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Original Article

Exact and Numerical Solution for Fractional Differential Equation based on Neural Network

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Abstract: In this paper a fractional differential equations based on Riemann-Liouville fractional derivatives are solved exactly. The solution is obtained in terms of H-functions and it is finite for all times. Moreover, by using the neural network method, we have estimated the numerical solution for some special equations.

Keywords: Fractional calculus, Riemann-Liouville fractional operators, exact solution, fractional differential equations, neural network, hypergeometric function

1. INTRODUCTION

The class of fractional differential equations of various types plays important roles and tools not only in mathematics but also in physics, control systems, dynamical systems and engineering to create the mathematical modeling of many physical phenomena. Naturally, such equations required to be solved. Many studies on fractional calculus and fractional differential equations have appeared, involving different operators such as Riemann-Liouville operators, Erdélyi-Kober operators, Weyl-Riesz operators, Caputo operators and Grünwald-Letnikov operators. The existence of positive solution and multi-positive solutions for nonlinear fractional differential equation are established and studied [1-4]. Moreover, by using the concepts of the subordination and superordination of analytic functions, the existence of analytic solutions for fractional differential equations in complex domain are suggested and posed in [5-8]. In addition, a generalization of fractional operators in the unit disk is imposed in [9]. One of the most frequently used tools in the theory of fractional calculus is furnished by the Riemann-Liouville operators [10]. The Riemann-Liouville fractional derivative could hardly pose the physical interpretation of the initial conditions required for the initial value problems involving fractional differential equations. Moreover, this operator possesses advantages of fast convergence, higher stability and higher accuracy to derive different types of numerical algorithms [11].

Our aim is to find the exact solution for different kind of fractional differential equations in sense of Riemann-Liouville fractional derivative, in terms of H-functions. Numerical solution for some equations are introduced by using the neural network.

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2. PRELIMINARIES

Definition 2.1. The fractional (arbitrary) order integral of the function f of order $\alpha > 0$ is defined by

$$I_a^\alpha f(t) = \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)d\tau.$$

When $a = 0$, we write $I_a^\alpha f(t) = f(t) * \phi_\alpha(t)$, where (*) denoted the convolution product

$\phi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, t > 0$ and $\phi_\alpha(t) = 0, t \le 0$ and $\phi_\alpha \to \delta(t)$ as $\alpha \to 0$ where $\delta(t)$ is the delta function.

Definition 2.2. The fractional (arbitrary) order derivative of the function f of order $0 \le \alpha < 1$ is defined by

Remark 2.1. From Definition 2.1 and Definition 2.2, we have

$$D^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}, \mu > -1; 0 < \alpha < 1$$

and

$$I^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{\mu+\alpha}, \mu > -1; \alpha > 0.$$

The goal of this work is to find the exact solution for different kind of fractional differential equations, in terms of H-functions. We consider the non-linear fractional differential equation

$$D_0^\alpha u(t) = f(t, u(t)), \tag{1}$$

where $0 < \alpha < 1$, subject to the initial values

$$[D_0^{\alpha-1}u(t)]_{t=0} = [D_0^{-\beta}u(t)]_{t=0} = [I_0^\beta u(t)]_{t=0} = 0, \ \beta = 1-\alpha. \tag{2}$$

Where $f(t, u(t) : [0, T] \times \mathrm{R} \to \mathrm{R}$ is a continuous function. Also the exact solution for the linear case

$$D^\alpha u(t) - \lambda u(t) = f(t).$$

Furthermore, The multi-term fractional differential equation

$$D^\alpha u(t) = f(t, u, D^{\alpha_1}u(t), ..., D^{\alpha_n}u(t)) \tag{3}$$

subject to the initial condition (2) is solved.

By using Laplace technique where $f : [0, T] \times \mathrm{R}^{n+1} \to \mathrm{R}$ is a continuous function and $0 < \alpha_i < \alpha < 1$ for all $i = 1, ..., n$. Also, the non-constant coefficients fractional differential equation

$$D^\alpha u(t) - \sum_{i=1}^n g_i(t, u) D^{\alpha_i} u(t) = f(t, u), \tag{4}$$

subject to the initial condition (2) is solved exactly in terms of H-functions. Finally, we find the exact solution for the mixed equation

$$D^\alpha f(x, t) = c \frac{\partial^2 f(x, t)}{\partial x^2} + g(x, t), \tag{5}$$

where c denotes the fractional diffusion constant, with the integral initial condition

$$I^\beta f(x, 0) = f_{0,\alpha}\delta(x). \tag{6}$$

For this purpose we need the following concepts.

