag{1.9}$$

If, for convenience, we write

$$H_{p,q,s}(\alpha_1) = H_p(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s), \tag{1.10}$$

then one can easily verify from the definition (1.8) that (see [14])

$$z(H_{p,q,s}(\alpha_1)f(z))' = \alpha_1 H_{p,q,s}(\alpha_1+1)f(z) -(\alpha_1+p)H_{p,q,s}(\alpha_1)f(z). \tag{1.11}$$

The linear operator $H_{p,q,s}(\alpha_1)$ was investigated recently by Liu and Srivastava [14] and Aouf [3]. Some interesting subclasses of analytic functions associated with the generalized hypergeometric function, were considered recently by (for example) Dziok and Srivastava [6, 7], Gangadharan et al [8], Liu [12].

We note that:

(i) $H_{p,2,1}(a,1;c)f(z) = L_p(a,c)f(z)$ $(f(z)\in\sum_p,$ $a>0,\ c>0)$ (see Liu and Srivastava [13]);

(ii) $H_{p,2,1}(n+p,p;p)f(z) = D^{n+p-1}f(z) = \dfrac{1}{z^p(1-z)^{n+p}} * f(z)$ $(n>-p,\ p\in\mathbb{N})$ (see Aouf [1] and Uralegaddi and Somanatha [16]);

(iii) $H_{p,2,1}(\nu,1;\nu+1)f(z) = \mathbf{F}_{\nu,p}(f)(z)$ $(\nu>0,\ p\in\mathbb{N})$ (see Aouf [1], Uralegaddi and Somanatha [16] and Yang [17]).

Making use of the operator $H_{p,q,s}(\alpha_1)f(z)$, we say that a function $f(z)\in\sum_p$ is in the class $M_{p,q,s}(\alpha_1;\gamma)$ if it satisfies the following inequality:

$$\mathrm{Re}\left\{\frac{H_{p,q,s}(\alpha_1+1)f(z)}{H_{p,q,s}(\alpha_1)f(z)} - (p+1)\right\} < -\frac{p(\alpha_1-1)+\gamma}{\alpha_1} \tag{1.12}$$

or, equivalently, to

$$\mathrm{Re}\left\{\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)f(z)}\right\} < -\gamma$$

$$(\alpha_1,...,\alpha_q \in \mathbb{R} \text{ and } \beta_1,...,\beta_s \in \mathbb{R}\setminus \mathbb{Z}_0^-;\ p\in\mathbb{N};\ q,s\in\mathbb{N}_0;\ q\le s+1;\ 0\le\gamma<p;\ z\in U). \tag{1.13}$$

We note the following interesting relationship with some of the special function classes which were investigated recently:

(i) $M_{p,2,1}(n+1;0) = M_n$ $(n\in\mathbb{N}_0)$ (see Aouf [2]) ;

(ii) $M_{1,2,1}(n+1;\alpha) = M_n(\alpha)$ $(n\in\mathbb{N}_0;\ 0\le\alpha<1)$ (see Aouf and Hossen [4]).

Also, we note that:

(i) $M_{p,2,1}(n+p;\gamma) = M_p(n;\gamma)$ $(n>-p)$

$$= \mathrm{Re}\left\{\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} - (p+1)\right\} < -\frac{pn+\gamma}{n+1}; \tag{1.14}$$

(ii) $M_{p,2,1}(a,c;\gamma) = M_p(a,c;\gamma)$ $(a,c>0)$

$$= \mathrm{Re}\left\{ \frac{L_p(a+1,c)f(z)}{L_p(a,c)f(z)} - (p+1) \right\} < -\frac{p(a-1)+\gamma}{a}. \tag{1.15}$$

In this paper along with other things we shall show that a function $f(z) \in \sum_p$, which satisfies the condition (1.12) is meromorphic multivalent starlike in U^*. More preceisely it is proved that for the classes $M_{p,q,s}(\alpha_1;\gamma)$ of functions in $\sum_p$,

$$M_{p,q,s}(\alpha_1+1;\gamma) \subset M_{p,q,s}(\alpha_1;\gamma) \tag{1.16}$$

holds. If $q=2,\ s=1,\ \alpha_1=\beta_1=1$ and $\alpha_2=1$, then $M_{p,2,1}(1;\gamma)=\sum_p^*(\gamma)$ is the class of meromorphic multivalent starlike functions of order γ $(0 \le \gamma < p)$. The starlikeness of members of $M_{p,q,s}(\alpha_1;\gamma)$ is a consequence of (1.16). Further for $\mu > 0$, let

