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Original Article

Stability for Univalent Solutions of Complex Fractional Differential Equations

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Abstract: In this paper, we consider the Hyers-Ulam stability for the following fractional differential equations in sense of Srivastava-Owa fractional operators (derivative and integral) defined in the unit disk:

$$D_z^{\beta} f(z) = G(f(z), D_z^{\alpha} f(z), z), \quad 0 < \alpha < 1 < \beta \leq 2,$$

in a complex Banach space. Furthermore, a generalization of the admissible functions in complex Banach spaces is imposed and applications are illustrated.

Keywords: Analytic function; unit disk; Hyers-Ulam stability; admissible functions; fractional calculus; fractional differential equation; univalent function; convex function.

1. INTRODUCTION

A classical problem in the theory of functional equations is that: If a function f approximately satisfies functional equation E when does there exists an exact solution of E which f approximates. In 1940, S. M. Ulam [1] imposed the question of the stability of Cauchy equation and in 1941, D. H. Hyers solved it [2]. In 1978, Th. M. Rassias [3] provided a generalization of Hyers theorem by proving the existence of unique linear mappings near approximate additive mappings. The problem has been considered for many different types of spaces (see [4-6]). Recently, Li and Hua [7] discussed and proved the Hyers-Ulam stability of spacial type of finite polynomial equation, and Bidkham, Mezerji and Gordji [8] introduced the Hyers-Ulam stability of generalized finite polynomial equation. Finally, M.J. Rassias [9] imposed a Cauchy type additive functional equation and investigated the generalised Hyers-Ulam "product-sum" stability of this equation.

Fractional calculus can be considered as a generalization of classical calculus. Fractional differential equations have many applications in various area not only in mathematics but also in physics, computer sciences, mechanics and others. Stability analysis of the solution for these equations is a main central task in the study of fractional analysis. The authors and researchers investigated the stability for different kind of fractional derivatives such as Caputo derivatives, Miller-Ross sequential derivative and Riemann-Liouville derivative.

In this note, we shall study the stability of complex differential equation in sense of the Srivastava-Owa fractional operators (derivative and integral). The solutions are univalent in the unit disk. Recently, the author suggested and introduced the generalized Ulam stability for various types of fractional differential equations in the complex domain [10-14]. Furthermore, Ulam stability for fractional differential equations can be found in [15-17].

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2. METHODS

Let $U := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $\mathbb{H}$ denote the space of all analytic functions on U. Here we suppose that $\mathbb{H}$ as a topological vector space endowed with the topology of uniform convergence over compact subsets of U. Also for $a \in \mathbb{C}$ and $m \in \mathbb{N}$, let $\mathbb{H}[a,m]$ be the subspace of $\mathbb{H}$ consisting of functions of the form

$$f(z) = a + a_m z^m + a_{m+1} z^{m+1} + \dots, \quad z \in U.$$

Let $\mathcal{A}$ be the class of functions f, analytic in U and normalized by the conditions $f(0) = f'(0) - 1 = 0$ A function $f \in \mathcal{A}$ is called univalent $(\mathcal{S})$ if it is one-one in U. A function $f \in \mathcal{A}$ is called convex if it satisfies the following inequality

$$\Re\{\frac{zf''(z)}{f'(z)} + 1\} > 0, (z \in U).$$

We denoted this class $\mathcal{C}$.

In [18], Srivastava and Owa, posed definitions for fractional operators (derivative and integral) in the complex z-plane $\mathbb{C}$ as follows:

Definition 2.1. The fractional derivative of order α is defined, for a function $f(z)$ by

$$D_z^\alpha f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\alpha} d\zeta,$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $log(z-\zeta)$ to be real when $(z-\zeta) > 0$

Definition 2.2 The fractional integral of order $\alpha > 0$ is defined, for a function $f(z)$ by

$$I_z^\alpha f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} d\zeta ; \alpha > 0,$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane $(\mathbb{C})$ containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $log(z-\zeta)$ to be real when $(z-\zeta) > 0$

Remark 2.1

$$D_z^\alpha z^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} z^{\mu-\alpha}, \mu > -1$$

and

$$I_z^\alpha z^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} z^{\mu+\alpha}, \mu > -1.$$

More details on fractional derivatives and their properties and applications can be found in [19,20].

We next introduce the generalized Hyers-Ulam stability depending on the properties of the fractional operators.

