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Research Article

Solution of Sitnikov Restricted Four Body Problem when All the Primaries are Oblate Bodies: Circular Case

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Abstract: In the present paper we have studied the Sitnikov problem when extended to four body problem with all the primaries as oblate bodies. Here all the three primaries are moving in the circular orbit in the x-y plane and fourth particle is moving along the z-axis. First we found the condition to maintain the equilateral configuration of the system, then we determined the equation of motion of the system. Using the Floquet theory, we found the Stability region of the motion depending on oblateness parameter.

Keywords: Sitnikov problem, Stability, Floquet theory, oblateness

1. INTRODUCTION

In the present paper we have studied the Sitnikov restricted four body problem when all the primaries are moving in circular orbits around their centre of mass with the assumption that all the primaries are oblate bodies. The Sitnikov problem is a special case of the restricted three body problem where the two primaries of equal masses ($m_1 = m_2 = m = 1/2$) are moving in circular or elliptic orbits around the centre of mass under Newtonian force of attraction and the third body of mass m_3 (the mass of the third body is much less than the masses of the primaries) moves along the line which is passing through the centre of mass of the primaries and is perpendicular to the plane of motion of the primaries. It was Pavanini [1], who originally introduced this problem by taking the circular case. MacMillan [2] found the exact solution which can be expressed in terms of Jacobi elliptic function. Sitnikov [3] has studied the existence of oscillating motion of the three body problem. Sitnikov problem is studied by many scientists, i.e., Perdios et al [4], Liu and Sun [5], Hagel [6], Belbruno et al [7], Faruque [8], Soulis et al [9, 10], Perdois [11], Boutis and Papadakis [12].

In this paper we have studied the Sitnikov problem when extended to four body problem by taking all the primaries as oblate bodies. We have found out the equation of motion when all the primaries are oblate bodies and are moving in circular orbits around their centre of mass. Then we have investigated the stability of motion of m_4 and discussed the results so obtained.

2. EQUATION OF MOTION

The system consists of three primaries with equal masses $(m_1 = m_2 = m_3 = \frac{1}{3})$ and all the primaries are oblate bodies. The fourth body have a mass (m_4) which is much less than the masses of the primaries. We have shown that all of the primaries are at the vertices of an equilateral triangle under certain conditions. The fourth body is confined to a motion perpendicular to the plane of motion of the three primaries, which are equally far away from the barycentre of the system. All the primaries are moving in circular orbits around their center of mass "O" which is taken as origin. The fourth body is moving along the line perpendicular to the plane of motion of the primaries and passing through the centre of mass. In such a system the motion of the fourth body is one dimensional.

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When all the primaries are oblate bodies then the equilateral triangle configuration does not remain the same. To make the equilateral configuration, we impose the conditions which are:

1. Principal axes of all the oblate bodies are parallel to the synodic axes.
2. The masses of all the bodies are equal.

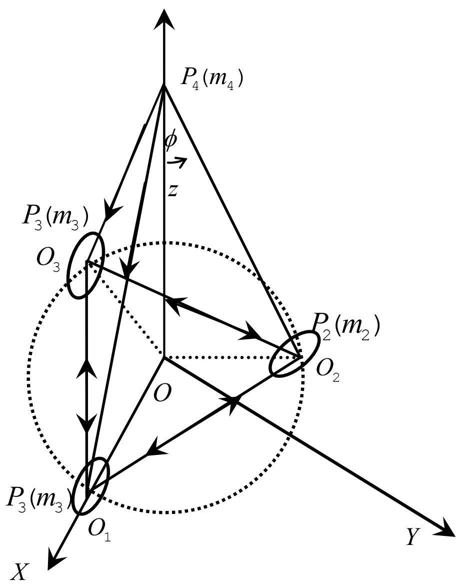

Fig.1 Sitnikov Four body configuration: Circular case.

Let the axes be a_i, b_i, c_i, then as the bodies are oblate $a_i = b_i$ where $i = 1, 2, 3$. Hence the moment of inertia of the oblate spheroid m_1, m_2 and m_3 about the principal axes of the body are:

$$A_1 = A_2 = A_3 = \frac{a^2 + c^2}{5}$$

$$B_1 = B_2 = B_3 = \frac{a^2 + c^2}{5}$$

$$C_1 = C_2 = C_3 = \frac{2a^2}{5}$$

The direction cosines of the principal axes of m_1, m_2 and m_3 are:

$$\lambda_{11}, \mu_{11}, \upsilon_{11} = 1, 0, 0;$$
$$\lambda_{21}, \mu_{21}, \upsilon_{21} = 0, 1, 0;$$
$$\lambda_{31}, \mu_{31}, \upsilon_{31} = 0, 0, 1;$$

$$\lambda_{12}, \mu_{12}, \upsilon_{12} = 1, 0, 0;$$
$$\lambda_{22}, \mu_{22}, \upsilon_{22} = 0, 1, 0; \quad \text{and}$$
$$\lambda_{32}, \mu_{32}, \upsilon_{32} = 0, 0, 1;$$

$$\lambda_{13}, \mu_{13}, \upsilon_{13} = 1, 0, 0;$$
$$\lambda_{23}, \mu_{23}, \upsilon_{23} = 0, 1, 0; \quad \text{respectively.}$$
$$\lambda_{33}, \mu_{33}, \upsilon_{33} = 0, 0, 1;$$