Definition 2.3 The function $F(s)$ on the complex variable s defined by

$$F(s) = \angle\{f(t); s\} = \int_0^\infty e^{-st} f(t)dt$$

is called the Laplace transform of the function $f(t)$.

Definition 2.4 The Mellin transform of the function $f(t)$ is

$$M\{f(t)\}(s) = \int_0^\infty t^{s-1} f(t)dt.$$

Definition 2.5 By $Fox's\,H-$ functions we mean a generalized hypergeometric function, defined by means of the Mellin-Barnes type contour integral

$$H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_j, A_j)_1^p \\ (b_k, B_k)_1^q \end{matrix} \right] = \frac{1}{2i\pi} \int_{c'} \frac{\prod_{k=1}^m \Gamma(b_k - sB_k) \prod_{j=1}^n \Gamma(1 - a_j + sA_j)}{\prod_{k=m+1}^q \Gamma(1 - b_k + sB_k) \prod_{j=n+1}^p \Gamma(a_j - sA_j)} z^s ds$$

or

$$H_{p,q}^{m,n}\left[z\,|\,\begin{matrix}(a_j,A_j)_1^p\\(b_k,B_k)_1^q\end{matrix}\right]=\frac{1}{2i\pi}\int_{c'}\frac{\prod_{k=1}^{m}\Gamma(b_k+sB_k)\prod_{j=1}^{n}\Gamma(1-a_j-sA_j)}{\prod_{k=m+1}^{q}\Gamma(1-b_k-sB_k)\prod_{j=n+1}^{p}\Gamma(a_j+sA_j)}z^{-s}ds,$$

$z\neq 0$ where c' is a suitable contour in $\mathbb{C}$, the orders (m,n,p,q) are integers such that $0\le m\le q, 0\le n\le p,$ and the parameters $a_j\in\mathbb{R}, A_j>0,\ j=1,...,p\ b_k\in\mathbb{R}, B_k>0,\ k=1,...,q$ are such that $A_j(b_k+l)\neq B_k(a_j-l'-1), l, l'=0,1,2,....$

Definition 2.6 The Fourier transformation for one dimension is defined as

$$F\{f(r)\}(q)=\int_{R^d}e^{iqr}f(r)dr.$$

3. THE EXACT SOLUTION

The Laplace transform of equation (1) yields

$$s^{\alpha}U(s)=F(s,U(s)). \tag{7}$$

To invert the Laplace transform it is convenient use the relation

$$M\{f(t)\}(s)=\frac{M\{\angle\{f(t)\}(u)\}(1-s)}{\Gamma(1-s)} \tag{8}$$

between the Laplace transform and Mellin transform of the function $f(t)$. But

$$\begin{aligned}M\{s^{\alpha}U(s)\}(1-v)&=M\{F(s,U(s))(1-v)\\&=M\{\int_0^{\infty}e^{-st}f(t,u(t))dt\}(1-v)\\&=F(v)G(1-v),\end{aligned}$$

where $F(v)$ is the Mellin transform of the function f and $G(1-v)$ is the Mellin transform of the function $g(st)=e^{-st}$. Hence

$$M\{u(t)\}(v)=\frac{F(v)G(1-v)}{\Gamma(1-v)}.$$

Now inverting Mellin transform and comparing this with the definition of general $H-$ function, allows one to identity the $H-$ function parameters for the first fraction as: $m=0, n=1, q=1, p=1, b_1=0, B_1=1, a_1=A_1=0.$

Then we obtain

$$u(t)=H_{1,1}^{0,1}\left[F(v)G(1-v)t\,|\,\begin{matrix}(0,0)\\(0,1)\end{matrix}\right]. \tag{9}$$

In the same way, we can show that the exact solution for equation (3), takes the form (9). For the equation (4), we have

$$M\{u(t)\}(v)=\frac{F(v)G(1-v)}{\Gamma(1-v)}+[G(1-v)]^2\sum_{i=1}^{n}G_i(v)\times\frac{\Gamma(v+\alpha_i)}{\Gamma(v)\Gamma(1-v)}U(1-v-\alpha_i).$$