Definition 2.3 Let p be a real number. We say that

$$\sum_{n=0}^{\infty} a_n z^{n+\alpha} = f(z) \tag{1}$$

has the generalized Hyers-Ulam stability if there exists a constant $K > 0$ with the following property:

for every $\varepsilon > 0, w \in \overline{U} = U \bigcup \partial U$, if

$$|\sum_{n=0}^{\infty} a_n w^{n+\alpha}| \le \varepsilon(\sum_{n=0}^{\infty} \frac{|a_n|^p}{2^n})$$

then there exists some $z \in \overline{U}$ that satisfies equation (1) such that

$$|z^i - w^i| \le \varepsilon K,$$

$$(z, w \in \overline{U}, \quad i \in \mathbb{N})$$

In the present paper, we study the generalized Hyers-Ulam stability for holomorphic solutions of the fractional differential equation in complex Banach spaces X and Y

$$D_z^\beta f(z) = G(f(z), D_z^\alpha f(z); z), \quad 0 < \alpha < 1 < \beta \le 2, \tag{2}$$

where $G: X^2 \times U \to Y$ and $f: U \to X$ are holomorphic functions such that $f(0) = \Theta$ (Θ is the zero vector in X).

3. RESULTS

In this section we present extensions of the generalized Hyers-Ulam stability to holomorphic vector-valued functions. Let X, Y represent complex Banach space. The class of admissible functions $\mathsf{G}(X, Y)$, consists of those functions $g: X^2 \times U \to Y$ that satisfy the admissibility conditions:

$$\| g(r, ks; z) \| \geq 1, when \quad \| r \| = 1, \| s \| = 1, \tag{3}$$

$(\quad z \in U, k \geq 1)$.

We need the following results:

Lemma 3.1. [21] If $f: D \to X$ is holomorphic, then $\| f \|$ is a subharmonic of $z \in D \subset \mathsf{C}$. It follows that $\| f \|$ can have no maximum in D unless $\| f \|$ is of constant value throughout D.

Lemma 3.2. [22, p. 88] If the function $f(z)$ is in the class S, then

$$| D_z^{\alpha+n} f(z) | \leq \frac{(n+\alpha+|z|)\Gamma(n+\alpha+1)}{(1-|z|)^{n+\alpha+2}},$$

$(z \in U; n \in \mathsf{N}_0 = \mathsf{N} \cup \{0\}; 0 \leq \alpha < 1)$.

Lemma 3.3. [18, p. 225] If the function $f(z)$ is in the class C, then

$$| D_z^{\alpha+n} f(z) | \leq \frac{\Gamma(n+\alpha+1)}{(1-|z|)^{n+\alpha+1}},$$

$(z \in U; n \in \mathsf{N}_0; 0 \leq \alpha < 1)$.

Theorem 3.1. Let $G \in \mathsf{G}(X, Y)$ and $f: U \to X$ be a holomorphic vector-valued function defined in the unit disk U, with $f(0) = \Theta$. If $f \in \mathsf{S}$, then

$$\| G(f(z), D_z^{\alpha} f(z); z) \| < 1 \Rightarrow \| f(z) \| < 1. \tag{4}$$

Proof. Since $f \in \mathsf{S}$, then from Lemma 3.2, we observe that

$$| D_z^{\alpha} f(z) | \leq \frac{(\alpha+|z|)\Gamma(\alpha+1)}{(1-|z|)^{\alpha+2}}.$$

Assume that f dose not satisfie $\| f(z) \| < 1$ for $z \in U$. Thus, there exists a point $z_0 \in U$ for which $\| f(z_0) \| = 1$ According to Lemma 3.1, we have

$$\max_{|z| \leq |z_0|} \| f(z) \| = \| f(z_0) \| = 1.$$

and

$$\max_{|z| \leq |z_0|} \| f(z) \| = \| f(z_0) \| = 1.$$

Consequently, we obtain

$$\| f(z_0) \| = \frac{(1-|z|)^{\alpha+2}}{(\alpha+|z|)\Gamma(\alpha+1)} \| D_{z_0}^{\alpha} f(z_0) \| = 1.$$

We put $k := \frac{(\alpha+|z|)\Gamma(\alpha+1)}{(1-|z|)^{\alpha+2}} \geq 1$, for some $0 < \alpha < 1$ and $z \in U$; hence from equation (3), we deduce

$$\| G(f(z_0), D_{z_0}^{\alpha} f(z_0); z_0) \| = \| G(f(z_0), k[D_{z_0}^{\alpha} f(z_0)/k]; z_0) \| \geq 1,$$

which contradicts the hypothesis in (4), we must have $\| f \| < 1$.

