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Research Article

Starlikeness of Generalized Srivastava-Owa Fractional Operators

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Abstract: In the geometric function theory much attention is paid to various fractional operators (differential and integrals) mapping the class of univalent functions and its subclasses into themselves. The classical definitions of fractional operators and their generalizations have fruitfully been applied in obtaining, for example, the characterization properties, coefficient estimates, distortion inequalities and convolution structures for various subclasses of analytic functions and the works in the research monographs. Here, we introduce a generalization for the well known fractional operators in the unit disk i.e. U:={z: |z|<1 } due to H. M. Srivastava and S. Owa. Some analytic and geometric properties are obtained.

Keywords: Fractional calculus; fractional differential operator; fractional integral operator; Srivastava-Owa fractional operators; unit disk; analytic function; univalent; starlike function

1. INTRODUCTION

Srivastava and Owa [1] imposed definitions for fractional operators (derivative and integral) in the complex z-plane $\mathbb{C}$ as follows:

Definition 1.1 The fractional derivative of order α is defined, for a function $f(z)$ by

$$D_z^\alpha f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\alpha} d\zeta$$
$$0 \le \alpha < 1,$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 1.2 The fractional integral of order α is defined, for a function $f(z)$,

$$I_z^\alpha f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} d\zeta;$$

Where α>0, the function $f(z)$ is analytic in simply-connected region of the complex z-plane ($\mathbb{C}$) containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

In [2], Ibrahim derived a formula for the generalized fractional integral, consider for natural $n \in \mathbb{N} = \{1,2,...\}$ and real μ, the n-fold integral of the form

$$I_z^{\alpha,\mu} f(z) = \int_0^z \zeta_1^\mu d\zeta_1 \int_0^{\zeta_1} \zeta_2^\mu d\zeta_2 \dots \int_0^{\zeta_{n-1}} \zeta_n^\mu f(\zeta_n) d\zeta_n. \quad (1)$$

Employing the Dirichlet technique yields

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$$\int_0^z \zeta_1^{\mu} d\zeta_1 \int_0^{\zeta_1} \zeta^{\mu} f(\zeta) d\zeta$$
$$= \int_0^z \zeta^{\mu} f(\zeta) d\zeta \int_{\zeta}^z \zeta_1^{\mu} d\zeta_1$$
$$= \frac{1}{\mu+1} \int_0^z (z^{\mu+1} - \zeta^{\mu+1}) \zeta^{\mu} f(\zeta) d\zeta.$$

Repeating the above step $n-1$ times we have

$$\int_0^z \zeta_1^{\mu} d\zeta_1 \int_0^{\zeta_1} \zeta_2^{\mu} d\zeta_2 ... \int_0^{\zeta_{n-1}} \zeta_n^{\mu} f(\zeta_n) d\zeta_n$$
$$= \frac{(\mu+1)^{1-n}}{(n-1)!} \int_0^z (z^{\mu+1} - \zeta^{\mu+1})^{n-1} \times \zeta^{\mu} f(\zeta) d\zeta$$

which implies the fractional operator type

$$I_z^{\alpha,\mu} f(z) = \frac{(\mu+1)^{1-\alpha}}{\Gamma(\alpha)} \int_0^z (z^{\mu+1} - \zeta^{\mu+1})^{\alpha-1} \zeta^{\mu} f(\zeta) d\zeta. \tag{2}$$

where $\alpha > 0$ and $\mu \neq -1$ are real numbers and the function $f(z)$ is analytic in simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z^{\mu+1} - \zeta^{\mu+1})^{-\alpha}$ is removed by requiring $\log(z^{\mu+1} - \zeta^{\mu+1})$ to be real when $(z^{\mu+1} - \zeta^{\mu+1}) > 0.$ When $\mu = 0,$ we arrive at the standard Srivastava-Owa fractional integral, which is used to define the Srivastava-Owa fractional derivatives. It was shown that

$$I_z^{\alpha,\mu} z^{\nu} = \frac{z^{\alpha(\mu+1)+\nu}}{(\mu+1)^{\alpha}} \frac{\Gamma(\frac{\nu+\mu+1}{\mu+1})}{\Gamma(\alpha + \frac{\nu+\mu+1}{\mu+1})},$$

where α>0, μ≥0 and ν>-1. Corresponding to the generalized fractional integrals (2), we define the generalized differential operator