The $H-$function parameters for the first term as $m=0, n=p=q=1, a_1=A_1=0, b_1=B_1=0,$ and for the second term as $m=n=p=1, q=2, a_1=A_1=0, b_1=\alpha_i, B_1=B_2=1, b_2=0,$ then we obtain

$$u(t)=H_{1,1}^{0,1}\left[F(v)G(1-v)t\,|\,\begin{matrix}(0,0)\\(0,1)\end{matrix}\right]+\sum_{i=1}^{n}H_{1,2}^{1,1}\left[G_i(v)[G(1-v)]^2U(1-v-\alpha_i)t\,|\,\begin{matrix}(0,0)\\(\alpha_i,1),(0,1)\end{matrix}\right].$$

4. THE MIXED PROBLEM

One of the main applications of the fractional calculus (integration and differentiation of arbitrary order)is the modeling of the processes. In this section, by applying Fourier transform with respect to x and Laplace with respect to t we shall provide the exact solution for the mixed problem (5-6). Fourier and Laplace transformation of equation (5-6) yields

$$f(q,u)=\frac{f_{0,\alpha}}{cq^2+u^{\alpha}}+G(q,u), \tag{10}$$

where q is the Fourier transform parameter and u is the Laplace transform parameter. To obtain $f(x,t)$, first invert the Fourier transform in equation(10) using the formula

$$(2\pi)^{-d/2}\int_{\mathbb{R}^d}e^{iqr}\left(\frac{|r|}{m}\right)^{1-d/2}K_{(d-2)/2}(m/|r|)dr=\frac{1}{q^2+m^2}$$

where $K_{(d-2)/2}$ is a Bessel function of order $(d-2)/2,$ which leads to:

$$f(x,u) = f_{0,\alpha}(2\pi)^{-d/2}\left(\frac{|x|}{\sqrt{c}}\right)^{1-(d/2)}u^{\alpha(d-2)/4}$$

$$K_{(d-2)/2}\left(\frac{|x|}{\sqrt{c}}u^{\alpha/2}\right)+G(x,u).$$

Setting $\lambda = \alpha/2, \nu = (d-2)/2$ *and* $\mu = \alpha(d-2)/4$ and using the general relation

$$M\{x^a g(yx^b)\}(s) = \frac{1}{b}y^{-(s+a)/b}g\left(\frac{s+a}{b}\right),\ y,b>0,$$

implies

$$M\{f(x,u)\}(s) = \frac{f_{0,\alpha}}{\lambda}(2\pi c)^{-d/2}\left(\frac{|x|}{\sqrt{c}}\right)^{1-(d/2)-(s+\mu)/\lambda}$$

$$M\{K_\nu(u)\}((s+\mu)/\lambda)+M\{G(x,u)\}(s). \qquad (11)$$

The Mellin transform of the Bessel function is

$$M\{K_\nu(u)\}(s) = 2^{s-2}\Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right)$$

Substituting this in equation (11), using (8), and restoring the original variables then we have

$$M\{f(x,t)\}(s) = \frac{f_{0,\alpha}}{\alpha[(|x|)^2\pi]^{d/2}}\left(\frac{|x|}{2\sqrt{c}}\right)^{2(1-\frac{1+s}{\alpha})}$$

$$\frac{\Gamma\left(\frac{d}{2}+\frac{1-s}{\alpha}-1\right)\Gamma\left(\frac{1-s}{\alpha}\right)}{\Gamma(1-s)}+\frac{G(1-s)}{\Gamma(1-s)},$$

where $G(1-s)$ denotes the Mellin transform of the function $G(x,u)$ Now inverting Mellin transform and comparing this with the general $H-$function allows one to identify the $H-$function parameters as $m=0, n=2, p=2, q=1, A_1 = A_2 = 1/\alpha, a_1 = 2-\frac{d}{2}-\frac{1}{\alpha}, a_2 = 1-\frac{1}{\alpha}, b_1 = 0$, and $B_1 = 1$, if $\frac{\alpha d}{2}-\alpha+1>0$, for the first term. And for the second term , setting $m=0, n=1, q=1, p=1, b_1 = 0, B_1 = 1, a_1 = A_1 = 0$. Then the result becomes

$$f(x,t) = \frac{f_{0,\alpha}}{\alpha[(|x|)^2\pi]^{d/2}}\left(\frac{|x|}{2\sqrt{c}}\right)^{2(1-1/\alpha)}H_{2,1}^{0,2}$$

$$\left[\left(\frac{2\sqrt{c}}{|x|}\right)^{2/\alpha}t\,\middle|\,\begin{matrix}((2-\frac{d}{2}-\frac{1}{\alpha}),\frac{1}{\alpha}),((1-\frac{1}{\alpha}),\frac{1}{\alpha})\\(0,1)\end{matrix}\right]$$

$$+H_{1,1}^{0,1}\left[G(1-s)t\,\middle|\,\begin{matrix}(0,0)\\(0,1)\end{matrix}\right].$$

