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Research Article

Hadamard-type Inequalities through *h*-Convexity

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Abstract: Some new inequalities of Hadamard-type for $h-$convex mappings with general point of line segment are established. Two Lemmas permitting us to establish new inequalities connected with lower and upper part of the celebrated Hermite-Hadamard inequality, are pointed out.

Keywords: Hermite-Hadamard inequality, convex, $s-$convex and $h-$convex functions.

1. INTRODUCTION

Let $f:[a,b]\to R$ be a convex function. Then

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2} \tag{1.1}$$

refer as the Hermite-Hadamard inequality [3, 8, 9]. In recent years, many authors established several inequalities connected to this famous integral inequality (1.1). For recent results, refinements, counterparts, generalizations and new Hermite-Hadamard type inequalities see the papers [2–7].

Definition 1: [3]Let I be an interval of real numbers. The function $f:I\to R$ is said to be convex if for all $x,y\in I$ and $t\in[0, 1]$, one has the inequality:

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y).$$

If the above inequality is reversed then f is said to be concave.

Dragomir and Agarwal [4] obtained inequalities for differentiable convex function which are connected with the Hermite-Hadamard inequality (1.1), one of them is pointed out as:

Theorem 1: Let $f:I\subseteq R\to R$ be differentiable function on I^o where $a,b\in I$ with $a<b$. If $|f'|$ is convex on $[a,b]$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq\frac{b-a}{8}\left[\,|f'(a)|+|f'(b)|\,\right] \tag{1.2}$$

Kirmaci [6] gave the following results:

Theorem 2: Let $f:I\subseteq R\to R$ be differentiable function on I^o where $a,b\in I$ with $a<b$. If $|f'|$ is convex on $[a,b]$, then

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-f\left(\frac{a+b}{2}\right)\right| \leq\frac{b-a}{8}\left[\,|f'(a)|+|f'(b)|\,\right] \tag{1.3}$$

Theorem 3: Let $f:I\subseteq R\to R$ be differentiable function on I^o where $a,b\in I$ with $a<b$ and $p>1$. If $|f'|^q$ is convex on $[a,b]$, then

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|\leq\frac{b-a}{16}\left(\frac{4}{p+1}\right)^{1/p} \left[\left(|f'(a)|^q+3|f'(b)|^q\right)^{1/q}+\left(3|f'(a)|^q+|f'(b)|^q\right)^{1/q}\right] \tag{1.4}$$

where $1/p+1/q=1$.

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Kirmaci et al [7] established the following new Hadamard-type inequality for concave functions:

Theorem 4: Let $f: I \subset R \to R$ be differentiable function on I^o where $a, b \in I$ with $a < b$. If $|f'|^q$, $q > 1$ is concave on $[a, b]$, then

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}$$

$$\left[\left|f'\left(\frac{3a+b}{4}\right)\right| + \left|f'\left(\frac{a+3b}{4}\right)\right|\right] \tag{1.5}$$

Alomari et al [2] established the following Hadamard-type inequality for concave functions:

Theorem 5: Let $f: I \subset R \to R$ be differentiable function on I^o where $a, b \in I$ with $a < b$. If $|f'|^q$, $q > 1$ is concave on $[a, b]$, then

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}$$

$$\left[\left|f'\left(\frac{3a+b}{4}\right)\right| + \left|f'\left(\frac{a+3b}{4}\right)\right|\right] \tag{1.6}$$

Definition 2: A function $f: I \subseteq R \to R$ is said to be Godunova-Levin function or f belongs to class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0,1)$, the inequality holds [3]:

$$f(tx + (1-t)y) \le \frac{1}{t} f(x) + \frac{1}{1-t} f(y).$$

Definition 3: A function $f: R \to [0, +\infty]$ is said to be a $P-$function or that f belongs to the class $P(I)$, if for all $x, y \in R$ and $t \in (0,1)$ [3]. we have :

$$f(tx + (1-t)y) \le f(x) + f(y).$$

Definition 4: [5] Let s be a real number, $s \in (0, 1]$. A function $f: [0, \infty) \to [0, \infty)$ is said to be $s-$function(in the second sense) or f belongs to the class K_s^2, if for all $x, y \in [0, \infty)$ and $t \in [0,1]$, the following inequality holds:

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y).$$

Varošanec [13] defined a large class of non-negative functions, the so-called $h-$convex functions. This class contains several well-known classes of functions such as non-negative convex functions, *s*-convex in the second sense, Godunova-Levin functions and *P*-functions. The definition of an *h*-convex function is stated below:

Definition 5: Let $h: J \subseteq R \to R$ be a non-negative function. A non-negative function $f: I \subseteq R \to R$ is said to be $h-$convex function (or f belongs to class $SX(h, I)$) if for all $x, y \in I$ and $t \in (0,1)$, the following inequality holds:

$$f(tx + (1-t)y) \le h(t)f(x) + h(1-t)f(y).$$

If the inequality is reversed then f is said to be $h-$concave function (or $f \in SV(h, I)$).