Let a_1, b_1, c_1 be the direction cosines of O_1O_2 relative to the principal axes of m_1, then

$$a_1^2 = \frac{3l^2}{4r^2}, \; b_1^2 = \frac{l^2}{4r^2}, \; c_1^2 = 0.$$

Let a_2, b_2, c_2 be the direction cosines O_1O_2 relative to the principal axes of m_2, then

$$a_2^2 = \frac{3l^2}{4r^2}, \; b_2^2 = \frac{l^2}{4r^2}, \; c_2^2 = 0.$$

Thus the potential between the two bodies $m_1 m_2$,

$$-V = \frac{Gm_1m_2}{r} + \frac{Gm_1}{r^3}\left[\frac{1}{2}\left(A_1 + B_1 + C_1\right) - \frac{3}{2}\left(A_1a_1^2 + B_1b_1^2 + C_1c_1^2\right)\right] + \frac{Gm_2}{r^3}\left[\frac{1}{2}\left(A_2 + B_2 + C_2\right) - \frac{3}{2}\left(A_2a_2^2 + B_2b_2^2 + C_2c_2^2\right)\right]$$

Clemency & Brouwer [13]

$$= \frac{Gm_1m_2}{r} + \frac{GA\left(m_1 + m_2\right)}{r^3}$$

Where,

$$A = \left[\frac{a^2 - c^2}{10}\right]. \tag{1}$$

Equation of motion of m_1 is

$$\left[-n^2 + \left(\frac{1}{3\rho_{12}^3} + \frac{6A}{\rho_{12}^5}\right) + \left(\frac{1}{3\rho_{13}^3} + \frac{6A}{\rho_{13}^5}\right)\right] r_1 u_1 - \left(\frac{1}{3\rho_{12}^3} + \frac{6A}{\rho_{12}^5}\right) r_2 u_2 - \left(\frac{1}{3\rho_{13}^3} + \frac{6A}{\rho_{13}^5}\right) r_3 u_3 = 0.$$

The equation of motion of m_2 is

$$-\left(\frac{1}{3\rho_{12}^3}+\frac{6A}{\rho_{12}^5}\right)r_1u_1$$
$$+\left[-n^2+\left(\frac{1}{3\rho_{12}^3}+\frac{6A}{\rho_{12}^5}\right)+\left(\frac{1}{3\rho_{23}^3}+\frac{6A}{\rho_{23}^5}\right)\right]r_2u_3$$
$$-\left(\frac{1}{3\rho_{23}^3}+\frac{6A}{\rho_{23}^5}\right)r_3u_3=0.$$

The equation of motion of m_3 is

$$-\left(\frac{1}{3\rho_{13}^3}+\frac{6A}{\rho_{13}^5}\right)r_1u_1-\left(\frac{1}{3\rho_{23}^3}+\frac{6A}{\rho_{23}^5}\right)r_2u_2$$
$$+\left[-n^2+\left(\frac{1}{3\rho_{13}^3}+\frac{6A}{\rho_{13}^5}\right)+\left(\frac{1}{3\rho_{23}^3}+\frac{6A}{\rho_{23}^5}\right)\right]r_3u_3=0.$$

Where ρ_y is the vectors from m_i to m_j.

When the body is oblate the equilateral configuration does not remain the same.

Let us suppose $\rho_{12}=l+\lambda_1$, $\rho_{13}=l+\lambda_2$ and $\rho_{23}=l+\lambda_3$ where λ_1, λ_2 and λ_2 are very-very small and l is the length of the sides of equilateral triangle.

Thus the equation of motion of m_1 becomes

$$\left[-n^2+\left\{\frac{1}{3l^2}\left(1-\frac{3\lambda_1}{l}\right)+\frac{6A}{l^5}\left(1-\frac{5\lambda_1}{l}\right)\right\}\right.$$
$$\left.+\left\{\frac{1}{3l^3}\left(1-\frac{3\lambda_2}{l}\right)+\frac{6A}{l^5}\left(1-\frac{5\lambda_2}{l}\right)\right]r_1u_1$$
$$-\left[\frac{1}{3l^2}\left(1-\frac{3\lambda_1}{l}\right)+\frac{6A}{l^5}\left(1-\frac{5\lambda_1}{l}\right)\right]r_2u_2$$
$$-\left[\frac{1}{3l^2}\left(1-\frac{3\lambda_2}{l}\right)+\frac{6A}{l^5}\left(1-\frac{5\lambda_2}{l}\right)\right]r_3u_3=0,$$

the equation of motion of m_2 becomes

$$-\left[\frac{1}{3l^2}\left(1-\frac{3\lambda_1}{l}\right)+\frac{6A}{l^5}\left(1-\frac{5\lambda_1}{l}\right)\right]r_1u_1$$
$$+\left[-n^2+\left\{\frac{1}{3l^2}\left(1-\frac{3\lambda_1}{l}\right)+\frac{6A}{l^5}\left(1-\frac{5\lambda_1}{l}\right)\right\}\right.$$
$$\left.+\left\{\frac{1}{3l^3}\left(1-\frac{3\lambda_3}{l}\right)+\frac{6A}{l^5}\left(1-\frac{5\lambda_3}{l}\right)\right\}\right]r_2u_2$$
$$-\left[\frac{1}{3l^2}\left(1-\frac{3\lambda_3}{l}\right)+\frac{6A}{l^5}\left(1-\frac{5\lambda_3}{l}\right)\right]r_3u_3=0,$$

