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Original Article

Hall Effects on Rayleigh-Stokes Problem for Heated Second Grade Fluid

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Abstract: The aim of the present paper is to discuss the influence of Hall current on the flows of second grade fluid. Two illustrative examples have been considered (i) Stokes first problem for heated flat plate (ii) The Raleigh-Stokes problem for a heated edge. Expressions for velocity and temperature distributions are obtained. The results for hydrodynamic fluid can be obtained as the limiting cases.

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1. INTRODUCTION

Recently, the Rayleigh Stokes problem for a flat plate and an edge has acquired a special status. The solution of Stokes first problem for a Newtonian fluid is obtained employing similarity transformations in [1, 2]. But for the same problem in second grade fluid such similarity transformations are not useful [3]. In general, the governing equations of second grade fluid are one order higher than the Navier-Stokes equation and to obtain an analytic solution is not so easy. Also for a unique solution one needs an extra condition. For the detail of this issue, I may refer the readers to the references [4-7]. In study [8] Bandelli et al. discussed the Stokes first problem using Laplace transformation treatment. It is shown that the resulting solution does not satisfy the initial condition. Fetecau and Zierep [9] removed this difficulty by using Fourier Sine transform technique. Christov and Christov [10] have given comments on [9] by showing that solution of [9] is incorrect and have given the correct solution. Heat transfer analysis on the unidirectional flows of a second grade fluid is examined by Bandelli [11]. In continuation Fetecau and Fetecau [12,13] analyzed the temperature distribution in second grade and Maxwell fluids for laminar flow on a heated flat plate and in a heated edge. The purpose of the present investigation is to extend the analysis of reference [12] for Hall effects. The corresponding results of Newtonian fluid can be recovered by choosing $\alpha = 0$. In absence of Hall effects, the results can be obtained by letting $\boldsymbol{B}_0 = 0$.

2. BASIC EQUATIONS

For second grade fluid the Cauchy stress tensor $\boldsymbol{T}$ is

$$T = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2, \tag{1}$$

where p is the scalar pressure, $\boldsymbol{I}$ is the identity tensor, μ is the coefficient of viscosity, $\alpha_i \, (i=1,2)$ are the material parameters of second grade fluid and $\boldsymbol{A}_i \, (i=1,2)$ are the first two Rivlin-Ericksen tensors defined through

$$A_1 = (grad\, V) + (grad\, V)^T \tag{2}$$

$$A_2 = \frac{dA_1}{dt} + A_1 (grad V) + (grad V)^T A_1 \tag{3}$$

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in which t is the time. The issue regarding the signs of α_1 and α_2 is controversy. For detailed analysis relevant to this issue, one may refer the readers to the references [14, 15]. The equations governing the MHD flow of heated fluid are:

Continuity equation:

$$div V = 0. \tag{4}$$

Equation of motion:

$$\rho \frac{dV}{dt} = div T + (J \times B). \tag{5}$$

Energy equation:

$$\rho \frac{de}{dt} = T .(grad V) - div q + \rho r. \tag{6}$$

Maxwell equations:

$$div B = 0, \qquad Curl B = \mu_m J, \qquad Curl E = -\frac{\partial B}{\partial t}. \tag{7}$$

Generalized Ohm's law:

$$J + \frac{w_e \tau_e}{B_0}(J \times B) = \sigma (E + V \times B). \tag{8}$$

In above equations J is the current density, $B(=B_0 + b)$ is the total magnetic field, B_0 is the applied magnetic field, b is the induced magnetic field, σ is the electrical conductivity of the fluid, E is the electric field, μ_m is the magnetic permeability, $\frac{d}{dt}$ is the material derivative, $e(=c\theta)$ is the internal energy, ρ is the fluid density, c is the specific heat, θ is the temperature, $q(= -grad\theta)$ is the heat flux vector, r is the radial heating, w_e and τ_e are the cyclotron frequency and collision time of electron respectively. It is assumed that $E = 0$ and $b = 0$. Further $w_e \tau_e \approx O(1)$ and $w_i \tau_i << 1$ (where w_i and τ_i are cyclotron frequency and collision time for ions respectively). Under the aforementioned assumptions, Eq (5) becomes

$$\frac{dV}{dt} = \frac{1}{\rho} div T - \frac{\sigma B_0^2 (1 + i\phi) V}{\rho (1 + \phi^2)}, \tag{9}$$

where $\phi = w_e \tau_e$ is the Hall parameter.