Definition 1.3 The generalized fractional derivative of order α is defined, for a function $f(z)$ by

$$D_z^{\alpha,\mu} f(z) := \frac{(\mu+1)^{\alpha}}{\Gamma(1-\alpha)} \frac{d}{dz} [\int_0^z \frac{\zeta^{\mu} f(\zeta)}{(z^{\mu+1} - \zeta^{\mu+1})^{\alpha}} d\zeta]; \tag{3}$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z^{\mu+1} - \zeta^{\mu+1})^{-\alpha}$ is removed by requiring $\log(z^{\mu+1} - \zeta^{\mu+1})$ to be real when $(z^{\mu+1} - \zeta^{\mu+1}) > 0.$ We have

$$D_z^{\alpha,\mu} z^{\nu} = \frac{(\mu+1)^{\alpha-1} \Gamma(\frac{\nu}{\mu+1} + 1)}{\Gamma(\frac{\nu}{\mu+1} + 1 - \alpha)} z^{(1-\alpha)(\mu+1)+\nu-1}.$$

Recently, various results, as convolution and inclusion properties, distortion theorem, extreme points, coefficient estimates etc., are proposed by many authors for the operators due to Srivastava involving the Wright function, generalized hypergeometric function and Meijer's G-functions. These operators are Dziok-Srivastava, Srivastava-Wright, Cho-Kwon-Srivastava operator, Cho-Saigo-Srivastava operator, Jung-Kim-Srivastava and Srivastava-Owa operators (see [3-11]).

In the recent work, we shall concern about a class of analytic functions [12]: for positive constant k

$$\mathcal{A}_k = \{f \in \mathcal{H} : f(z) = z + \sum_{n=1}^{\infty} a_{k+n} z^{k+n}\}.$$

It is clear that $\mathcal{A}_1 = \mathcal{A}$ the normalized class with

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

A function $f(z) \in \mathcal{A}$ is said to be starlike of order β in U if it satisfies $\Re\{\frac{zf'(z)}{f(z)}\} > \beta$ for some $0 \leq \beta < 1.$

We need the following lemmas in the sequel which can be found in [12].

Lemma 1.1 Let $f(z) \in \mathcal{A}_k$ and let $0 \leq \beta < 1.$ If f satisfies

$$|zf''(z)| < \frac{k(k+1)(1-\beta)}{k+1-\beta}$$

then $f(z)$ is starlike of order β in $U.$

Lemma 1.2 Let $f(z) \in \mathcal{A}_k$ and $0 \leq \beta < k.$ If f satisfies

$$|zf''(z) - \beta(f'(z) - 1)| < k - \beta$$

then $f(z)$ is starlike in $U.$

Lemma 1.3 Let $f(z) \in \mathcal{A}_k$, $0 \leq \eta < 1$ and

$0 \le \beta < k$. If f satisfies

$$|zf''(z) - \beta(f'(z)-1)| < \frac{(k+1)(1-\eta)(k-\beta)}{k+1-\eta}$$

then $f(z)$ is starlike in U.