and the equation of motion of m_3 becomes

$$-\left[\frac{1}{3l^2}\left(1-\frac{3\lambda_1}{l}\right)+\frac{6A}{l^5}\left(1-\frac{5\lambda_1}{l}\right)\right]r_1u_1$$
$$-\left[\frac{1}{3l^2}\left(1-\frac{3\lambda_2}{l}\right)+\frac{6A}{l^5}\left(1-\frac{5\lambda_2}{l}\right)\right]r_2u_2$$
$$+\left[-n^2+\left\{\frac{1}{3l^2}\left(1-\frac{3\lambda_1}{l}\right)+\frac{6A}{l^5}\left(1-\frac{5\lambda_1}{l}\right)\right\}\right.$$
$$\left.+\left\{\frac{1}{3l^3}\left(1-\frac{3\lambda_2}{l}\right)+\frac{6A}{l^5}\left(1-\frac{5\lambda_2}{l}\right)\right\}\right]r_3u_3=0.$$

We have neglected the higher order terms in λ_i and $\lambda_i A$, $(i=1,2,3)$.

Since the centre of mass has been taking as the origin, we have

$$m_1r_1u_1+m_2r_2u_2+m_3r_3u_3=0.$$

Applying the same process as used by McCuskey [14], we have the non-trivial solution when the determinant is zero. Thus, we get

$$\frac{1}{9l^2}\left(l^3-n^3\right)^2\left(36A+l^8-l^5n^2\right.$$
$$\left.-2l^7\lambda_1-2l^7\lambda_2-2l^7\lambda\right)=0.$$

And hence, $n^2=l^3$.

If we put the value of the n^2 in the second factor, we get

$$-18A+l^7\lambda_1+l^7\lambda_2+l^7\lambda_3=0.$$

Suppose that the configuration remains the same, then, we must have

$$\lambda_1=\lambda_2=\lambda_3=\lambda$$

$$\Rightarrow\quad \lambda=\frac{6A}{l^7}.$$

Taking the unit of length such that $l = 1$ (This corresponds to the length of the side of the equilateral triangle in the classical case. McCaskey [14]), we get

$$\lambda = 6A . \tag{2}$$

Taking the unit of mass such that $m_1 = m_2 = m_3 = \frac{1}{3}$,

The unit of time such that $G = 1$.

Then the Equation of motion of m_4 is

$$\frac{d^2z}{dt^2} = -\frac{z}{r^3} - \frac{9Az}{r^5} + \frac{15Az^2}{r^7},$$

where

$$r = \sqrt{z^2 + \frac{(1+6A)^2}{3}}.$$

Neglecting A^2, we get

$$r = \sqrt{z^2 + \frac{(1+12A)}{3}}.$$

Hence

$$\frac{d^2z}{dt^2} = -\frac{z}{\left[z^2 + \frac{(1+6A)^2}{3}\right]^{\frac{3}{2}}} - \frac{9Az}{\left[z^2 + \frac{(1+6A)^2}{3}\right]^{\frac{5}{2}}} + \frac{15Az^3}{\left[z^2 + \frac{(1+6A)^2}{3}\right]^{\frac{7}{2}}}. \tag{3}$$

Writing the general Equation of motion in synodic axes

$$\begin{aligned} \ddot{x} - 2n\dot{y} &= \Omega_x, \\ \ddot{y} + 2n\dot{x} &= \Omega_y, \\ \ddot{z} &= \Omega_z. \end{aligned} \tag{4}$$

Where

$$\Omega = \frac{1}{2}\left(x^2 + y^2\right) + m\left[\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right) + A\left(\frac{1}{r_1^3} + \frac{1}{r_2^3} + \frac{1}{r_3^3}\right) - 3Az^2\left(\frac{1}{r_1^5} + \frac{1}{r_2^5} + \frac{1}{r_3^5}\right)\right],$$

$$r_i = \sqrt{\left(x - x_i\right)^2 + \left(y - y_i\right)^2 + z^2}, \quad (i = 1, 2, 3)$$

$$\left(x_1, y_1\right) = \left(\frac{1+6A}{\sqrt{3}}, 0\right);$$

$$\left(x_2, y_2\right) = \left(-\frac{1+6A}{2\sqrt{3}}, \frac{1+6A}{2}\right);$$

$$\left(x_3, y_3\right) = \left(-\frac{1+6A}{2\sqrt{3}}, -\frac{1+6A}{2}\right).$$

The corresponding Jacobi integral can be written as

$$C = 2\Omega - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2),$$

Where C is the Jacobian constant. Zero velocity surfaces for this problem are shown in the Fig 2(a), 2(b) and 2(c) for different oblateness parametre.