For $g(x,t)\equiv 0$, and $\alpha = 1$, equation (5) becomes the classical diffusion equation, and for $\alpha = 2$ it becomes the classical wave equation, For $0<\alpha<1$, we have the so-called ultraslow diffusion, and values $1<\alpha<2$ correspond to so-called intermediate processes.

5. ARTIFICIAL NEURAL NETWORK

Neural network (NN) was first introduced by McCulloch and Pitts in 1943, since the introduction it has been widely used in different real world classification tasks in industry, business, and science [12]. Neural network emulates the functionality of human brains in which the neurons (nerves cell) communicate with each other by sending messages among them. Artificial neural network (ANN) represents the mathematical model of these biological neurons. It is a parallel distributed information processing structure consisting of a number of nonlinear processing units, which can be trained to recognize features and to identify incomplete data [13]. Neural network has great mapping capabilities or pattern association thus exhibiting generalization, robustness, high fault tolerance, and high speed parallel information processing.

In this work a standard back-propagation neural network (NN) is used to estimate the exact solution for the following mixed fractional differential equation.

$$D_t^\alpha u + 6uu_x + u_{xxx} = 0 \qquad t>0 \quad 0<\alpha\leq 1 \qquad (12)$$

subject to the initial condition

$$u(x,0) = \frac{1}{2}sech^2\left(\frac{1}{2}x\right)$$

The exact solution, for the special case $\alpha = 1$

$$u(x,t) = \frac{1}{2}sech^2\left(\frac{1}{2}(x-t)\right)$$

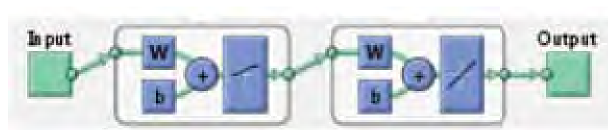

Fig. 1. Neural network structure.