3. THE FIRST PROBLEM OF STOKES FOR A HEATED FLAT PLATE WITH HALL CURRENT

Let a second grade fluid, at rest, fill the space above an infinitely extended plate in (y, z)-plane. When time $t = 0^+$, the plate starts suddenly to slide, in its own plane, with velocity V. Let T(t) and f(x) denote the temperature of the plate for $t \geq 0$ and the temperature of the fluid at the moment $t = 0$. The velocity and temperature fields are

$$V = v(x,t)\hat{j}, \qquad \theta = \theta(x,t). \tag{10}$$

where $\hat{j}$ is a unit vector in the y-direction. The continuity equation (4) is identically satisfied. Furthermore, Equations (6) and (9) give

$$(\upsilon + \alpha \frac{\partial}{\partial t}) \frac{\partial^2 v(x,t)}{\partial x^2} - \frac{\sigma B_0^2 (1 + i\phi) v(x,t)}{\rho (1 + \phi^2)} = \frac{\partial v(x,t)}{\partial t}, \qquad x > 0, \quad t > 0, \tag{11}$$

$$\beta \frac{\partial^2 \theta(x,t)}{\partial x^2} + g(x,t) = \frac{\partial \theta(x,t)}{\partial t}, \qquad x > 0, \quad t > 0, \tag{12}$$

where $\upsilon = \mu / \rho$ is the kinematic viscosity,

$$\alpha = \frac{\alpha_1}{\rho}, \quad \beta = \frac{k}{\rho c}$$

$$g(x,t) = (\frac{\upsilon}{c}) \left[\frac{\partial v(x,t)}{\partial x} \right]^2 + \frac{r(x,t)}{\rho c}.$$

The relevant initial and boundary conditions are

$$v(x,0) = 0, \quad x > 0, \quad v(0,t) = V(t), \quad t > 0, \tag{13}$$

$$\theta(x,0) = f(x), \quad x > 0; \quad \theta(0,t) = T(t), \quad t \geq 0, \tag{14}$$

$$v(x,t), \quad \frac{\partial v(x,t)}{\partial x}, \quad \theta(x,t), \quad \frac{\partial \theta(x,t)}{\partial x} \to 0 \quad as \quad x \to \infty. \tag{15}$$

By Fourier Sine transform, the solution for v is

$$v(x,t)=\frac{2}{\pi}\int_0^\infty \frac{\upsilon\xi(1+\phi^2)}{\left\{\sigma B_0^2(1+i\phi)+\upsilon\xi^2(1+\phi^2)\right\}}$$

$$\left[V(t)-V(0)\exp\left\{-\frac{\left\{\sigma B_0^2(1+i\phi)+\upsilon\xi^2(1+\phi^2)\right\}}{(1+\phi^2)(1+\alpha\xi^2)}t\right\}\right]\sin\xi x d\xi$$

$$-\frac{2}{\pi}\int_0^\infty \left[\begin{array}{l}\dfrac{\xi\left\{\upsilon(1+\phi^2)-\alpha\sigma B_0^2(1+i\phi)\right\}}{(1+\alpha\xi^2)\left\{\sigma B_0^2(1+i\phi)+\upsilon\xi^2(1+\phi^2)\right\}}\\ \displaystyle\int_0^t V'(\tau)\exp\left\{-\frac{\left\{\sigma B_0^2(1+i\phi)+\upsilon\xi^2(1+\phi^2)\right\}}{(1+\phi^2)(1+\alpha\xi^2)}(t-\tau)\right\}d\tau\end{array}\right]$$

$$\sin\xi x d\xi. \tag{16}$$

If the plate moves with constant velocity *V*, then $V'(\tau)=0$ and the above equation simplifies to:

$$v(x,t)=\frac{2}{\pi}\int_0^\infty \frac{\upsilon\xi(1+\phi^2)}{\left\{\sigma B_0^2(1+i\phi)+\upsilon\xi^2(1+\phi^2)\right\}}$$

$$\left[V(t)-V(0)\exp\left\{-\frac{\left\{\sigma B_0^2(1+i\phi)+\upsilon\xi^2(1+\phi^2)\right\}}{(1+\phi^2)(1+\alpha\xi^2)}t\right\}\right]\sin\xi x d\xi. \tag{17}$$