Remark 1: It is noted that if:

- $h(t) = t$, then all the non-negative convex functions belong to the class $SX(h, I)$ and all non-negative concave functions belong to the class $SV(h, I)$.
- $h(t) = \frac{1}{t}$, then $SX(h, I) = Q(I)$.
- $h(t) = 1$, then $SX(h, I) \supseteq P(I)$.
- $h(t) = t^s$, where $s \in (0,1)$, then $SX(h, I) \supseteq K_s^2$.

Some interesting and important inequalities for $h-$convex functions can be found in [11,12].

In [12], Sarikaya et. al. established a new Hadamard-type inequality for $h-$convex functions.

Theorem 6: Let $f \in SX(h, I)$, $a, b \in I$ with $a < b$ and $f \in L[a, b]$, then

$$\frac{1}{2h(1/2)} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\le [f(a) + f(b)]\int_0^1 h(t)dt$$

In this paper new results of Hadamard-type on the basis of two established lemmas for $h-$convex functions will be presented.

2. MAIN RESULTS

Lemma 1: Let $f: I \subset R \to R$ be differentiable function on I^o where $a, b \in I$ with $a < b$. If $f' \in L[a, b]$ and $\lambda, \mu \in [0, \infty)$ with $\lambda + \mu \ne 0$, then

$$\frac{\mu f(a)+\lambda f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx=$$
$$\frac{b-a}{(\lambda+\mu)^2}\left[\int_0^\mu(-t)\,f'\left(\frac{\lambda+t}{\lambda+\mu}a+\frac{\mu-t}{\lambda+\mu}b\right)dt+\right.$$
$$\left.\int_0^\lambda t\ f'\left(\frac{\lambda-t}{\lambda+\mu}a+\frac{\mu+t}{\lambda+\mu}b\right)dt\right] \qquad (2.1)$$

Proof: It suffices to note that

$$I_1=\int_0^\mu(-t)\,f'\left(\frac{\lambda+t}{\lambda+\mu}a+\frac{\mu-t}{\lambda+\mu}b\right)dt$$
$$=\frac{b-a}{(\lambda+\mu)^2}f\left(\frac{\lambda+t}{\lambda+\mu}a+\frac{\mu-t}{\lambda+\mu}b\right)\Bigg|_0^\mu-\frac{1}{\lambda+\mu}\int_0^\mu f\left(\frac{\lambda+t}{\lambda+\mu}a+\frac{\mu-t}{\lambda+\mu}b\right)dt$$
$$=\frac{\mu f(a)}{\lambda+\mu}-\frac{1}{b-a}\int_a^{\frac{\lambda a+\mu b}{\lambda+\mu}}f(x)dx. \qquad (2.2)$$

and similarly we get

$$I_2=\int_0^\lambda t\,f'\left(\frac{\lambda-t}{\lambda+\mu}a+\frac{\mu+t}{\lambda+\mu}b\right)dt$$
$$=\frac{\lambda f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_{\frac{\lambda a+\mu b}{\lambda+\mu}}^b f(x)dx. \qquad (2.3)$$

By adding identities (2.2) and (2.3), we get identity (2.1).

Remark 2: For $\lambda=\mu$, Lemma 1 reduces to [1, Lemma 2.1].

In the following theorems, we shall propose some new upper bounds for the right-hand side of Hadamard's inequality for $h-$convex mappings.