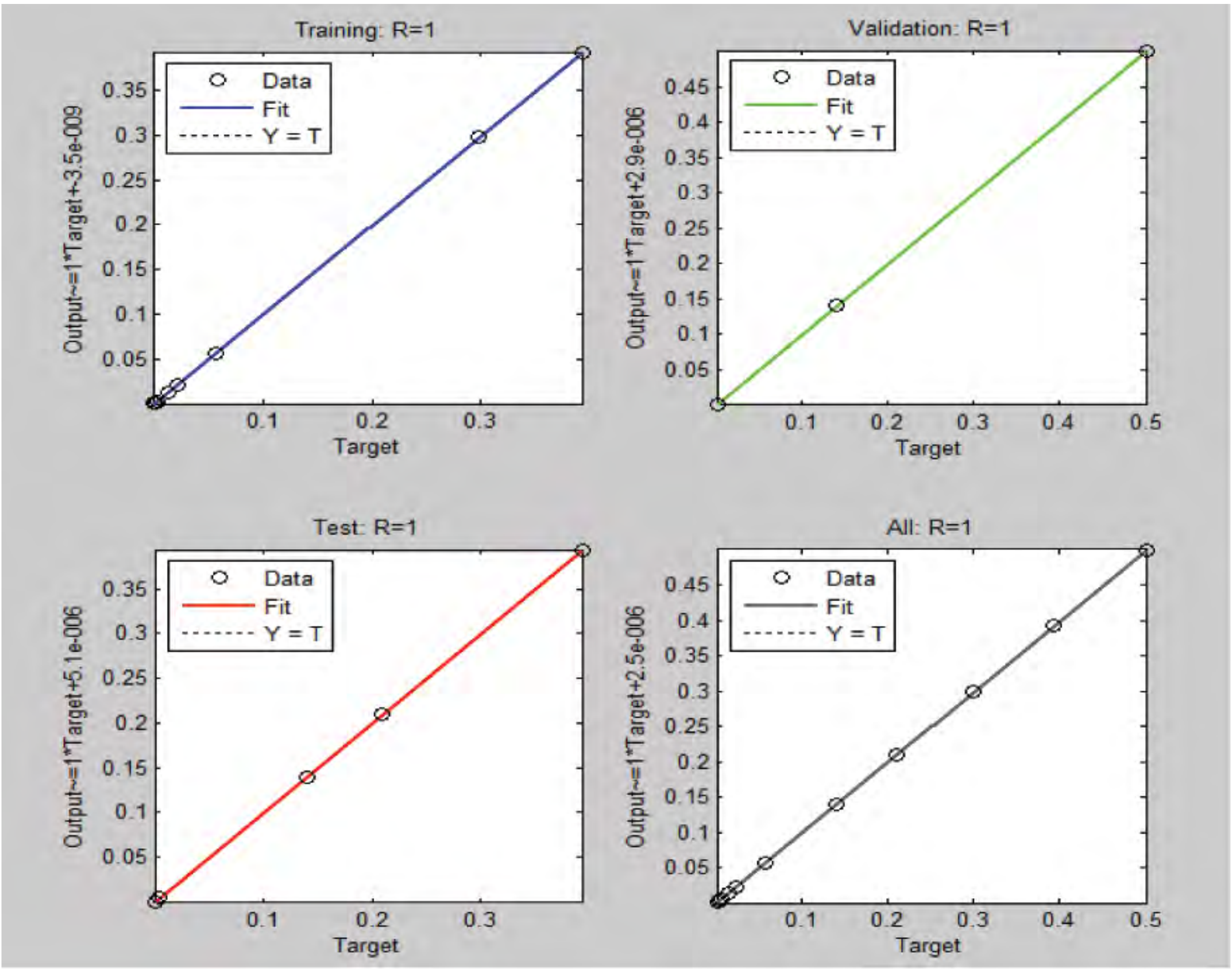

Fig. 2. Regressions analysis for t=0.

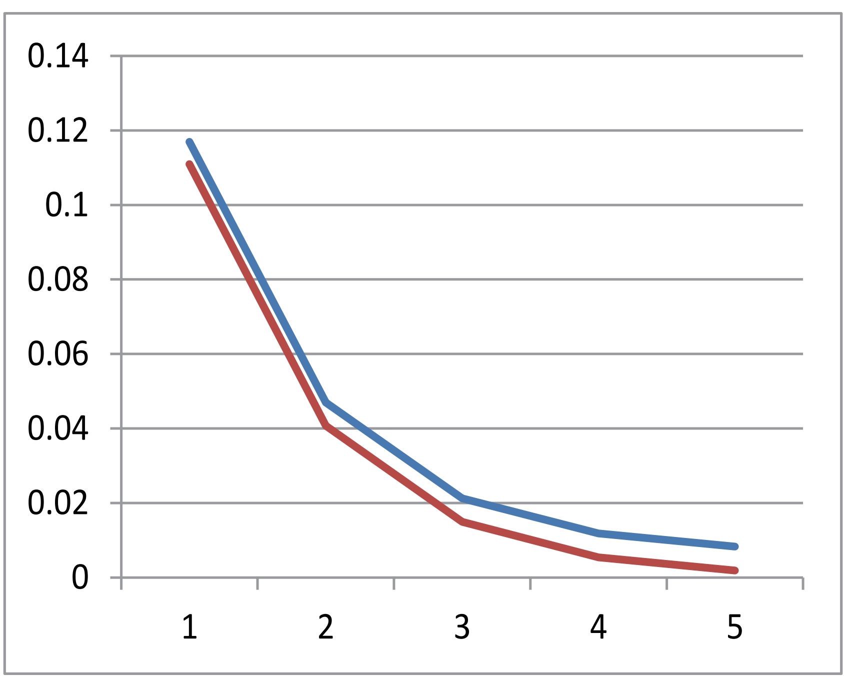

Fig. 3. Neural network estimated exact solution v. the exact solution for t=0.