2. MAIN RESULTS

We introduce the following extension operator

$$\Phi^{\alpha,\mu} f(z) := \frac{\Gamma(\frac{1}{\mu+1}+1-\alpha)}{(\mu+1)^{\alpha-1}\Gamma(\frac{1}{\mu+1}+1)} z^{\alpha-\mu+\mu\alpha} D_z^{\alpha,\mu} f(z)$$

$$= \frac{\Gamma(\frac{1}{\mu+1}+1-\alpha)}{(\mu+1)^{\alpha-1}\Gamma(\frac{1}{\mu+1}+1)} z^{\alpha-\mu+\mu\alpha} D_z^{\alpha,\mu} f(z)$$

$$+\sum_{n=1}^{\infty} a_{n+k} z^{n+k}) = \frac{\Gamma(\frac{1}{\mu+1}+1-\alpha)}{(\mu+1)^{\alpha-1}\Gamma(\frac{1}{\mu+1}+1)} z^{\alpha-\mu+\mu\alpha}$$

$$\times[\frac{(\mu+1)^{\alpha-1}\Gamma(\frac{1}{\mu+1}+1)}{\Gamma(\frac{1}{\mu+1}+1-\alpha)} z^{-\alpha+\mu-\mu\alpha+1}$$

$$+\sum_{n=1}^{\infty} \frac{\Gamma(\frac{n+k}{\mu+1}+1)}{\Gamma(\frac{n+k}{\mu+1}+1-\alpha)} a_{n+k} z^{n+k-\alpha+\mu-\mu\alpha}] = z$$

$$+\sum_{n=1}^{\infty} \frac{\Gamma(\frac{1}{\mu+1}+1-\alpha)}{\Gamma(\frac{1}{\mu+1}+1)} \times \frac{\Gamma(\frac{n+k}{\mu+1}+1)}{\Gamma(\frac{n+k}{\mu+1}+1-\alpha)} a_{n+k} z^{n+k} := \tag{4}$$

$$z + \sum_{n=1}^{\infty} \phi_{n+k}^{\alpha,\mu} a_{n+k} z^{n+k}$$

Obviously, when $k=1, \mu=0$ we have the extension fractional differential operator defined in [13] ([14] for recent work), which contains the Carlson and Shaffer operator.

Theorem 2.1 Let $f \in \mathrm{A}_k$. If for positive constant k

$$(k+1)\phi_{1+k}^{\alpha,\mu} |a_{1+k}| < \frac{1}{2}$$

and

$$\sum_{n=2}^{\infty} \phi_{n+k}^{\alpha,\mu} (n+k)(n+k-1) |a_{n+k}| \le \phi_{1+k}^{\alpha,\mu} |a_{1+k}|$$

then the operator $\Phi^{\alpha,\mu} f(z)$ is starlike of order β where

$$\beta < \frac{1-2(k+1)\phi_{1+k}^{\alpha,\mu} |a_{1+k}|}{1-2\phi_{1+k}^{\alpha,\mu} |a_{1+k}|}.$$

Proof. By applying Lemma 1.1, we impose

$$|z(\Phi^{\alpha,\mu} f(z))''| = |\sum_{n=1}^{\infty} \phi_{n+k}^{\alpha,\mu} (n+k)(n+k-1) a_{n+k} z^{n+k}|$$

$$\le \sum_{n=1}^{\infty} \phi_{n+k}^{\alpha,\mu} (n+k)(n+k-1) |a_{n+k}|$$

$$= \phi_{1+k}^{\alpha,\mu} k(1+k) |a_{1+k}| + \sum_{n=2}^{\infty} \phi_{n+k}^{\alpha,\mu} (n+k)(n+k-1) |a_{n+k}|$$

$$\le 2\phi_{1+k}^{\alpha,\mu} k(1+k) |a_{1+k}|$$

$$< \frac{k(k+1)(1-\beta)}{k+1-\beta}.$$

Thus $\Phi^{\alpha,\mu} f(z)$ is starlike of order β.