$$F(z) = \frac{\mu}{z^{\mu+p}} \int_0^z t^{\mu+p-1} f(t)\,dt, \tag{1.17}$$

it is shown that $F(z) \in M_{p,q,s}(\alpha_1;\gamma)$ whenever $f(z) \in M_{p,q,s}(\alpha_1;\gamma)$. Also it shown that if $f(z) \in M_{p,q,s}(\alpha_1;\gamma)$ then

$$F(z) = \frac{n+1}{z^{n+p+1}} \int_0^z t^{n+p} f(t)\,dt \tag{1.18}$$

belongs to $M_{p,q,s}(\alpha_1+1;\gamma)$ for $F(z) \ne 0$ in U^*. Some known results Bajpaj [5], Goel and Sohi [10], Ganigi and Uralegaddi [9], Aouf and Hossen [4] and Aouf [2] are extended.

2. PROPERTIES OF THE CLASS $M_{p,q,s}(\alpha_1;\gamma)$

Unless otherwise mentioned, we assume throughout this paper that :

$\alpha_1,\ldots,\alpha_q \in R$ and

$\beta_1,\ldots,\beta_s \in R \setminus Z_0^-,\ p \in N,\ q,s \in N_0, q \le s+1,$

$\alpha_1 > 0,\quad 0 \le \gamma < p;\ z \in U.$

In proving our main results, we shall need the following lemma due to Jack [11].

Lemma. (Jack [11]) *Suppose* $w(z)$ be a nonconstant analytic function in U with $w(0)=0$. If $|w(z)|$ attains its maximum value at apoint $z_0 \in U$ on the circle $|z| = r < 1$, then $z_0 w'(z_0) = \zeta w(z_0)$, where $\zeta \ge 1$ is some real number.

Theorem 1. $M_{p,q,s}(\alpha_1+1;\gamma) \subset M_{p,q,s}(\alpha_1;\gamma)$.

Proof. Let $f(z) \in M_{p,q,s}(\alpha_1+1;\gamma)$. Then

$$\mathrm{Re}\left\{ \frac{H_{p,q,s}(\alpha_1+2)f(z)}{H_{p,q,s}(\alpha_1+1)f(z)} - (p+1) \right\} < -\frac{p\alpha_1+\gamma}{\alpha_1}. \tag{2.1}$$

We have to show that (2.1) implies the inequality

$$\mathrm{Re}\left\{ \frac{H_{p,q,s}(\alpha_1+1)f(z)}{H_{p,q,s}(\alpha_1)f(z)} - (p+1) \right\} < -\frac{p(\alpha_1-1)+\gamma}{\alpha_1}. \tag{2.2}$$

Define $w(z)$ in $U = \{z : z \in C \text{ and } |z| < 1\}$ by

$$\frac{H_{p,q,s}(\alpha_1+1)f(z)}{H_{p,q,s}(\alpha_1)f(z)} - (p+1) = -\left\{ \frac{p(\alpha_1-1)+\gamma}{\alpha_1} + \frac{p-\gamma}{\alpha_1} \cdot \frac{1-w(z)}{1+w(z)} \right\}. \tag{2.3}$$

Clearly w is regular and $w(0)=0$. Equation (2.3) may be written as

$$\frac{H_{p,q,s}(\alpha_1+1)f(z)}{H_{p,q,s}(\alpha_1)f(z)} = \frac{\alpha_1 + (\alpha_1+2p-2\gamma)w(z)}{\alpha_1(1+w(z))}. \tag{2.4}$$

Differentiating (2.4) logarithmically and using the identity (1.11), we obtain

$$\frac{H_{p,q,s}(\alpha_1+2)f(z)}{H_{p,q,s}(\alpha_1+1)f(z)} - (p+1) + \frac{p\alpha_1+\gamma}{\alpha_1+1} = \frac{\alpha_1 + (\alpha_1+2p-2\gamma)w(z)}{(\alpha_1+1)(1+w(z))} - \frac{\alpha_1+p-\gamma}{\alpha_1+1} + \frac{2zw'(z)}{(\alpha_1+1)[1+w(z)][\alpha_1+(\alpha_1+2p-2\gamma)w(z)]} \tag{2.5}$$

that is

$$\frac{H_{p,q,s}(\alpha_1+2)f(z)}{H_{p,q,s}(\alpha_1+1)f(z)} - (p+1) + \frac{p\alpha_1+\gamma}{\alpha_1+1}$$

$$=\frac{p-\gamma}{\alpha_1+1}\left\{\begin{array}{l}-\frac{1-w(z)}{1+w(z)}\\+\frac{2zw'(z)}{[1+w(z)][\alpha_1+(\alpha_1+2p-2\gamma)w(z)]}\end{array}\right\}. \quad (2.6)$$