Corollary 3.1. Assume the problem (2). If $G \in \mathsf{G}(X, Y)$ is a holomorphic univalent vector-valued function defined in the unit disk U then

$$\| G(f(z), D_z^{\alpha} f(z); z) \| < 1 \Rightarrow \| I_z^{\beta} G(f(z), D_z^{\alpha} f(z); z) \| < 1. \tag{5}$$

Proof. By univalency of G, the fractional differential equation (2) has at least one holomorphic univalent solution f. Thus according to Remark 1.1, the solution $f(z)$ of the problem (2) takes the form

$$f(z) = I_z^{\beta} G(f(z), D_z^{\alpha} f(z); z).$$

Therefore, in virtue of Theorem 3.1, we obtain the assertion (5).

Theorem 3.2. Let $G \in \mathsf{G}(X, Y)$ be holomorphic univalent vector-valued functions defined in the unit disk U then the equation (2) has the generalized Hyers-Ulam stability for $z \to \partial U$.

Proof. Assume that

$$G(z) := \sum_{n=0}^{\infty} \varphi_n z^n, \quad z \in U$$

therefore, by Remark 1.2, we have

$$G(z) := \sum_{n=0}^{\infty} \varphi_n z^n, \quad z \in U$$

Also, $z \to \partial U$ and thus $|z| \to 1$ According to Theorem 3.1, we have

$$\| f(z) \| < 1 = |z|.$$

Let $\varepsilon > 0$ and $w \in \overline{U}$ be such that

$$\left| \sum_{n=1}^{\infty} a_n w^{n+\beta} \right| \le \varepsilon \left(\sum_{n=1}^{\infty} \frac{|a_n|^p}{2^n} \right).$$

We will show that there exists a constant K independent of ε such that

$$|w^i - u^i| \le \varepsilon K, \quad w \in \overline{U}, u \in U$$

and satisfies (1). We put the function

$$f(w) = \frac{-1}{\lambda a_i} \sum_{n=1, n \ne i}^{\infty} a_n w^{n+\beta}, \qquad a_i \ne 0, 0 < \lambda < 1, \tag{6}$$

thus, for $w \in \partial U$, we obtain

$$\begin{aligned} |w^i - u^i| &= |w^i - \lambda f(w) + \lambda f(w) - u^i| \\ &\le |w^i - \lambda f(w)| + \lambda |f(w) - u^i| \\ &< |w^i - \lambda f(w)| + \lambda |w^i - u^i| \\ &= \left| w^i + \frac{1}{a_i} \sum_{n=1, n\ne i}^{\infty} a_n w^{n+\beta} \right| + \lambda |w^i - u^i| \\ &= \frac{1}{|a_i|} \left| \sum_{n=1}^{\infty} a_n w^{n+\beta} \right| + \lambda |w^i - u^i|. \end{aligned}$$

Without loss of generality, we consider $|a_i| = \max_{n \ge 1}(|a_n|)$ yielding

$$\begin{aligned} |w^i - u^i| &\le \frac{1}{|a_i|(1-\lambda)} \left| \sum_{n=1}^{\infty} a_n w^{n+\beta} \right| \\ &\le \frac{\varepsilon}{|a_i|(1-\lambda)} \left(\sum_{n=0}^{\infty} \frac{|a_n|^p}{2^n} \right) \\ &\le \frac{\varepsilon |a_i|^{p-1}}{(1-\lambda)} \left(\sum_{n=0}^{\infty} \frac{1}{2^n} \right) \\ &\le \frac{2\varepsilon |a_i|^{p-1}}{(1-\lambda)} \\ &:= K\varepsilon. \end{aligned}$$

This completes the proof.

In the same manner of Theorem 3.1, and by using Lemma 3.3, we have the following result:

Theorem 3.3. Let $G \in \mathsf{G}(X, Y)$ and $f : U \to X$ be a holomorphic vector-valued function defined in the unit disk U, with $f(0) = \Theta$. If $f \in \mathsf{C}$, then

$$\| G(f(z), D_z^{\alpha} f(z); z) \| < 1 \Rightarrow \| f(z) \| < 1. \tag{7}$$

4. APPLICATIONS

In this section, we introduce some applications of functions to achieve the generalized Hyers-Ulam stability.