For B_0 = 0, we get the results of reference [12] as

$$v(x,t)=V\left[1-\frac{2}{\pi}\int_0^\infty \exp\left\{\frac{-\upsilon\xi^2}{1+\alpha\xi^2}t\right\}\frac{\sin\xi x}{\xi}d\xi\right]. \tag{18}$$

When $\alpha\to 0$ the above equation yields

$$v(x,t)=V\left[1-Erf\left(\frac{x}{2\sqrt{\upsilon t}}\right)\right], \tag{19}$$

where Erf (x) is the error function of Guass. Employing the same methodology as for v we obtain

$$\theta(x,t)=T\left[1-Erf\left(\frac{x}{2\sqrt{\beta t}}\right)\right]$$

$$+\int_0^t T'(\tau)\left[1-Erf\left(\frac{x}{2\sqrt{\beta(t-\tau)}}\right)\right]d\tau+$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \exp(-\beta\xi^2 t)\sin(x\xi)$$

$$\left[f_s(\xi)+\int_0^t g_s(\xi,\tau)\exp(\beta\xi^2\tau)d\tau\right]d\xi, \tag{20}$$

$$T=\lim_{t\to 0}T(t)$$

At rest, the temperature distribution is the same in presence of Hall currents as for a second grade fluid and for a Newtonian one. Further if the radiant heating $r(x,t)$ is negligible quantity the relation (20) takes the same form as in [12], i.e.,

$$\theta(x,t)=T\left[1-Erf\left(\frac{x}{2\sqrt{\beta t}}\right)\right]$$

$$+\int_0^t T'(\tau)\left[1-Erf\left(\frac{x}{2\sqrt{\beta(t-\tau)}}\right)\right]d\tau+$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \sin(x\xi)f_s(\xi)\exp(-\beta\xi^2 t)d\xi. \tag{21}$$

From the above results we see that, if $t\to\infty$, then $\theta(x,t)\to T(\infty)$.

Moreover, if the initial temperature of the fluid is zero and the plate is kept to the constant temperature T, Eq. (21) gives

$$\theta(x,t)=T\left[1-Erf\left(\frac{x}{2\sqrt{\beta t}}\right)\right], \tag{22}$$

and $\theta(x,t)\to T \quad as \quad t\to\infty.$

4. THE RAYLEIGH-STOKES PROBLEM FOR HEATED EDGE WITH HALL CURRENT

Consider a second grade fluid at rest occupying the space of the first dial of rectangular edge (x $\geq$ 0, $-\infty<y<\infty,\ z\geq 0)$. For t = 0^+, the extended edge is impulsively brought to the constant speed V. The walls of the edge have temperature $T(t)$. The velocity and temperature fields are

$$\boldsymbol{V}=v(x,z,t)\hat{\boldsymbol{j}}, \tag{23}$$

$$\theta=\theta(x,z,t).$$

Equations (6) and (9) here are of the following forms:

$$\left(\nu+\alpha\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)v(x,z,t)$$

$$-\frac{\sigma B_0^2(1+i\phi)}{\rho(1+\phi^2)}v(x,z,t)=\frac{\partial}{\partial t}v(x,z,t), \tag{24}$$

$$x>0,\quad z>0,\quad t>0,$$

$$\beta\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\theta(x,z,t)\right] \tag{25}$$
$$+g(x,z,t)=\frac{\partial}{\partial t}\theta(x,z,t),$$
$$x>0,\quad z>0,\quad t>0,$$

where

$$g(x,t)=\left(\frac{v}{c}\right)\left\{\begin{aligned}&\left[\frac{\partial v(x,z,t)}{\partial x}\right]^2\\ &+\left[\frac{\partial v(x,z,t)}{\partial z}\right]^2\end{aligned}\right\}+\frac{r(x,z,t)}{\rho c}.$$