Theorem 2.2 Let $f \in \mathrm{A}_k$. If for positive constant k

$$(k+1)\phi_{1+k}^{\alpha,\mu} |a_{1+k}| < \frac{1}{2}$$

and

$$\sum_{n=2}^{\infty} \phi_{n+k}^{\alpha,\mu} (n+k)(n+k-1-\beta) |a_{n+k}|$$

$$\le (k+1)(k-\beta)\phi_{1+k}^{\alpha,\mu} |a_{1+k}|$$

then the operator $\Phi^{\alpha,\mu} f(z)$ is starlike of order β where $\beta < k$.

Proof. By employing Lemma 1.2, we get

$$|z(\Phi^{\alpha,\mu} f(z))'' - \beta(\Phi^{\alpha,\mu} f(z)' - 1)|$$

$$= |\sum_{n=1}^{\infty} \phi_{n+k}^{\alpha,\mu} (n+k)(n+k-1-\beta) a_{n+k} z^{n+k}|$$

$$\le \sum_{n=1}^{\infty} \phi_{n+k}^{\alpha,\mu} (n+k)(n+k-1-\beta) |a_{n+k}|$$

$$= \phi_{1+k}^{\alpha,\mu} (k+1)(k-\beta) |a_{1+k}|$$

$$+ \sum_{n=2}^{\infty} \phi_{n+k}^{\alpha,\mu} (n+k)(n+k-1-\beta) |a_{n+k}|$$

$$\le 2\phi_{1+k}^{\alpha,\mu} (k+1)(k-\beta) |a_{1+k}|$$

$$< k-\beta.$$

Thus $\Phi^{\alpha,\mu} f(z)$ is starlike of order β.

Theorem 2.3 Let $f \in \mathcal{A}_k$. If for positive constant k

$$(k+1)\phi_{1+k}^{\alpha,\mu} \mid a_{1+k} \mid < \frac{1}{2}$$

and

$$\sum_{n=2}^{\infty} \phi_{n+k}^{\alpha,\mu} (n+k)(n+k-1-\eta) \mid a_{n+k} \mid \leq (k+1)(k-\eta)\phi_{1+k}^{\alpha,\mu} \mid a_{1+k} \mid$$

then the operator $\Phi^{\alpha,\mu} f(z)$ is starlike of order β where

$$\beta < \frac{1-2(k+1)\phi_{1+k}^{\alpha,\mu} \mid a_{1+k} \mid}{1-2\phi_{1+k}^{\alpha,\mu} \mid a_{1+k} \mid}.$$

Proof. By using Lemma 1.3, we get

$$\begin{aligned} &\mid z(\Phi^{\alpha,\mu} f(z))'' - \eta(\Phi^{\alpha,\mu} f(z)' - 1) \mid \\ &= \mid \sum_{n=1}^{\infty} \phi_{n+k}^{\alpha,\mu} (n+k)(n+k-1-\eta) a_{n+k} z^{n+k} \mid \\ &\leq \sum_{n=1}^{\infty} \phi_{n+k}^{\alpha,\mu} (n+k)(n+k-1-\eta) \mid a_{n+k} \mid \\ &= \phi_{1+k}^{\alpha,\mu} (k+1)(k-\eta) \mid a_{1+k} \mid \\ &+ \sum_{n=2}^{\infty} \phi_{n+k}^{\alpha,\mu} (n+k)(n+k-1-\eta) \mid a_{n+k} \mid \\ &\leq 2\phi_{1+k}^{\alpha,\mu} (k+1)(k-\eta) \mid a_{1+k} \mid \\ &< \frac{(k+1)(1-\beta)(k-\eta)}{k+1-\beta}. \end{aligned}$$

Thus $\Phi^{\alpha,\mu} f(z)$ is starlike of order β.

3. CONCLUSIONS

An extention fractional differential operator are defined in the unit disk for some class of analytic functions taking the form $\mathcal{A}_k = \{f \in \mathcal{H} : f(z) = z + \sum_{n=1}^{\infty} a_{k+n} z^{k+n}\}$. .Furtherm ore, starlikeness conditions are imosed depending on results due to Kuroki & Owa.

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