We claim that $|w(z)|<1$ in U. For otherwise (by Jack's Lemma) there exists $z_0\in U$ such that

$$z_0w'(z_0)=\zeta w(z_0) \quad (2.7)$$

where $|w(z_0)|=1$ and $\zeta\geq 1$. From (2.6) and (2.7), we obtain

$$\frac{H_{p,q,s}(\alpha_1+2)f(z_0)}{H_{p,q,s}(\alpha_1+1)f(z_0)}-(p+1)+\frac{p\alpha_1+\gamma}{\alpha_1+1}$$

$$=\frac{p-\gamma}{\alpha_1+1}\left\{\begin{array}{l}-\frac{1-w(z_0)}{1+w(z_0)}\\+\frac{2\zeta w(z_0)}{[1+w(z_0)][\alpha_1+(\alpha_1+2p-2\gamma)w(z_0)]}\end{array}\right\}. \quad (2.8)$$

Thus

$$\operatorname{Re}\left\{\frac{H_{p,q,s}(\alpha_1+2)f(z_0)}{H_{p,q,s}(\alpha_1+1)f(z_0)}-(p+1)+\frac{p\alpha_1+\gamma}{\alpha_1+1}\right\}$$

$$\geq\frac{p-\gamma}{2(\alpha_1+1)(\alpha_1+p-\gamma)}>0$$

which contradicts (2.1). Hence $|w(z)|<1$ in U and from (2.3) it follows that $f(z)\in M_{p,q,s}(\alpha_1;\gamma)$.

Putting $q=2,\ s=1, \alpha_1=n+p\ (n>-p)$ and $\alpha_2=\beta_1=p\ (p\in\mathrm{N})$ in Theorem 1, we obtain the following corollary.

Corollary 1. $M_p(n+1;\gamma)\subset M_p(n;\gamma)$.

Putting $q=2,\ s=1, \alpha_1=a>0,\ \alpha_2=1$ and $\beta_1=c>0$ in Theorem 1, we obtain the following corollary.

Corollary 2. $M_p(a+1,c;\gamma)\subset M_p(a,c;\gamma)$.

Theorem 2. Let $f(z)\in\sum_p$ satisfy the condition

$$\operatorname{Re}\left\{\frac{H_{p,q,s}(\alpha_1+1)f(z)}{H_{p,q,s}(\alpha_1)f(z)}-(p+1)\right\}$$
$$<\frac{(p-\gamma)-2(p\alpha_1-p+\gamma)(c+p-\gamma)}{2\alpha_1(c+p-\gamma)}. \quad (2.9)$$

Then

$$F(z)=\frac{\mu}{z^{\mu+p}}\int_0^z t^{\mu+p-1}f(t)dt\ (\mu>0) \quad (2.10)$$

belongs to $M_{p,q,s}(\alpha_1;\gamma)$.

Proof. From the definition of $F(z)$, we have

$$z\left(H_{p,q,s}(\alpha_1)F(z)\right)'=\mu H_{p,q,s}(\alpha_1)f(z)$$
$$-(\mu+p)H_{p,q,s}(\alpha_1)F(z) \quad (2.11)$$

using (2.11) and the identity (1.11), the condition (2.9) may be written as

$$\operatorname{Re}\left\{\frac{(\alpha_1+1)\frac{H_{p,q,s}(\alpha_1+2)F(z)}{H_{p,q,s}(\alpha_1+1)F(z)}-(\alpha_1+1-\mu)}{\alpha_1-\left(\alpha_1-\mu\frac{H_{p,q,s}(\alpha_1)F(z)}{H_{p,q,s}(\alpha_1+1)F(z)}\right)}\right\}$$
$$<\frac{(p-\gamma)-2(p\alpha_1-p+\gamma)(\mu+p-\gamma)}{2\alpha_1(\mu+p-\gamma)}. \quad (2.12)$$