Theorem 7: Let $|f'|$ be $h-$convex function and the assumptions of Lemma 1 hold, then

$$\left|\frac{\mu f(a)+\lambda f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq(b-a)\int_0^1\left\{\left|f'(a)\left(\frac{\lambda}{\lambda+\mu}-t\right)\right|+\right.$$
$$\left.\left|f'(b)\left(\frac{\mu}{\lambda+\mu}-t\right)\right|\right\}h(t)\,dt \qquad (2.4)$$

Proof: By using the property of modulus and from Lemma 1, we have

$$\left|\frac{\mu f(a)+\lambda f(b)}{\lambda+\mu}+\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq\frac{b-a}{(\lambda+\mu)^2}\left[\int_0^\mu t\left|f'\left(\frac{\lambda+t}{\lambda+\mu}a+\frac{\mu-t}{\lambda+\mu}b\right)\right|dt\right.$$
$$\left.+\int_0^\lambda t\left|f'\left(\frac{\lambda-t}{\lambda+\mu}a+\frac{\mu+t}{\lambda+\mu}b\right)\right|dt\right] \qquad (2.5)$$

Since $|f'|$ is $h-$convex

$$\left|\frac{\mu f(a)+\lambda f(b)}{\lambda+\mu}+\frac{1}{b-a}\int_a^b f(x)dx\right|\leq$$
$$\frac{b-a}{(\lambda+\mu)^2}\left[\int_0^\mu t\left\{\begin{matrix}|f'(a)|h\left(\frac{\lambda+t}{\lambda+\mu}\right)+\\ |f'(b)|h\left(\frac{\mu-t}{\lambda+\mu}\right)\end{matrix}\right\}dt+\right.$$
$$\left.\int_0^\lambda t\left\{|f'(a)|h\left(\frac{\lambda-t}{\lambda+\mu}\right)+|f'(b)|h\left(\frac{\mu+t}{\lambda+\mu}\right)\right\}dt\right]$$
$$=(b-a)\left[|f'(a)|\int_{\frac{\lambda}{\lambda+\mu}}^1\left(t-\frac{\mu}{\lambda+\mu}\right)h(t)\,dt\right.$$
$$+|f'(b)|\int_0^{\frac{\mu}{\lambda+\mu}}\left(\frac{\mu}{\lambda+\mu}-t\right)h(t)\,dt$$
$$+|f'(a)|\int_0^{\frac{\lambda}{\lambda+\mu}}\left(\frac{\lambda}{\lambda+\mu}-t\right)h(t)\,dt$$
$$\left.+|f'(b)|\int_{\frac{\mu}{\lambda+\mu}}^1\left(t-\frac{\lambda}{\lambda+\mu}\right)h(t)\,dt\right]$$

Hence the proof is completed.

Remark 3: For $\lambda=\mu$, and convex function inequality (2.4) reduces to inequality (1.2).

Theorem 8: Let $|f'|^q$ be $h-$convex function and the assumptions of Lemma 1 hold, then

$$\left|\frac{\mu f(a)+\lambda f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq\frac{b-a}{(p+1)^{1/p}}\left[\left(\frac{\mu}{\lambda+\mu}\right)^{1+\frac{1}{p}}\left(\int_0^{\frac{\mu}{\lambda+\mu}}\left\{|f'(a)|^q h(1-t)+|f'(b)|^q h(t)\right\}dt\right)^{1/q}\right.$$

$$\left.+\left(\frac{\lambda}{\lambda+\mu}\right)^{1+\frac{1}{p}}\left(\int_{\frac{\mu}{\lambda+\mu}}^{1}\{|f'(a)|^q h(1-t)+|f'(b)|^q h(t)\}dt\right)^{1/q}\right] \qquad (2.6)$$

Proof: By applying Hölder's inequality and $h-$convexity on $|f'|^q$, we have

$$\int_0^{\mu} t\left|f'\left(\frac{\lambda+t}{\lambda+\mu}a+\frac{\mu-t}{\lambda+\mu}b\right)\right|dt\leq$$

$$\left(\int_0^{\mu} t^p\,dt\right)^{1/p}\left(\int_0^{\mu}\left|f'\left(\frac{\lambda+t}{\lambda+\mu}a+\frac{\mu-t}{\lambda+\mu}b\right)\right|^q dt\right)^{1/q}$$

$$\leq\frac{\mu^{1+1/p}(\lambda+\mu)^{1/q}}{(p+1)^{1/p}}\left(\begin{array}{l}|f'(a)|^q\int_0^{\frac{\lambda}{\lambda+\mu}}h(1-t)dt+\\|f'(b)|^q\int_0^{\frac{\lambda}{\lambda+\mu}}h(t)dt\end{array}\right)^{1/q}$$

Analogously

$$\int_0^{\lambda} t\left|f'\left(\frac{\lambda-t}{\lambda+\mu}a+\frac{\mu+t}{\lambda+\mu}b\right)\right|dt$$

$$\leq\frac{\lambda^{1+1/p}(\lambda+\mu)^{1/q}}{(p+1)^{1/p}}\left(\begin{array}{l}|f'(a)|^q\int_0^{\frac{\lambda}{\lambda+\mu}}h(1-t)dt\\+|f'(b)|^q\int_0^{\frac{\lambda}{\lambda+\mu}}h(t)dt\end{array}\right)^{1/q}$$

Putting the above inequalities in (2.5) we get (2.6).