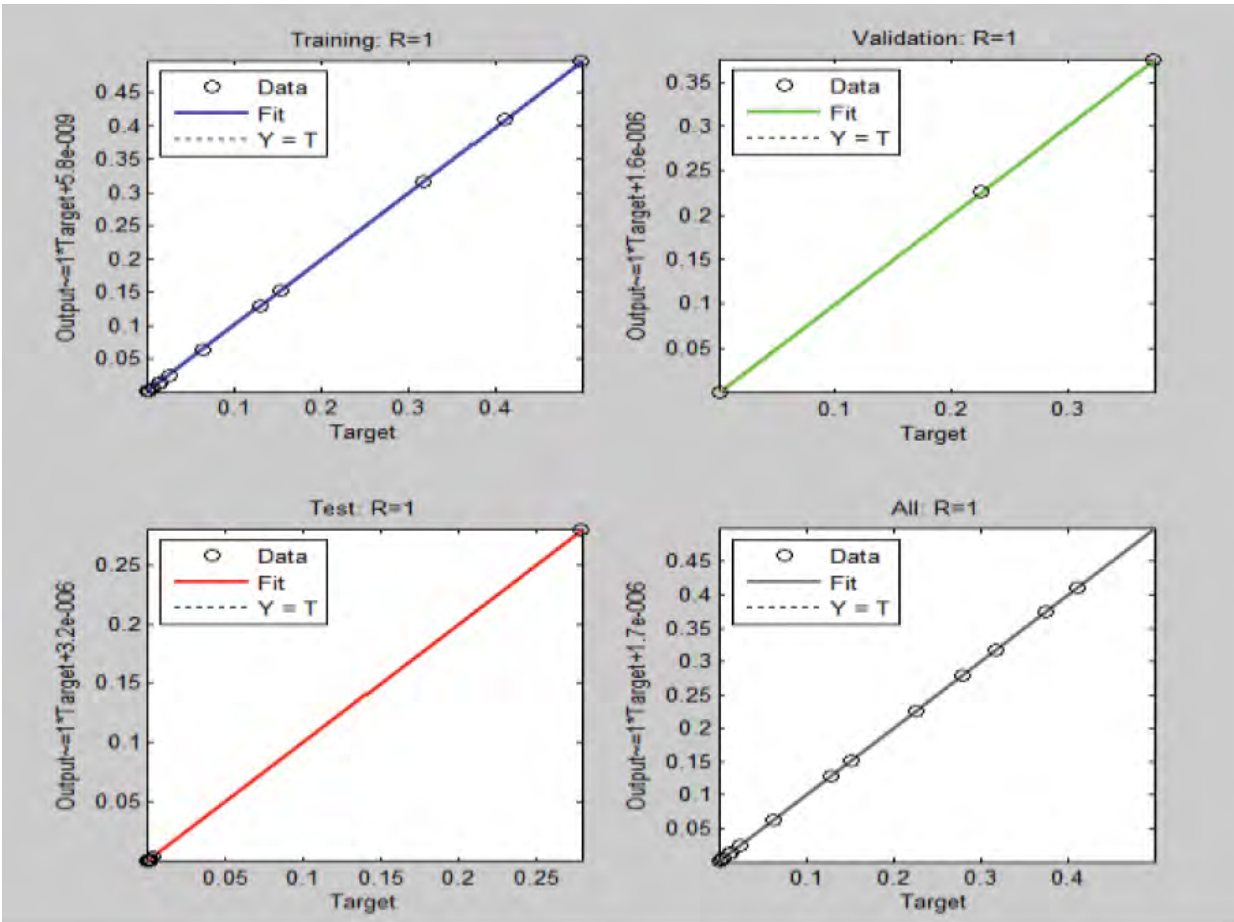

Fig. 4. Regressions analysis for t=0.1.