We have to prove that implies the inequality

$$\operatorname{Re}\left\{\frac{H_{p,q,s}(\alpha_1+1)F(z)}{H_{p,q,s}(\alpha_1)F(z)}-(p+1)\right\}<-\frac{p(\alpha_1-1)+\gamma}{\alpha_1}.$$

Define $w(z)$ in U by

$$\frac{H_{p,q,s}(\alpha_1+1)F(z)}{H_{p,q,s}(\alpha_1)F(z)}-(p+1)=$$
$$-\left\{\frac{p(\alpha_1-1)+\gamma}{\alpha_1}+\frac{p-\gamma}{\alpha_1}.\frac{1-w(z)}{1+w(z)}\right\}. \quad (2.13)$$

Clearly w is regular and $w(0)=0$. The quation (2.13) may be written as

$$\frac{H_{p,q,s}(\alpha_1+1)F(z)}{H_{p,q,s}(\alpha_1)F(z)}$$
$$=\frac{\alpha_1+(\alpha_1+2p-2\gamma)w(z)}{\alpha_1(1+w(z))}. \quad (2.14)$$

Differentiating (2.14) logarithmically and using the identity (1.11), we obtain

$$\frac{(\alpha_1+1)H_{p,q,s}(\alpha_1+2)F(z)}{H_{p,q,s}(\alpha_1+1)F(z)} - \frac{\alpha_1 H_{p,q,s}(\alpha_1+1)F(z)}{H_{p,q,s}(\alpha_1)F(z)} - 1 = \frac{2(p-\gamma)zw'(z)}{[1+w(z)][\alpha_1+(\alpha_1+2p-2\gamma)w(z)]}. \tag{2.15}$$

The above equation may be written as

$$\frac{(\alpha_1+1)\frac{H_{p,q,s}(\alpha_1+2)F(z)}{H_{p,q,s}(\alpha_1+1)F(z)}-(\alpha_1+1-\mu)}{\alpha_1-(\alpha_1-\mu)\frac{H_{p,q,s}(\alpha_1)F(z)}{H_{p,q,s}(\alpha_1+1)F(z)}}-(p+1) = \frac{H_{p,q,s}(\alpha_1+1)F(z)}{H_{p,q,s}(\alpha_1)F(z)}-(p+1) + \left[\frac{2(p-\gamma)zw'(z)}{[1+w(z)][\alpha_1+(\alpha_1+2p-2\gamma)w(z)]}\right] \cdot \left[\frac{1}{\alpha_1-(\alpha_1-\mu)\frac{H_{p,q,s}(\alpha_1)F(z)}{H_{p,q,s}(\alpha_1+1)F(z)}}\right] \tag{2.16}$$

which, by using (2.13) and (2.14), reduces to

$$\frac{(\alpha_1+1)\frac{H_{p,q,s}(\alpha_1+2)F(z)}{H_{p,q,s}(\alpha_1+1)F(z)}-(\alpha_1+1-\mu)}{\alpha_1-(\alpha_1-\mu)\frac{H_{p,q,s}(\alpha_1)F(z)}{H_{p,q,s}(\alpha_1+1)F(z)}}-(p+1) = -\left\{\frac{p(\alpha_1-1)+\gamma}{\alpha_1}+\frac{p-\gamma}{\alpha_1}.\frac{1-w(z)}{1+w(z)}\right\} + \left[\frac{2(p-\gamma)zw'(z)}{\alpha_1[1+w(z)][\mu+(\mu+2p-2\gamma)w(z)]}\right]. \tag{2.17}$$

We claim that $|w(z)|<1$ in U. For otherwise (by Jack's Lemma) there exists $z_0 \in U$ such that

$z_0 w'(z_0) = \zeta w(z_0)$ (2.18)

where $|w(z_0)|=1$ and $\zeta \geq 1$. From (2.17) and

$$\frac{(\alpha_1+1)\frac{H_{p,q,s}(\alpha_1+2)F(z_0)}{H_{p,q,s}(\alpha_1+1)F(z_0)}-(\alpha_1+1-\mu)}{\alpha_1-(\alpha_1-\mu)\frac{H_{p,q,s}(\alpha_1)F(z_0)}{H_{p,q,s}(\alpha_1+1)F(z_0)}}-(p+1) = -\left\{\frac{p(\alpha_1-1)+\gamma}{\alpha_1}+\frac{p-\gamma}{\alpha_1}.\frac{1-w(z_0)}{1+w(z_0)}\right\} + \left[\frac{2(p-\gamma)\zeta w(z_0)}{\alpha_1[1+w(z_0)][\mu+(\mu+2p-2\gamma)w(z_0)]}\right]. \tag{2.19}$$