Corollary 1: For $\lambda=\mu$, and convex function inequality (2.6) reduces as:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq$$

$$\frac{b-a}{4(p+1)^{1/p}}\left[\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{1/q}+\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{1/q}\right]$$

Theorem 8 may be extended to be as follows:

Theorem 9: Let $|f'|^q$ be $h-$convex function and the assumptions of Lemma 1 hold, then

$$\left|\frac{\mu f(a)+\lambda f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq$$

$$\frac{b-a}{2^{1/p}}\left[\left(\int_0^{\frac{\mu}{\lambda+\mu}}\left(\frac{\mu}{\lambda+\mu}-t\right)\{|f'(a)|^q h(1-t)+|f'(b)|^q h(t)\}\right)^{1/q}+\right.$$

$$\left.\left(\int_{\frac{\mu}{\lambda+\mu}}^{1}\left(t-\frac{\mu}{\lambda+\mu}\right)\{|f'(a)|^q h(1-t)+|f'(b)|^q h(t)\}\right)^{1/q}\right] \qquad (2.7)$$

Proof. The proof of this theorem is same as proof of Theorem 8 and using facts that $\frac{\lambda^2}{(\lambda+\mu)^2}<1$ and $\frac{\mu^2}{(\lambda+\mu)^2}<1$. However, the details are left to the reader.

Now, we give the following Hadamard-type inequality for $h-$concave mappings.

Theorem 10: Let $|f'|^q$ be $h-$concave function and the assumptions of Lemma 1 hold, then

$$\left|\frac{\mu f(a)+\lambda f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq\frac{b-a}{(\lambda+\mu)^2}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{2h(1/2)}\right)^{1/q}\times$$
$$\left[\begin{array}{l}\mu^2\left|f'\left(\frac{1}{\mu}\left(\frac{\lambda+\mu}{2}a-\frac{\lambda^2}{2(\lambda+\mu)}a+\frac{\mu^2}{2(\lambda+\mu)}b\right)\right)\right|\\ +\lambda^2\left|f'\left(\frac{1}{\lambda}\left(\frac{\lambda+\mu}{2}b+\frac{\lambda^2}{2(\lambda+\mu)}a-\frac{\mu^2}{2(\lambda+\mu)}b\right)\right)\right|\end{array}\right] \quad (2.8)s$$

Proof: By applying Hölder's inequality, we have

$$\int_0^\mu t\left|f'\left(\frac{\lambda+t}{\lambda+\mu}a+\frac{\mu-t}{\lambda+\mu}b\right)\right|dt\leq$$

$$\left(\int_0^\mu t^p\,dt\right)^{1/p}\left(\int_0^\mu\left|f'\left(\frac{\lambda+t}{\lambda+\mu}a+\frac{\mu-t}{\lambda+\mu}b\right)\right|^q dt\right)^{1/q}$$

Since $|f'|^q$ is $h-$concave on $[a, b]$, we have

$$2h(1/2)\int_0^\mu\left|f'\left(\frac{\lambda+t}{\lambda+\mu}a+\frac{\mu-t}{\lambda+\mu}b\right)\right|^q dt \leq\left(\int_0^\mu t^0 dt\right)$$

$$\left|f'\left(\frac{1}{\int_0^\mu t^0 dt}\right)\int_0^\mu\left(\frac{\lambda+t}{\lambda+\mu}a+\frac{\mu-t}{\lambda+\mu}b\right)dt\right|^q$$

$$=\mu\left|f'\left(\frac{1}{\mu}\left(\frac{\lambda+\mu}{2}a-\frac{\lambda^2}{2(\lambda+\mu)}a+\frac{\mu^2}{2(\lambda+\mu)}b\right)\right)\right|^q$$