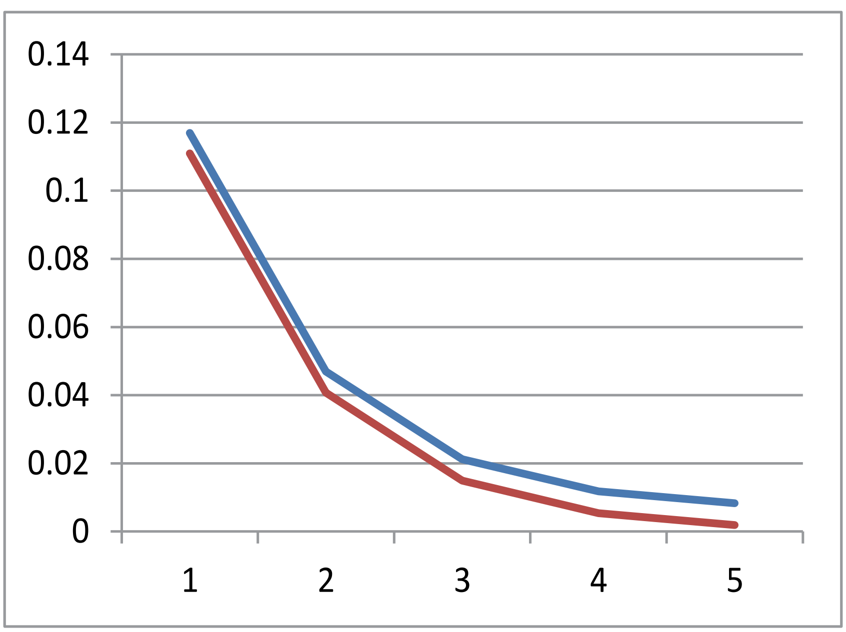

Fig. 5. Neural network estimated exact solution v. th exact solution for t=0.1.

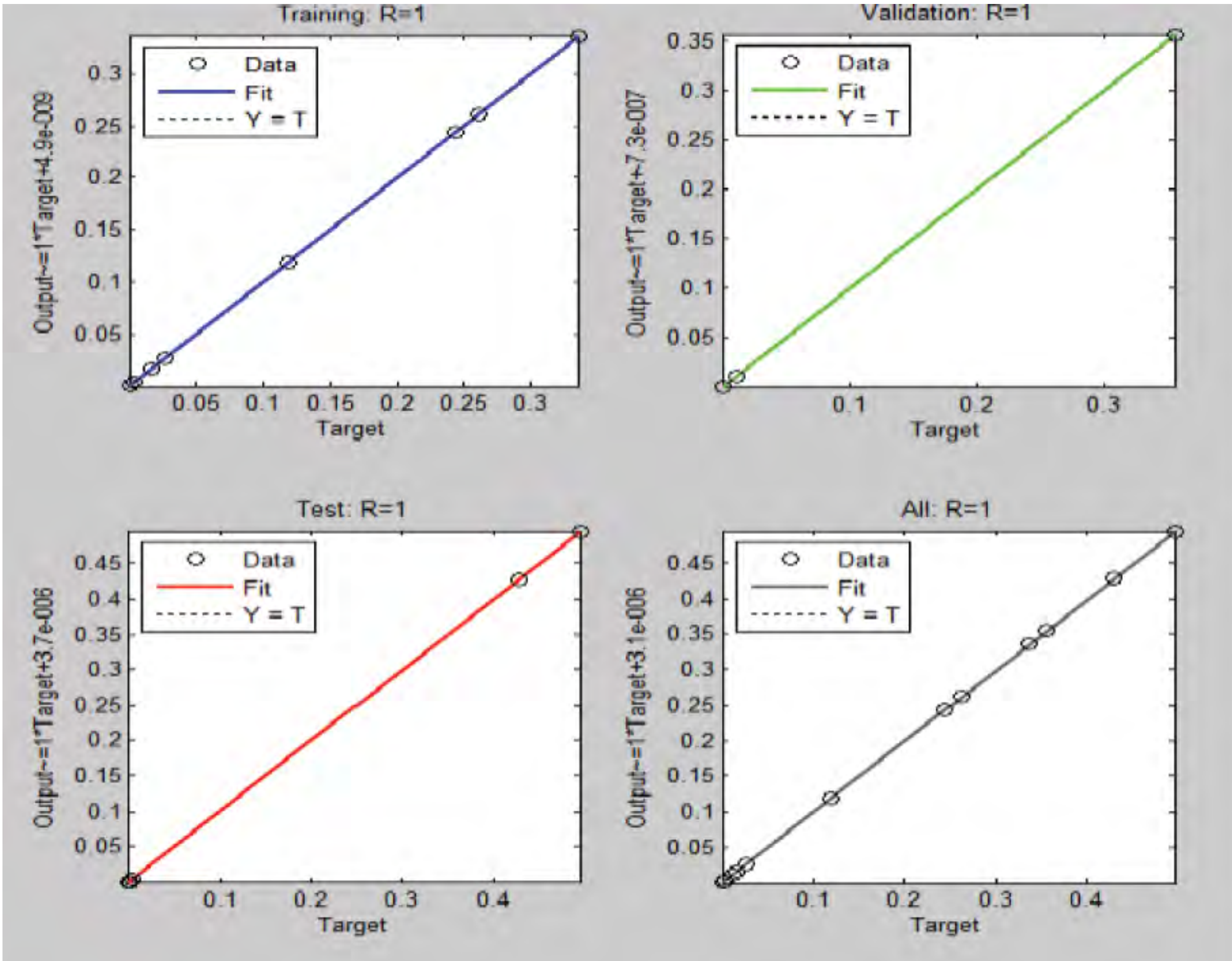

Fig. 6. Regressions analysis for t=0.1.