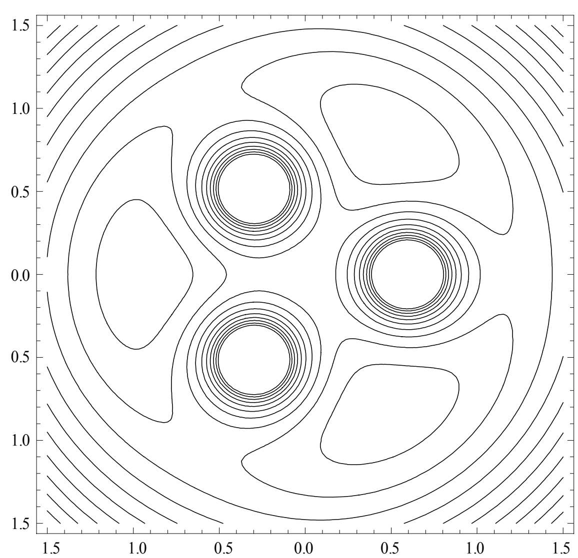

Fig. 2 (a) Surface of zero- velocity for A=0.005.

Table 1 Stability table for A=0.05.

z_{in}	A	T	p	q	Result
0.8010232	0.05	4.93737	-1.05698-4.12738i	-1.05698+4.12738i	Unstable
1.0475939	0.05	5.53443	-1.18304-0.12629i	-1.18304+0.12629i	Unstable
1.3562737	0.05	6.00899	-0.43893-0.76074i	-0.43893+0.76074i	Unstable
1.5447426	0.05	6.21043	-0.19218-0.96779i	-0.19218+0.96779i	Unstable
1.7768767	0.05	6.40107	-0.25376-0.78482i	-0.25376+0.78482i	Unstable
1.8868767	0.05	6.47516	-0.46357-0.44274i	-0.46357+0.44274i	Unstable
1.9235451	0.05	6.49797	-0.55226-0.19137i	-0.55226+0.19137i	Unstable
1.9000000	0.05	6.48342	-0.49499-0.37296i	-0.49499+0.37296i	Unstable
1.9300000	0.05	6.50189	-0.56781-0.09399i	-0.56781+0.09399i	Unstable
1.9320000	**0.05**	**6.50311**	**-0.57259-0.01680i**	**-0.57259+0.01680i**	**Unstable**
1.9325000	0.05	6.50341	-0.616801	-0.530759	Stable
1.9327500	0.05	6.50356	-0.628381	-0.520368	Stable
1.9330000	0.05	6.50371	-0.638073	-0.511865	Stable
1.9340000	0.05	6.50432	-0.668106	-0.486577	Stable
1.9350000	0.05	6.50492	-0.691447	-0.467968	Stable
1.9356451	0.05	6.50531	-0.704601	-0.457857	Stable
1.9469551	0.05	6.51209	-0.857111	-0.357457	Stable
1.9569551	0.05	6.51801	-0.949192	-0.308667	Stable
1.9622789	0.05	6.52115	-0.990633	-0.288784	Stable
1.9722697	0.05	6.52697	-1.05792	-0.258238	Stable
1.9845324	0.05	6.53405	-1.12422	-0.258244	Stable
1.9902327	0.05	6.53731	-1.1491	-0.215945	Stable
2.0000000	0.05	6.66816	-1.18254	-0.195993	Stable
2.1500000	**0.05**	**6.54285**	**1.8315+1.12082i**	**1.8315-1.12082i**	**Unstable**
2.2500000	0.05	6.66816	6.30708	8.03711	Unstable

Table 2 Stability table for A=0.5.

z_{in}	A	T	p	q	Result
1.8868767	**0.5**	**4.33124**	**-3.60933-3.41153i**	**-3.60933+3.41153i**	**Unstable**
1.9000000	0.5	4.34297	-0.470723	1.60923	Stable
1.9235451	0.5	4.36364	-0.559094	1.60131	Stable
1.9356451	0.5	4.37410	-0.603030	1.59658	Stable
1.9469551	0.5	4.38376	-0.643225	1.59179	Stable
1.9902327	0.5	4.41982	-0.789642	1.59179	Stable
2.1500000	0.5	4.54133	-1.24247	1.46356	Stable
2.2000000	0.5	4.57629	-1.359120	1.4234	Stable
2.2195234	0.5	4.58905	-1.40058	1.40774	Stable
2.2199934	0.5	4.58936	-1.40158	1.40735	Stable
2.2239599	0.5	4.592	-1.41005	1.40404	Stable
2.2249599	0.5	4.59266	-1.41102	1.40438	Stable
2.2249999	0.5	4.59269	-1.41225	1.40317	Stable
2.2500000	0.5	4.60908	-1.46405	1.38206	Stable
2.2600000	0.5	4.61554	-1.48408	1.3735	Stable
2.2700000	0.5	4.62194	-1.50374	1.36487	Stable
2.2800000	0.5	4.62829	-1.52302	1.35618	Stable
2.2900000	0.5	4.63458	-1.54193	1.34743	Stable
2.3000000	0.5	4.64082	-1.56047	1.33863	Stable
2.3100000	0.5	4.64701	-1.57866	1.32977	Stable
2.3200000	0.5	4.65314	-1.59649	1.32086	Stable
2.3300000	0.5	4.65923	-1.61398	1.31191	Stable
2.5000000	0.5	4.07532	-1.86265	1.15383	Stable
2.5500000	0.5	4.82983	-1.90280	1.10594	Stable
2.6000000	0.5	4.80596	-1.9718	1.05771	Stable
2.6300057	0.5	4.8204	-2.00000	1.02862	Stable
2.6300570	**0.5**	**4.8205**	**-2.00004**	**1.028620**	**Unstable**
2.6433000	0.5	4.82669	-2.01969	1.007070	Unstable
2.6500000	0.5	4.82983	-2.01771	1.009300	Unstable
3.0005700	0.5	4.97428	-2.21686	0.674617	Unstable
3.5000000	0.5	5.12804	-2.28262	0.249255	Unstable