Therefore

$$\int_0^\mu\left|f'\left(\frac{\lambda+t}{\lambda+\mu}a+\frac{\mu-t}{\lambda+\mu}b\right)\right|^q dt$$
$$\leq\frac{\mu^2}{(p+1)^{1/p}}\left(\frac{1}{2h(1/2)}\right)^{1/q}$$
$$\left|f'\left(\frac{1}{\mu}\left(\frac{\lambda+\mu}{2}a-\frac{\lambda^2}{2(\lambda+\mu)}a+\frac{\mu^2}{2(\lambda+\mu)}b\right)\right)\right|$$

Analogously

$$\int_0^\lambda\left|f'\left(\frac{\lambda-t}{\lambda+\mu}a+\frac{\mu+t}{\lambda+\mu}b\right)\right|^q dt$$
$$\leq\frac{\lambda^2}{(p+1)^{1/p}}\left(\frac{1}{2h(1/2)}\right)^{1/q}$$
$$\left|f'\left(\frac{1}{\lambda}\left(\frac{\lambda+\mu}{2}b-\frac{\mu^2}{2(\lambda+\mu)}b+\frac{\lambda^2}{2(\lambda+\mu)}a\right)\right)\right|$$

Using these above inequalities in (2.5) we get (2.8).

Remark 4: For $\lambda=\mu$ and concave function, the inequality (2.8) reduces to inequality (1.5).

Lemma 2: Let assumptions of Lemma 1 be satisfied, then

$$\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{\mu a+\lambda b}{\lambda+\mu}\right)=$$
$$\frac{b-a}{(\lambda+\mu)^2}\left[\int_0^\lambda(t-\lambda)f'\left(\frac{\mu+t}{\lambda+\mu}a+\frac{\lambda-t}{\lambda+\mu}b\right)dt+\right.$$
$$\left.\int_0^\mu(\mu-t)f'\left(\frac{\mu-t}{\lambda+\mu}a+\frac{\lambda+t}{\lambda+\mu}b\right)dt\right] \quad (2.9)$$

Proof. The proof is consequence of integration by parts.

In the following theorems, we shall propose some new upper bounds for the left-hand side of Hadamard's inequality for *h*-convex mappings

Theorem 11: Let $|f'|$ be $h-$convex function and the assumptions of Lemma 1 hold, then

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{\mu a+\lambda b}{\lambda+\mu}\right)\right|=$$
$$(b-a)\left[|f'(a)|\left\{\int_0^{\frac{\mu}{\lambda+\mu}}t\,h(t)dt+\int_{\frac{\mu}{\lambda+\mu}}^1(1-t)\,h(t)dt\right\}+\right.$$
$$\left.|f'(b)|\left\{\int_0^{\frac{\lambda}{\lambda+\mu}}t\,h(t)dt+\int_{\frac{\lambda}{\lambda+\mu}}^1(1-t)\,h(t)dt\right\}\right] \quad (2.10)$$

Proof: The proof is similar as proof of Theorem 7.

Remark 5: For $\lambda=\mu$ and convex function, the inequality (2.10) reduces to inequality (1.3).

Theorem 12: Let $|f'|^q$ be $h-$convex function and the assumptions of Lemma 1 hold, then

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{\mu a+\lambda b}{\lambda+\mu}\right)\right|=$$

$$\frac{b-a}{(p+1)^{1/p}}\left[\left(\frac{\mu}{\lambda+\mu}\right)^{1+\frac{1}{p}}\left(\int_0^{\frac{\lambda}{\lambda+\mu}}\left\{\begin{array}{l}|f'(a)|^q h(1-t)\\+|f'(b)|^q h(t)\end{array}\right\}dt\right)^{1/q}+\right.$$

$$\left.\left(\frac{\lambda}{\lambda+\mu}\right)^{1+\frac{1}{p}}\left(\int_{\frac{\lambda}{\lambda+\mu}}^1\left\{\begin{array}{l}|f'(a)|^q h(1-t)\\+|f'(b)|^q h(t)\end{array}\right\}dt\right)^{1/q}\right] \quad (2.11)$$

Proof: The proof is similar as proof of Theorem 8.

Remark 6: For $\lambda=\mu$ and convex function, the inequality (2.11) reduces to inequality (1.4).