Thus

$$\mathrm{Re}\left\{\frac{(\alpha_1+1)\frac{H_{p,q,s}(\alpha_1+2)F(z_0)}{H_{p,q,s}(\alpha_1+1)F(z_0)}-(\alpha_1+1-\mu)}{\alpha_1-(\alpha_1-\mu)\frac{H_{p,q,s}(\alpha_1)F(z_0)}{H_{p,q,s}(\alpha_1+1)F(z_0)}}-(p+1)\right\} > \frac{(p-\gamma)-2(p\alpha_1-p+\gamma)(\mu+p-\gamma)}{2\alpha_1(\mu+p-\gamma)}$$

which contradicts (2.9). Hence $|w(z)|<1$ in U and from (2.13) it follows that $F(z)\in M_{p,q,s}(\alpha_1;\gamma)$.

Putting $q=2,\ s=1, \alpha_1=n+1\ (n>-1)$ and $\alpha_2=\beta_1=1$ in Theorem 2, we obtain the following corollary.

Corollary 3. Let $f(z)\in\sum_p$ satisfy the condition

$$\mathrm{Re}\left\{\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}-(p+1)\right\} < \frac{(p-\gamma)-2(pn+\gamma)(\mu+p-\gamma)}{2(n+1)(\mu+p-\gamma)}, \tag{2.20}$$

then $F(z)$ is given by (2.10) belongs to $M_p(n;\gamma)$.

Putting $q=2,\ s=1, \alpha_1=a>0,\ \alpha_2=1$ and $\beta_1=c>0$ in Theorem 2, we obtain the following corollary.

Corollary 4. Let $f(z)\in\sum_p$ satisfy the condition

$$\mathrm{Re}\left\{\frac{L_p(a+1,c)f(z)}{L_p(a,c)f(z)}-(p+1)\right\}$$

$$< \frac{(p-\gamma)-2(pa-p+\gamma)(\mu+p-\gamma)}{2a(\mu+p-\gamma)}, \quad (2.21)$$

then $F(z)$ is given by (2.10) belongs to $M_p(a,c;\gamma)$.

Theorem 3. If $f(z) \in M_{p,q,s}(\alpha_1;\gamma)$, then

$$F(z) = \frac{n+1}{z^{n+p+1}} \int_0^z t^{n+p} f(t)dt \quad (2.22)$$

belongs to $M_{p,q,s}(\alpha_1+1;\gamma)$ for $F(z) \neq 0$ in U^*.

Proof. We have

$$\mu H_{p,q,s}(\alpha_1) f(z) = \alpha_1 H_{p,q,s}(\alpha_1+1)F(z) - (\alpha_1-\mu)H_{p,q,s}(\alpha_1)F(z) \quad (2.23)$$

and

$$\mu H_{p,q,s}(\alpha_1+1) f(z) = (\alpha_1+1) H_{p,q,s}(\alpha_1+2)F(z) - (\alpha_1+1-\mu)H_{p,q,s}(\alpha_1+1)F(z). \quad (2.24)$$

Taking $\mu = \alpha_1$ in the above relations, we obtain

$$\frac{(\alpha_1+1) H_{p,q,s}(\alpha_1+2)F(z) - H_{p,q,s}(\alpha_1+1)F(z)}{\alpha_1 H_{p,q,s}(\alpha_1+1)F(z)} = \frac{H_{p,q,s}(\alpha_1+1)f(z)}{H_{p,q,s}(\alpha_1)f(z)} \quad (2.25)$$

which reduces to

$$\frac{(\alpha_1+1)}{\alpha_1} \frac{H_{p,q,s}(\alpha_1+2)F(z)}{H_{p,q,s}(\alpha_1+1)F(z)} - \frac{1}{\alpha_1} = \frac{H_{p,q,s}(\alpha_1+1)f(z)}{H_{p,q,s}(\alpha_1)f(z)}. \quad (2.26)$$

Thus

$$\operatorname{Re}\left\{ \frac{(\alpha_1+1)}{\alpha_1} \frac{H_{p,q,s}(\alpha_1+2)F(z)}{H_{p,q,s}(\alpha_1+1)F(z)} - \frac{1}{\alpha_1} - (p+1) \right\}$$

$$= \operatorname{Re}\left\{ \frac{H_{p,q,s}(\alpha_1+1)f(z)}{H_{p,q,s}(\alpha_1)f(z)} - (p+1) \right\} < -\frac{p(\alpha_1-1)+\gamma}{\alpha_1} \quad (2.27)$$

from which it follows that

$$\operatorname{Re}\left\{ \frac{H_{p,q,s}(\alpha_1+2)F(z)}{H_{p,q,s}(\alpha_1+1)F(z)} - (p+1) \right\} < -\frac{p\alpha_1+\gamma}{\alpha_1+1}.$$

Then $F(z) \in M_{p,q,s}(\alpha_1+1;\gamma)$. This complete the proof of Theorem 3.