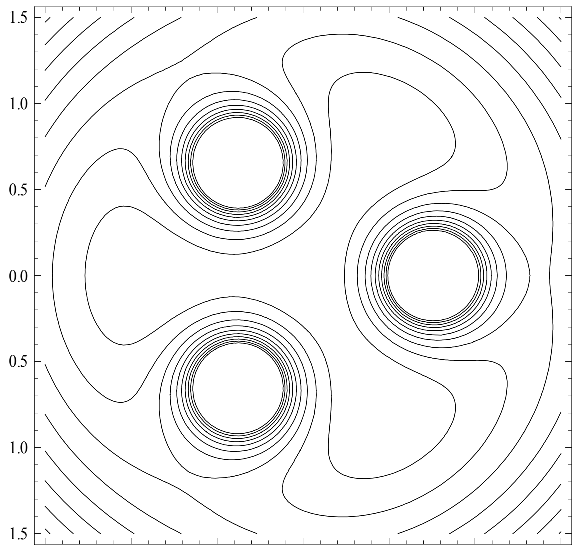

Fig. 2 (b) Surface of zero-velocity For A=0.05.

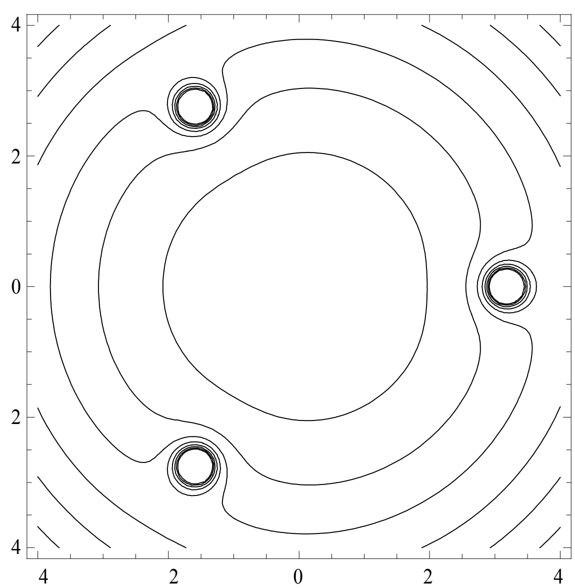

Fig. 2 (c) Surface of zero- velocity for A=0.75.

The Sitnikov motions can be obtained also from the equations of restricted four body problem as a special case, i.e. when

$$m_1 = m_2 = m_3 = \frac{1}{3} \text{ and}$$
$$x(t) = y(t) = 0.$$

$$C = 2\Omega - \dot{z}^2 \tag{5}$$

3. STABILITY

Writing the Equation of motion in phase space,

Let

$$\begin{aligned} &x = x_1,\ y = x_2,\ z = x_3,\\ &\dot{x} = x_4,\ \dot{y} = x_5,\ \dot{z} = x_6,\\ &\dot{x}_1 = x_4,\ \dot{x}_2 = x_5,\ \dot{x}_3 = x_6,\\ &\dot{x}_4 = 2x_5 + \Omega_x,\\ &\dot{x}_5 = -2x_4 + \Omega_y,\\ &\dot{x}_6 = \Omega_z, \end{aligned} \tag{6}$$

Thus the Equation of motion in phase space are

$$\dot{x}_i = f_i(x_i), \quad i = 1, 2, ..., 6. \tag{7}$$

Where

$$f_1 = x_4,$$
$$f_2 = x_5,$$
$$f_3 = x_6,$$

$$f_4 = 2x_5 + x_1 - \frac{1}{3}\left[\left[\left\{\frac{x_1 - \left(\frac{1+6A}{\sqrt{3}}\right)}{\left\{\left(x_1 - \left(\frac{1+6A}{\sqrt{3}}\right)\right)^2 + x_2^2 + x_3^3\right\}^{\frac{3}{2}}}\right.\right.\right.$$
$$+ \frac{x_1 + \left(\frac{1+6A}{2\sqrt{3}}\right)}{\left\{\left(x_1 + \left(\frac{1+6A}{2\sqrt{3}}\right)\right)^2 + \left(x_2 - \left(\frac{1+6A}{2}\right)\right)^2 + x_3^2\right\}^{\frac{3}{2}}}$$
$$\left. + \frac{x_1 + \left(\frac{1+6A}{2\sqrt{3}}\right)}{\left\{\left(x_1 + \left(\frac{1+6A}{2\sqrt{3}}\right)\right)^2 + \left(x_2 + \left(\frac{1+6A}{2}\right)\right)^2 + x_3^2\right\}^{\frac{3}{2}}}\right\}$$