Theorem 13: Let $|f'|^q$ be $h-$convex function and the assumptions of Lemma 1 hold, then

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{\mu a+\lambda b}{\lambda+\mu}\right)\right|=$$

$$\frac{b-a}{(\lambda+\mu)^{1/p}}\left[\left(\frac{\lambda^2}{2}\right)^{\frac{1}{p}}\left(\int_0^{\frac{\lambda}{\lambda+\mu}}t\left\{\begin{array}{l}|f'(a)|^q h(1-t)\\+|f'(b)|^q h(t)\end{array}\right\}dt\right)^{1/q}+\right.$$

$$\left.\left(\frac{\mu^2}{2}\right)^{\frac{1}{p}}\left(\int_{\frac{\lambda}{\lambda+\mu}}^1(1-t)\left\{\begin{array}{l}|f'(a)|^q h(1-t)\\+|f'(b)|^q h(t)\end{array}\right\}dt\right)^{1/q}\right] \quad (2.12)$$

Proof: The proof is similar as proof of Theorem 9.

Corollary 2: For $\lambda=\mu$, and convex function inequality (2.12) reduces as:

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|\le\frac{b-a}{8}$$

$$\left[\left(\frac{2|f'(a)|^q+|f'(b)|^q}{3}\right)^{1/q}+\left(\frac{|f'(a)|^q+2|f'(b)|^q}{3}\right)^{1/q}\right]$$

Theorem 14: Let $|f'|^q$ be h-concave function and the assumptions of Lemma 1 hold, then

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-f\left(\frac{\mu a+\lambda b}{\lambda+\mu}\right)\right|$$

$$\le\frac{b-a}{(1+p)^{1/p}}\left(\frac{1}{2h(1/2)}\right)^{1/q}$$

$$\left[\frac{\lambda^2}{(\lambda+\mu)^2}\left|f'\left(\frac{1}{\lambda}\left(\frac{\lambda+\mu}{2}a-\frac{\mu^2}{2(\lambda+\mu)}a+\frac{\lambda^2}{2(\lambda+\mu)}b\right)\right)\right|\right.$$

$$\left.+\frac{\mu^2}{(\lambda+\mu)^2}\left|f'\left(\frac{1}{\mu}\left(\frac{\lambda+\mu}{2}b+\frac{\mu^2}{2(\lambda+\mu)}a-\frac{\lambda^2}{2(\lambda+\mu)}b\right)\right)\right|\right] \quad (2.13)$$

Proof: The proof is similar as proof of Theorem 10.

Remark 7: For $\lambda=\mu$ and concave function, the inequality (2.13) reduces to inequality (1.6).

Remark 8: For $h(t)=\frac{1}{t},1$ and t^s at mid-point, the inequalities (2.4), (2.6), (2.7) and (2.8) provide new estimates of right-hand of Hermite-Hadamard inequality (1.1) for functions of $Q(I)$, $P(I)$ and K_s^2 and the inequalities (2.10), (2.11), (2.12) and (2.13) provide new estimates of left-hand Hermite-Hadamard inequality (1.1) for functions of $Q(I)$, $P(I)$ and K_s^2 respectively.

3. APPLICATIONS TO SPECIAL MEANS

Now, using the results of Section 2, we give some applications to special means of real numbers.

We shall consider the means for arbitrary two real numbers.

1. Arithmetic mean

$$A(a,b)=\frac{a+b}{2}; \qquad a,b\in R$$

2. Geometric mean

$$G(a,b)=\sqrt{ab}; \qquad a,b\in R_+$$

3. Logarithmic mean

$$L(a,b)=\frac{b-a}{\ln b-\ln a}\ ; \quad a,b\in R_+$$

Therefore, by applying the convex mapping $f(x)=e^x$, the following inequalities hold:

Proposition 1: Let $a, b \in I$, with $a<b$, we have

$$\left|A(a,b)-L(a,b)\right|\leq\frac{b-a}{2(p+1)^{1/p}} \times\frac{A\left(\left(\frac{3a^q+b^q}{4}\right)^{1/q},\left(\frac{a^q+3b^q}{4}\right)^{1/q}\right)}{L(a,b)}$$

Proof: The proof follows from Corollary 1.

Proposition 2: Let $a,b\in I$, with $a<b$, we have

$$\left|L(a,b)-G(a,b)\right|\leq\frac{b-a}{4} \times\frac{A\left(\left(\frac{2a^q+b^q}{3}\right)^{1/q},\left(\frac{a^q+2b^q}{3}\right)^{1/q}\right)}{L(a,b)}$$

Proof: The proof follows from Corollary 2.

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