Remarks:

(i) Taking $q=2,\ s=1, \alpha_1 = n+1\ (n>-1)$, $\alpha_2 = \beta_1 = 1$ and $\gamma = 0$, in all our results, we obtain the results obtained by Aouf [2];

(ii) Taking $q=2,\ s=1, \alpha_1 = n+1\ (n>-1)$ and $\alpha_2 = \beta_1 = p = 1$ in all our results, we obtain the results obtained by Aouf and Hossen [4];

(iii) Taking $q=2,\ s=1, \alpha_1 = n+1\ (n>-1)$, $\alpha_2 = \beta_1 = p = 1$ and $\gamma = 0$, in all our results, we obtain the results obtained by Ganigi and Uralegaddi [9].

3. REFERENCES

1. Aouf, M.K. New criteria for multivalent meromorphic starlike function of order alpha, Proc. Japan. *Acad. Ser.* A 69, no. 3: 66-70 (1993).
2. Aouf, M.K. A new criterion for meromorphic $p-$valent starlike functions, Analele Stiintifice Ale Univ. "AL.I.CUZA" IASI, 41: 101-107 (1995).
3. Aouf, M.K. Certain subclasses of meromorphically multivalent functions associated with generalized hypergeometric function, *Comput. Math. Appl.* 55: 494-509 (2008).
4. Aouf, M.K. & H.M. Hossen, New criteria for meromorphic univalent functions of order α. *Nihonkai Math. J.* 5: 1-11 (1994).
5. Bajpai, S.K. A note on a class of meromorphic univalent functions. *Rev. Roum. Math. Pures Appl.* 22: 295-297 (1977).
6. Dziok, J. & H.M. Srivastava. Classes of analytic functions associated with the generalized hypergeometric function, *Appl. Math.Comput.* 103: 1-13 (1999).
7. Dziok, J. & H.M. Srivastava, Certain subclasses of analytic functions associated with the generalized hypergeometric function, Integral Transforms Spec. Funct., 14: 7-18 (2003).
8. Gangadharan, A., T.N. Shanmugan & H.M. Srivastava, Generalized hypergeometric functions with k-uniformly convex functions. *Comput.*

Math. Appl. 44, (no. 12): 1515-1526 (2002).

9. Ganigi M.D. & B.A. Uralegaddi. New criteria for meromorphic univalent functions. *Bull. Math. Soc. Sci. Math. R. S. Roumanie (N. S.)* 33 (no. 1): 9-13 (1989).
10. Goel, R.M. & N.S. Sohi. On a class meromorphic functions. *Glas. Mat.* 17: 19-28 (1981).
11. I. S. Jack, Functions starlike and convex of order α. *J. London Math. Soc,* 3: 469-474 (1971).
12. Liu, J.-L. Strongly starlike functions associated with the Dziok-Srivastava operator. *Tamkang J. Math.* 35: (no. 1): 37-42 (2004).
13. Liu J.-L. & H.M. Srivastava, A linear operator and associated families of meromorphically multivalent functions. *J. Math. Anal. Appl.* 259: 566-581 (2000).
14. Liu J.-L. & H.M. Srivastava. Classes of meromorphically multivalent associated with the generalized hypergeometric function. *Math. Comput. Modell.* 39: 21-34 (2004).
15. Srivastava H.M. & P.W. Karlsson, Multiple Gaussian Hypergeometric Series, Halsted Press, Ellis Horwood Limited, Chichester, John Wiley and Sons, New York, Chichester, Brisbane, Toronto (1985).
16. Uralegaddi B.A. & C. Somanatha, Certain classes of meromorphic multivalent functions. *Tamkang J. Math.* 23: 223-231 (1992).
17. Yang, D. On a class of meromorphic starlike multivalent functions. *Bull. Inst. Math. Acad. Sinica* 24: 151-157 (1996).