$$+3A\left\{\frac{x_1 - \left(\frac{1+6A}{\sqrt{3}}\right)}{\left\{\left(x_1 - \left(\frac{1+6A}{\sqrt{3}}\right)\right)^2 + x_2^2 + x_3^3\right\}^{\frac{5}{2}}}\right.$$

$$+\frac{x_1+\left(\frac{1+6A}{2\sqrt{3}}\right)}{\left\{\left(x_1+\left(\frac{1+6A}{2\sqrt{3}}\right)\right)^2+\left(x_2-\left(\frac{1+6A}{2}\right)\right)^2+x_3^2\right\}^{\frac{5}{2}}}$$

$$\left.+\frac{x_1+\left(\frac{1+6A}{2\sqrt{3}}\right)}{\left\{\left(x_1+\left(\frac{1+6A}{2\sqrt{3}}\right)\right)^2+\left(x_2+\left(\frac{1+6A}{2}\right)\right)^2+x_3^2\right\}^{\frac{5}{2}}}\right\}$$

$$+15Ax_3^2\left\{\frac{x_1-\left(\frac{1+6A}{\sqrt{3}}\right)}{\left\{\left(x_1-\left(\frac{1+6A}{\sqrt{3}}\right)\right)^2+x_2^2+x_3^3\right\}^{\frac{5}{2}}}\right.$$

$$+\frac{x_1+\left(\frac{1+6A}{2\sqrt{3}}\right)}{\left\{\left(x_1+\left(\frac{1+6A}{2\sqrt{3}}\right)\right)^2+\left(x_2-\left(\frac{1+6A}{2}\right)\right)^2+x_3^2\right\}^{\frac{5}{2}}}$$

$$\left.\left.\left.+\frac{x_1+\left(\frac{1+6A}{2\sqrt{3}}\right)}{\left\{\left(x_1+\left(\frac{1+6A}{2\sqrt{3}}\right)\right)^2+\left(x_2+\left(\frac{1+6A}{2}\right)\right)^2+x_3^2\right\}^{\frac{5}{2}}}\right\}\right]\right],$$

$$f_5=-2x_4+x_2-\frac{1}{3}\left[\left[\left\{\frac{x_2}{\left\{\left(x_1-\left(\frac{1+6A}{\sqrt{3}}\right)\right)^2+x_2^2+x_3^3\right\}^{\frac{3}{2}}}\right.\right.\right.$$

$$+\frac{x_2-\left(\frac{1+6A}{2}\right)}{\left\{\left(x_1+\left(\frac{1+6A}{2\sqrt{3}}\right)\right)^2+\left(x_2-\left(\frac{1+6A}{2}\right)\right)^2+x_3^2\right\}^{\frac{3}{2}}}$$

$$\left.+\frac{x_2+\left(\frac{1+6A}{2}\right)}{\left\{\left(x_1+\left(\frac{1+6A}{2\sqrt{3}}\right)\right)^2+\left(x_2+\left(\frac{1+6A}{2}\right)\right)^2+x_3^2\right\}^{\frac{3}{2}}}\right\}$$

$$+3A\left\{\frac{x_2}{\left\{\left(x_1-\left(\frac{1+6A}{\sqrt{3}}\right)\right)^2+x_2^2+x_3^3\right\}^{\frac{5}{2}}}\right.$$

$$+\frac{x_2-\left(\frac{1+6A}{2}\right)}{\left\{\left(x_1+\left(\frac{1+6A}{2\sqrt{3}}\right)\right)^2+\left(x_2-\left(\frac{1+6A}{2}\right)\right)^2+x_3^2\right\}^{\frac{5}{2}}}$$

$$\left.+\frac{x_2+\left(\frac{1+6A}{2}\right)}{\left\{\left(x_1+\left(\frac{1+6A}{2\sqrt{3}}\right)\right)^2+\left(x_2+\left(\frac{1+6A}{2}\right)\right)^2+x_3^2\right\}^{\frac{5}{2}}}\right\}$$

$$+15Ax_3^2\left\{\frac{x_2}{\left\{\left(x_1-\left(\frac{1+6A}{\sqrt{3}}\right)\right)^2+x_2^2+x_3^3\right\}^{\frac{5}{2}}}\right.$$

$$+\frac{x_2-\left(\frac{1+6A}{2}\right)}{\left\{\left(x_1+\left(\frac{1+6A}{2\sqrt{3}}\right)\right)^2+\left(x_2-\left(\frac{1+6A}{2}\right)\right)^2+x_3^2\right\}^{\frac{5}{2}}}$$

$$\left.\left.\left.+\frac{x_2+\left(\frac{1+6A}{2}\right)}{\left\{\left(x_1+\left(\frac{1+6A}{2\sqrt{3}}\right)\right)^2+\left(x_2+\left(\frac{1+6A}{2}\right)\right)^2+x_3^2\right\}^{\frac{5}{2}}}\right\}\right]\right],$$

$$f_6=-\frac{1}{3}\left[\left[\left\{\frac{x_3}{\left\{\left(x_1-\left(\frac{1+6A}{\sqrt{3}}\right)\right)^2+x_2^2+x_3^3\right\}^{\frac{3}{2}}}\right.\right.\right.$$

