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Original Article

A New Class of Harmonic p-Valent Functions of Complex Order

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Abstract: In this paper, we define a class of p-valent harmonic functions and study some results as coefficient inequality, distortion theorem, extreme points, convolution conditions and convex combination.

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1. INTRODUCTION

A continuous complex-valued function f = u + iv defined in a simply connected complex domain B is said to be harmonic in B if both u and v are real harmonic in B. Let

$f = h + \bar{g}$

be defined in any simply connected domain, where h and g are analytic in B. A necessary and sufficient condition for f to be locally univalent and sense-preserving in B is that

$$|h'(z)| > |g'(z)|, z \in B \text{ (see [2])}. \tag{1.1}$$

Let $H(p)$ denote the class of functions of the form:

$$f = h + g$$

which are harmonic p-valent in the open unit disc U={z : z ∈ ℂ ,|z| < 1}, where

$$h(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=p}^{\infty} b_k z^k$$
$$(|b_p| < 1; p \in N = \{1,2,3,\dots\}). \tag{1.2}$$

Let $\overline{H}(p)$ denote the class of functions of the form:

$$f = h + \bar{g}, \tag{1.3}$$

Where

$$h(z) = z^p - \sum_{k=p+1}^{\infty} |a_k| z^k, g(z) = \sum_{k=p}^{\infty} |b_k| z^k, |b_p| < 1. \tag{1.4}$$

For $0 < \beta \leq 1, p \in N, b \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}, |b| \leq 1, z' = (\partial/(\partial\theta))(z = re^{i\theta}), 0 \leq r < 1, 0 \leq \theta < 2\pi$ *and* $f'(z) = \frac{\partial}{\partial\theta}\left(f(z)\right)$, let $S_H(b,p,\beta)$, let be the class of hamonic functions $f(z)$ of the form (1.2) such that

$$\left|\frac{1}{b}\left[\frac{zf'(z)}{z'f(z)} - p\right]\right| < \beta, \tag{1.5}$$

or, equivalently,

$$\text{Re}\left\{\frac{zf'(z)}{z'f(z)}\right\} > p - \beta|b|. \tag{1.6}$$

Also, let

$\bar{S}_H(b,p,\beta) = S_H(b,p,\beta) \cap \overline{H}(p)$.

We note that:

(i) $\bar{S}_H(p-\alpha,p,1) = TH(p,\alpha)$ (see Ahuja and Jahangiri [1]);

(ii) $\bar{S}_H(b,1,\beta) = \overline{H}S^*(b,\beta)$ (see Janteng [4]);

(iii) $\bar{S}_H(1-\alpha,1,1) = S_H^*(\alpha)$ $(0 \leq \alpha < 1)$ (*see Jahangiri* [3]);

(iv) $\bar{S}_H(1-\alpha,1,1) = S_H^*(0) = T_H^*$(*see Silverman* [5]).

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Also we note that:

(i) $\bar{S}_H\left((p-\alpha)cos\lambda e^{-i\lambda}, p, 1\right) = \overline{S_H^{\lambda}}(p,\alpha)$

$$= \left\{f(z) \in \bar{H}(p): \text{Re}\left\{e^{i\lambda}\frac{zf'(z)}{z'f(z)}\right\} \geq \alpha\cos\lambda \ \left(0 \leq \alpha < p; |\lambda| < \frac{\pi}{2}\right)\right\};$$

(ii) $\bar{S}_H(1, p, \beta) = \bar{S}_H(p, \beta)$

$$= \left\{f(z) \in \bar{H}(p): \left|\frac{zf'(z)}{z'f(z)} - p\right| < \beta\right\}.$$

In this paper we introduce a new classes $S_H(b,p,\beta)$ and $\bar{S}_H(b,p,\beta)$. We obtain also the coefficient inequality, distortion theorem, extreme points, convolution conditions and convex combination for functions in the class $\bar{S}_H(b,p,\beta)$.

2. COEFFICIENT ESTIMATE

Unless otherwise mentioned, we assume throughout this paper that $0 < \beta \leq 1, p \in N, b \in \mathbb{C}^*$, $|b| \leq 1, z' = (\partial/(\partial\theta))(z = re^{i\theta})$, $0 \leq r < 1, 0 \leq \theta < 2\pi, f'(z) = (\partial/(\partial\theta))f(z), z \in U$ and $f(z)$ is given by (1.3).

In the following theorem, we obtain the coefficient inequality for functions of the class $S_H(b,p,\beta)$.