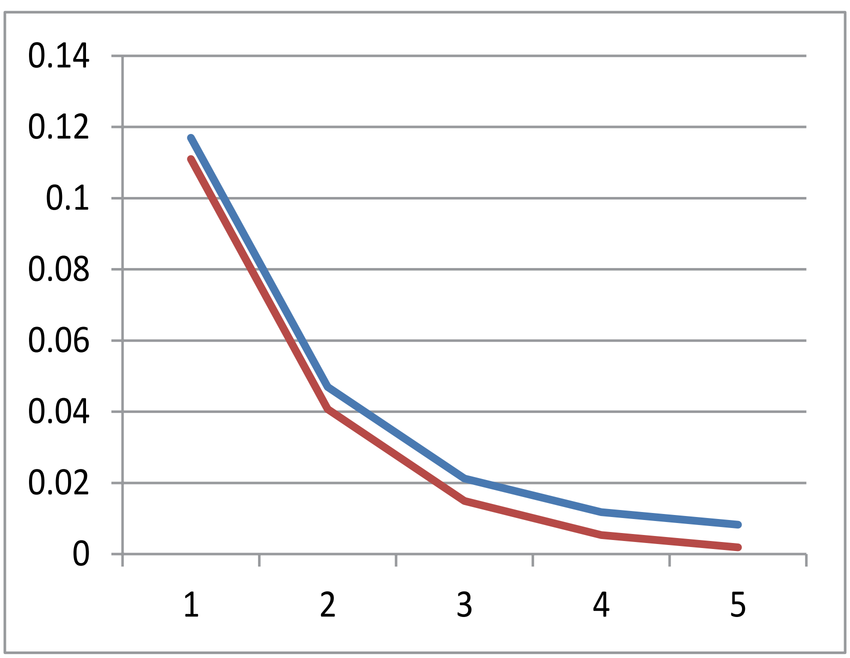

Fig. 7. Neural network estimated exact solution v. the exact solution for t=0.1.

6. TRAINING PHASE

The NN is trained to estimate the exact solution. The dataset contains 105 exact solutions for training, which are solved numerically for three values of *t* (*t*=0, *t*=0.1 and *t*=0.2), using Eq.(12). In the training phase of the NN, the weight matrices among the input and the hidden and output layers are initialized with random values. After repeatedly presenting data of the input samples and desired targets, we have compared the output with the desired outcome, followed by error measurement and weight adjustment. This pattern is repeated until the error rate of the output layer reaches a minimum value. This process is then repeated for the next input value, until all values of the input have been processed. The binary-sigmoid activation function is used. The value of this function ranges between 0 and 1. Whereas, the output layer neuron is estimated using the activation function that features the linear transfer function. The training algorithm used is Gradient descent with momentum back propagation. The exact solution is solved manually by using the giving equation and entered as training input data into the NN. The quality of the training sets that enters into the network determines efficiency of neural network. Fig. 2.4 & 6, show the regressions analysis of the trained network for different values of *t* in Eq. (12). The regressions analysis returns the correlation coefficient R. This coefficient equals to 1 between the output and the target for training; thus, both output and target are very close, which indicates good fit.

7. RESULTS AND DISCUSSION

The experimental results are presented to show the effectiveness of the proposed neural network. The training and testing phases were carried out on a 2.33 GHz Intel (R) Core TM 2Duo CPU 4 GB RAM on Windows 7 platform using MATLAB R2011a. The results of the proposed algorithm in this paper are compared with the exact solution. Fig. 3.5 and 7, show the estimated neural network exact solution (blue line) v. the exact solution (red line) for different values of *t* in Eq. (12). The figures show that for all values of *t* there are almost similar estimated exact solution.

8. CONCLUSIONS

This paper had proved that the fractional differential equations based on Riemann-Liouville fractional derivatives are solved exactly. The solution was obtained in terms of H-functions. The solution was proved to be finite for all times. Moreover, by using the neural network method, the numerical solution for some special equations has been estimated. The experimantal results have proved that for all values of *t* there are almost similar estimated exact solutions. The parallel processing property of trained neural network enabled to recognize features and to identify incomplete data, which resulted in reducing the differences between the estimated and the exact solution.

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