The corresponding initial and boundary conditions are

$$v(x,z,0)=0,\quad x>0,\quad z>0,$$

$$v(0,z,t)=v(x,0,t)=V,\qquad t>0, \tag{26}$$

$$\theta(x,z,0)=f(x,z),\quad x>0;\quad z>0;$$

$$\theta(0,z,t)=\theta(x,0,t)=T(t),\quad t\geq 0, \tag{27}$$

where the function $f(x,z)$ represents the temperature distribution of the fluid at the moment t = 0. Moreover, $v(x,z,t)$, $\theta(x,z,t)$ and their partial derivatives with respect to x and z have to tend to zero as $x^2+z^2\to\infty$.

Following the same method of solution as in section 3, we obtain

$$v(x,z,t)=\frac{4}{\pi^2}\int_0^\infty\int_0^\infty\frac{\upsilon(\xi^2+\eta^2)(1+\phi^2)}{\eta\xi\left\{\sigma B_0^2(1+i\phi)+\upsilon(\xi^2+\eta^2)(1+\varphi^2)\right\}}\times$$

$$\left[V(t)-V(0)\exp\left[-\left\{\frac{\left\{\sigma B_0^2(1+i\phi)+\upsilon(\xi^2+\eta^2)(1+\varphi^2)\right\}}{(1+\phi^2)\left\{1+\alpha(\xi^2+\eta^2)\right\}}t\right\}\right]\right]\times$$

$$\sin\xi x\sin\eta z\,d\xi\,d\eta+\frac{4}{\pi^2}\int_0^\infty\int_0^\infty\frac{(\xi^2+\eta^2)}{\eta\xi\left\{1+\alpha(\xi^2+\eta^2)\right\}}\times$$

$$\left[\begin{aligned}&\frac{\alpha\sigma B_0^2(1+i\phi)-\mu(1+\varphi^2)}{\sigma B_0^2(1+i\phi)+\mu(\xi^2+\eta^2)(1+\varphi^2)}\\ &\int_0^t V'(\tau)\exp\left[-\frac{\left\{\sigma B_0^2(1+i\phi)+\upsilon(\xi^2+\eta^2)(1+\varphi^2)\right\}}{(1+\phi^2)\left\{1+\alpha(\xi^2+\eta^2)\right\}}(t-\tau)\right]d\tau\end{aligned}\right]\times$$

$$\sin\xi x\sin\eta z\,d\xi\,d\eta. \tag{28}$$

When plate has constant velocity V, then $V'(\tau)=0$ and the above equation

reduces to

$$v(x,z,t)=\frac{4}{\pi^2}$$

$$\int_0^\infty\int_0^\infty\frac{\mu(\xi^2+\eta^2)(1+\phi^2)}{\eta\xi\left\{\sigma B_0^2(1+i\phi)+\mu(\xi^2+\eta^2)(1+\varphi^2)\right\}}\times$$

$$\left[V(t)-V\exp\left[-\left\{\frac{\left\{\sigma B_0^2(1+i\phi)+\mu(\xi^2+\eta^2)(1+\varphi^2)\right\}}{\rho(1+\phi^2)\left\{1+\alpha(\xi^2+\eta^2)\right\}}t\right\}\right]\right]\times$$

$$\sin\xi x\sin\eta z\,d\xi\,d\eta. \tag{29}$$

For B_0 = 0 above equation reduces to the result of [12].

$$v(x,z,t)=V\left[1-\frac{4}{\pi^2}\int_0^\infty\int_0^\infty\frac{\sin\xi x}{\xi}\times\frac{\sin\eta z}{\eta}\times\exp\left[-\frac{\nu(\xi^2+\eta^2)}{1+\alpha(\xi^2+\eta^2)}d\xi\,d\eta\right]\right]. \tag{30}$$