$$+\frac{x_3}{\left\{\left(x_1+\left(\frac{1+6A}{2\sqrt{3}}\right)\right)^2+\left(x_2-\left(\frac{1+6A}{2}\right)\right)^2+x_3^2\right\}^{\frac{3}{2}}}$$

$$\left.+\frac{x_3}{\left\{\left(x_1+\left(\frac{1+6A}{2\sqrt{3}}\right)\right)^2+\left(x_2+\left(\frac{1+6A}{2}\right)\right)^2+x_3^2\right\}^{\frac{3}{2}}}\right\}$$

$$+3A\left\{\frac{x_3}{\left\{\left(x_1-\left(\frac{1+6A}{\sqrt{3}}\right)\right)^2+x_2^2+x_3^3\right\}^{\frac{5}{2}}}\right.$$

$$+\frac{x_3}{\left\{\left(x_1+\left(\frac{1+6A}{2\sqrt{3}}\right)\right)^2+\left(x_2-\left(\frac{1+6A}{2}\right)\right)^2+x_3^2\right\}^{\frac{5}{2}}}$$

$$\left.+\frac{x_3}{\left\{\left(x_1+\left(\frac{1+6A}{2\sqrt{3}}\right)\right)^2+\left(x_2+\left(\frac{1+6A}{2}\right)\right)^2+x_3^2\right\}^{\frac{5}{2}}}\right\}$$

$$+15Ax_3^2\left\{\frac{x_2}{\left\{\left(x_1-\left(\frac{1+6A}{\sqrt{3}}\right)\right)^2+x_2^2+x_3^3\right\}^{\frac{5}{2}}}+\frac{x_2-\left(\frac{1+6A}{2}\right)}{\left\{\left(x_1+\left(\frac{1+6A}{2\sqrt{3}}\right)\right)^2+\left(x_2-\left(\frac{1+6A}{2}\right)\right)^2+x_3^2\right\}^{\frac{5}{2}}}+\frac{x_2+\left(\frac{1+6A}{2}\right)}{\left\{\left(x_1+\left(\frac{1+6A}{2\sqrt{3}}\right)\right)^2+\left(x_2+\left(\frac{1+6A}{2}\right)\right)^2+x_3^2\right\}^{\frac{5}{2}}}\right\}\Bigg]\Bigg].$$

Let

$$x_i(t) = \phi_i\left(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, t\right), \quad (i = 1, 2, ..., 6) \tag{8}$$

are unique solution of the Equation (7), where α_i's represent the initial conditions.

The known solution of the Equation (7) as the unperturbed motion is (8) and defining the solution of these equations in the neighborhood of the unperturbed motion, the latter motion can be expressed in the form

$$x_i(t) = \phi_i(t) + \xi_i(t), \quad \xi_i(t) << 1, \quad i = 1, 2, 3, ...6. \tag{9}$$

where the function $\phi_i(t)$, constitutes a known solution of the Equation (7) and the function $\xi_i(t)$, referred as the perturbation.

Introducing Equation (9) into Equation (7) and recalling that the functions $\phi_i(t)$, satisfying Equation (7), we obtain

$$\dot{\xi}_i(t) = f_i\left(\phi_1 + \xi_1, \phi_2 + \xi_2, ..., \phi_6 + \xi_6\right) - f_i\left(\phi_1, \phi_2, ..., \phi_6\right), \quad i = 1, 2, ..., 6 \tag{10}$$

which are known as differential equations of the perturbed motion. The origin $\xi_i = 0$ is a trivial solution of these equations. Expanding the Equation (10) about the origin, we obtain

$$\dot{\xi}_i = \sum_{j=1}^{6} f_{ij}(t)\,\xi_{ij}(t) + \varepsilon_i\left(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, t\right), \quad i = 1, 2, 3, ..., 6.$$

where

$$f_{ij}(t) = \left.\frac{\partial f_i(x_1, x_2, x_3, ..., x_6)}{\partial x_j}\right|_{x_i=\phi_i} \lim_{\delta x\to 0} \tag{11}$$

are either constant or periodic, and the function ε_i are power series in $\phi_1, \phi_2, \phi_3, ..., \phi_6$ containing terms of second and higher powers in these variables.

The perturbation ϕ_i generally results from small distribution of unknown sources. Let us assume that the perturbation is sufficiently small to permit the second order terms in ϕ_i to be approximates as

$$\dot{\xi}_i = \sum_{i=1}^{6} f_{ij}(t)\,\xi_{ij}(t), \; i = 1, 2, ..., 6. \tag{12}$$

Equation (12) is known as the variational equations of Poincare. Equation (12) constitutes a set of simultaneous linear differential equations with periodic coefficients. To study the stability of these types of equations we use the Floquet's theory.