Example 4.1. Consider the function

$$G : X^2 \times U \to \mathsf{R}$$

by

$$G(r, s; z) = a(\| r \| + \| s \|)^n + b|z|^2, \quad n \in \mathsf{R}_+$$

with $a \ge 0.5$, $b \ge 0$ and $G(\Theta, \Theta, 0) = 0$ Our aim is to employ Theorem 3.1, this holds because

$$\begin{aligned} \| G(r, ks; z) \| &= a(\| r \| + k\| s \|)^n + b|z|^2 \\ &= a(1+k)^n + b|z|^2 \ge 1, \end{aligned}$$

when $\| r \| = \| s \| = 1, z \in U$. Thus by Theorem 3.1, yields : If $a \ge 0.5$, $b \ge 0$ and $f : U \to X$ is a holomorphic univalent vector-valued function defined in U, with $f(0) = \Theta$, then

$$\begin{aligned} & a(\| f(z) \| + \| D_z^{\alpha} f(z) \|)^n + b|z|^2 < 1 \\ & \Rightarrow \| f(z) \| < 1. \end{aligned}$$

Consequently, $\| I_z^{\alpha} G(f(z), D_z^{\alpha} f(z); z) \| < 1$, thus in view of Theorem 3.2, f has the generalized Hyers-Ulam stability.

Example 4.2. Assume the function $G : X^2 \to X$ by

$$G(r, s; z) = G(r, s) = r e^{\| s \|^m - 1}, \quad m \ge 1$$

with $G(\Theta, \Theta) = \Theta$. By applying Theorem 3.1, we need to show that $G \in \mathsf{G}(X, X)$. Since

$$\| G(r, ks) \| = \| r e^{\| ks \|^m - 1} \| = e^{k^m - 1} \ge 1,$$

when $\| r \| = \| s \| = 1, z \in U$. Therefore by Theorem 3.1 implies : For $f : U \to X$ is a holomorphic univalent vector-valued function defined in U, with $f(0) = \Theta$, then,

$$\| f(z)e^{\| D_z^{\alpha} f(z) \|^{m}-1} \| < 1 \Rightarrow$$
$$\| f(z) \| < 1$$

Consequently, $\| I_z^{\alpha} G(f(z), D_z^{\alpha} f(z); z) \| < 1$; thus in view of Theorem 3.2, f has the generalized Hyers-Ulam stability.

Example 4.3. Let $a, b : U \to \mathbb{C}$ satisfy

$$| a(z) + \mu b(z) | \geq 1,$$

for every $\mu \geq 1$, $\nu > 1$ and $z \in U$. Consider the function $G : X^2 \to Y$ by

$$G(r, s; z) = a(z)r + \mu b(z)s,$$

with $G(\Theta, \Theta) = \Theta$. Now for $\| r \| = \| s \| = 1$, we have

$$\| G(r, \mu s; z) \| = | a(z) + \mu b(z) | \geq 1$$

and thus $G \in \mathcal{G}(X, Y)$. If $f : U \to X$ is a holomorphic univalent vector-valued function defined in U, with $f(0) = \Theta$, then

$$\| a(z) f(z) + b(z) D_z^{\alpha} f(z) \| < 1$$
$$\Rightarrow \| f(z) \| < 1.$$

According to Theorem 3.2, f has the generalized Hyers-Ulam stability.

Example 4.4. Let $\lambda : U \to \mathbb{C}$ be a function such that

$$\Re\left(\frac{1}{\lambda(z)}\right) > 0,$$

for every $z \in U$. Consider the function $G : X^2 \to Y$ by

$$G(r, s; z) = r + \frac{s}{\lambda(z)},$$

with $G(\Theta, \Theta) = \Theta$. Now for $\| r \| = \| s \| =$ we have

$$\| G(r, ks; z) \| = | 1 + \frac{k}{\lambda(z)} | \geq 1, \quad k \geq 1$$

and thus $G \in \mathcal{G}(X, Y)$. If $f : U \to X$ is a holomorphic univalent vector-valued function defined in U, with $f(0) = \Theta$, then

$$\| f(z) + \frac{D_z^{\alpha} f(z)}{\lambda(z)} \| < 1 \Rightarrow \| f(z) \| < 1.$$

Hence in view of Theorem 3.2, f has the generalized Hyers-Ulam stability.

5. CONCLUSIONS

Ulam stability of fractional differential equation is defined and studied. The applications are imposed by employing the concept of addimisible functions in the unit disk. This class of functions is generalized to include the fractional differential operator in sense of the Srivastaava-Owa operators.

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