The expression of temperature is

$$\theta(x,z,t)=T\left[1-Erf\left(\frac{x}{2\sqrt{\beta t}}\right)Erf\left(\frac{z}{2\sqrt{\beta t}}\right)\right]$$
$$+\int_0^t T'(\tau)\left[1-Erf\left(\frac{x}{2\sqrt{\beta(t-\tau)}}\right)Erf\left(\frac{z}{2\sqrt{\beta(t-\tau)}}\right)\right]d\tau$$

$$+\frac{2}{\pi}\int_0^\infty\int_0^\infty\exp\left\{-\beta(\xi^2+\eta^2)\right\}\sin(x\xi)\sin(z\eta)$$
$$\left[f_s(\xi,\eta)+\int_0^t g_s(\xi,\eta,\tau)\times e\,\mathrm{xp}\left\{\beta(\xi^2+\eta^2)\tau\right\}d\tau\right]d\xi\,d\eta. \tag{31}$$

in which $f_s(\xi,\eta)$ *and* $g_s(\xi,\eta,t)$ are the double Fourier sine transforms of the functions

$f(x,z)$ *and* $g(x,z,t)$ with respect to the variables x and z. When $\alpha\to 0$, relations (30) and (31) reduce again to those resulting from the Navier-Stokes fluids. Thus, we recover the universal profile of velocity [2].

$$v(x,z,t)=V\left[1-Erf\left(\frac{x}{2\sqrt{\nu t}}\right)Erf\left(\frac{z}{2\sqrt{\nu t}}\right)\right], \tag{32}$$

in which only similarity variables $x/\sqrt{\nu t}$ *and* $z/\sqrt{\nu t}$ occur. For $z \to \infty$, $v(x,z,t)$ goes to $v(x,t)$ given by (19). The expression for $\theta(x,z,t)$ is

$$\theta(x,z,t) = T\left[1 - Erf\left(\frac{x}{2\sqrt{\beta t}}\right)Erf\left(\frac{z}{2\sqrt{\beta t}}\right)\right] + \int_0^t T'(\tau)\left[1 - Erf\left(\frac{x}{2\sqrt{\beta(t-\tau)}}\right)Erf\left(\frac{z}{2\sqrt{\beta(t-\tau)}}\right)\right]d\tau, \quad (33)$$

which for $z \to \infty$ goes to

$$\theta(x,z,t) = T\left[1 - Erf\left(\frac{x}{2\sqrt{\beta t}}\right)\right] + \int_0^t T'(\tau)\left[1 - Erf\left(\frac{x}{2\sqrt{\beta(t-\tau)}}\right)\right]d\tau. \quad (34)$$

If the edge is maintained to the constant temperature T, Eq. (33) takes the form

$$\theta(x,z,t) = T\left[1 - Erf\left(\frac{x}{2\sqrt{\beta t}}\right)Erf\left(\frac{z}{2\sqrt{\beta t}}\right)\right], \quad (35)$$

and $\theta(x,z,t) \to T$ *as* $t \to \infty$.

5. CONCLUSIONS

In this paper, the exact solutions for laminar flow of an electrically conducting non-Newtonian fluid are obtained. The velocity field and the temperature distribution in a second grade fluid on heated flat plate and on a heated edge in the presence of Hall current are determined. These solutions are obtained using simple and double Fourier sine transforms and presented as a sum of steady state and transient solutions. For large values of time, when transient disappear, these solutions reduce to steady-state solutions. Direct calculations show that $\theta(x,t)$ and $\theta(x,z,t)$ of Eqs. (20) and (31) as well $v(x,t)$ and $v(x,z,t)$ of Eqs. (16) and (30) satisfy the corresponding partial differential equations together with the initial and boundary conditions.

Putting B_0 = 0 in Eqs. (17) and (29), we obtain the results of [12]. If we put $\alpha \to 0$ in (17), (20), (30) and (31), we find corresponding solutions for the Navier-Stokes fluid. In a fluid at rest the temperature distribution is the same whether it is second grade or not. If the radiant heating is negligible $\theta(x,t)$ and $\theta(x,z,t)$ become $T(\infty)$ *as* $t \to \infty$ or T if the plate and the edge are maintained to the constant temperature. The corresponding results in absence of Hall current can be obtained by choosing B_0 = 0.

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