Now, we let $\varphi(t)$ be the solution of the system (12), corresponding to the initial condition $\varphi(t) = [e_i]$, $i = 1, 2, ..., 6$. Here $[e_i]$ is a unit vector whose typical vector $\{e_i\}$ is defined as having all of its components zero except for the i^th components which are equal to 1. This is simply a column of the identity matrix. $\varphi(t)$ is then said to be the fundamental set or linear basis. The 6×6 matrix $[\varphi(t)]$ with its columns consisting of a set of linearly independent solutions of the system (12) is called the fundamental matrix satisfying the matrix equation

$$\dot{\varphi}_{ij}(t) = \sum_{K=1}^{6} f_{ij}(t)\,\varphi_{ij}, \; i = 1, 2, ..., 6 \tag{13}$$

$[\varphi(t)]$ is a fundamental matrix of then, $[\varphi(t + T)]$ is also a fundamental matrix. Here T is the period of then, there exist a non-singular constant matrix $[C]$ such that

$$[\varphi(t + T)] = [\varphi(t)]\,[C], \tag{14}$$

where the matrix $[C]$ is sometimes referred to as the monodromy matrix of the fundamental matrix

$[\varphi(t)]$. But a matrix $[R]$ can be found which satisfies

$$[C] \quad e^{T[R]}. \tag{15}$$

So that from the Equation (14) and (15), we have

$$[\varphi(t+T)] = [\varphi(t)]\, e^{T[R]}. \tag{16}$$

Next, let us define the matrix $[Q(t)]$ by

$$[Q(t)] \quad [\varphi(t)]\, e^{\ T[R]}. \tag{17}$$

Now from the Equation (16),

$$\begin{aligned}[Q(t+T)] &= [\varphi(t+T)]\, e^{-(t+T)\,[R]}\\ &\ [\varphi(t)]\, e^{T[R]}.e^{-(t+T)\,[R]}\\ &\ [\varphi(t)]\, e^{-t[R]}\\ &\ [Q(t)]\,.\end{aligned} \tag{18}$$

Hence $[Q(t)]$ is a periodic matrix with period T. Let $t = t_0 = 0$, then

$$[\varphi(t)] \quad [\varphi(0)]\,[C]$$

or

$$[C] \quad [\varphi(0)]^{-1} \,.\, [\varphi(T)] \tag{19}$$

or,

$$C \quad [\varphi(T)] \text{ as } [\varphi(0)] \quad 1.$$

Thus from the equation (15)

$$[\varphi(T)] = e^{T[R]} = C$$

or,

$$T[R] \quad \log[C]$$

$$\Rightarrow [R] \quad \frac{1}{T}\log[C] \tag{20}$$

Now, the characteristic polynomial associated with the matrix $|f_{ij}|$ is defined by the characteristic determinant

$$\big|[C] - \lambda_i[I]\big| = 0\,; \tag{21}$$

Where $i = 1, 2, ..., 6$.

And $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_6$ are characteristic multipliers for the system (12).

The eigen values of $[R]$, denoted by $\rho_1, \rho_2, \rho_3, \ldots, \rho_6$ are called the characteristic exponents associated with the periodic matrix $|f_{ij}|$ and are related to the characteristic multipliers by

$$\lambda_i = e^{T\rho_i}, \qquad i \quad 1, 2, ..., 6. \tag{22}$$

Whereas the characteristic multipliers λ_i are uniquely defined and the real parts of the characteristic exponents ρ_j are defined uniquely by

$$\rho_i \quad \frac{1}{T}\left[\log|\lambda_j| + i \arg \lambda_j\right], \quad j \quad 1, 2, 3, ..., 6. \tag{23}$$

The imaginary parts are determined up to an integral multiple of $\frac{2\pi}{T}$. This enables us to reach the followings conclusions:

If all the characteristic exponents have negative real parts, all solutions of Equation (12) are asymptotically stable,

$$\lim_{t\to\infty}\{\varphi(t)\} = 0, \quad \mathrm{Re}(\rho_i) < 0,$$

$$\mathrm{Re}\,|\lambda_i| < 1, \qquad j = 1, 2, ..., 6.$$

Here $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_6$ are the characteristic multipliers of the system (12) and so its complex conjugate. From the Equation (21) the determinant of the characteristic polynomial is

$$\begin{aligned}&(\lambda - 1)^2(\lambda^4 + A\lambda^3 + B\lambda^2\\ &+A\lambda + 1) = 0,\end{aligned} \tag{24}$$

Where

$$A \quad 2 - T_r\,[\varphi(T)]$$

$$\mathbf{B} = \frac{1}{2}\left[A^2 - \left(T_r\left[\varphi^2[T] - 2\right]\right)\right]$$

And T_r is the trace of matrix $[\varphi[T]]$. The Equation (24) can be written as

$$(\lambda - 1)^2 \left(\lambda^2 + p\lambda + 1\right)\left(\lambda^2 + q\lambda + 1\right) = 0$$

where

$$p + q = A, \ \ pq = \mathrm{B} - 2.$$

The stability of the system (12) is given by the conditions

$$|p| \le 2 \text{ And } |q| \le 2.$$

(George Katsiaris, 1971) [15]

We have solved the system of the Equation (14) for different $z_{initial}$ applying initial conditions as stated before and for different oblateness parametre, we have found the different set of characteristic polynomials and consequently different set of characteristic multipliers from which we have evaluated p and q. The results are shown in the Table-1 and Table-2.

4. CONCLUSIONS

We have find out the Equation of motion of Sitnikov problem where all the primaries are oblate spheroid. Then we have shown the stability of the motion of the problem. We have observed that when we increase the oblateness parameter the stability interval increases. When A=0.05 the motion is stable for $z_{initial} \in$ [1.932500, 2.000000] and when A=0.5, the motion is stable for $z_{initial} \in$ [1.900000, 2.6300057].

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