Thereom 1. Let $f = h + \bar{g}$, where h and g are given by (1.2). Furthermore, let

$$\sum_{k=p+1}^{\infty} \frac{k-p+\beta|b|}{\beta|b|}|a_k| + \sum_{k=p}^{\infty} \frac{k+p-\beta|b|}{\beta|b|}|b_k| \leq 1, \tag{2.1}$$

then $f(z) \in S_H(b,p,\beta)$.

Proof. We only need to show that if (2.1) holds then the condition (1.6) is satisfied. Since $Re\, w > \delta$ if and only if $|1-\delta+w| > |1+\delta-w|$, it suffices to show that

$$|(1-p+\beta|b|)z'f(z) + zf'(z)| - |(1+p-\beta|b|)z'f(z) - zf'(z)| > 0.$$

Substituting for $z'f(z)$ and $zf'(z)$, we obtain

$$|(1-p+\beta|b|)z'f(z) + zf'(z)| - |(1+p-\beta|b|)z'f(z) - zf'(z)|$$

$$= \left|(1+\beta|b|)z^p + \sum_{k=p+1}^{\infty}(k-p+1+\beta|b|)a_k z^k - \sum_{k=p}^{\infty}(k+p-1-\beta|b|)b_k \bar{z}^k\right|$$

$$- \left|(1-\beta|b|)z^p - \sum_{k=p+1}^{\infty}(k-p-1+\beta|b|)a_k z^k + \sum_{k=p}^{\infty}(k+p+1-\beta|b|)b_k \bar{z}^k\right|$$

$$\geq (1+\beta|b|)|z|^p - \sum_{k=p+1}^{\infty}(k-p+1+\beta|b|)|a_k||z|^k - \sum_{k=p}^{\infty}(k+p-1-\beta|b|)|b_k||z|^k$$

$$-(1-\beta|b|)|z|^p - \sum_{k=p+1}^{\infty}(k-p-1+\beta|b|)|a_k||z|^k - \sum_{k=p}^{\infty}(k+p+1-\beta|b|)|b_k||z|^k$$

$$= 2\beta|b||z|^p - 2\sum_{k=p+1}^{\infty}(k-p+\beta|b|)|a_k||z|^k - 2\sum_{k=p}^{\infty}(k+p-\beta|b|)|b_k||z|^k$$

$$> 2\beta|b|\left\{1 - \sum_{k=p+1}^{\infty}\frac{k-p+\beta|b|}{\beta|b|}|a_k| - \sum_{k=p}^{\infty}\frac{k+p-\beta|b|}{\beta|b|}|b_k|\right\}. \tag{2.2}$$

This last expression is non-negative by (2.1), which completes the proof of Theorem 1. The harmonic p-valent function

$$f(z)=z^p+\sum_{k=p+1}^{\infty}\frac{\beta|b|}{k-p+\beta|b|}X_k z^k+\sum_{k=p}^{\infty}\frac{\beta|b|}{k+p-\beta|b|}\overline{Y_k z^k}, \tag{2.3}$$

where $\sum_{k=p+1}^{\infty}|X_k|+\sum_{k=p}^{\infty}|Y_k|=1$, show that the coefficient bound given by (2.1) is sharp. This is because

$$\sum_{k=p+1}^{\infty}\frac{k-p+\beta|b|}{\beta|b|}|a_k|+\sum_{k=p}^{\infty}\frac{k+p-\beta|b|}{\beta|b|}|b_k|$$

$$=\sum_{k=p+1}^{\infty}\frac{k-p+\beta|b|}{\beta|b|}\cdot\frac{\beta|b|}{k-p+\beta|b|}|X_k|$$

$$+\sum_{k=p}^{\infty}\frac{k+p-\beta|b|}{\beta|b|}\cdot\frac{\beta|b|}{k+p-\beta|b|}|Y_k|$$

$$=\sum_{k=p+1}^{\infty}|X_k|+\sum_{k=p}^{\infty}|Y_k|=1.$$

Now, we need to prove that the condition (2.1) is also necessary for functions of the form (1.3) to be in the class $\bar{S}_H(b,p,\beta)$.

Thereom 2. Let $f=h+\bar{g}$, where h and g are given by (1.4). then $f(z)\in\bar{S}_H(b,p,\beta)$ if and only if

$$\sum_{k=p+1}^{\infty}\frac{k-p+\beta|b|}{\beta|b|}|a_k|+\sum_{k=p}^{\infty}\frac{k+p-\beta|b|}{\beta|b|}|b_k|\le 1. \tag{2.4}$$

Proof. Since $\bar{S}_H(b,p,\beta)\subset S_H(b,p,\beta)$, we only need to prove the "only if" part of this theorem. Let $f(z)\in\bar{S}_H(b,p,\beta)$, then

$$\operatorname{Re}\left\{\frac{zf'(z)}{z'f(z)}\right\}>p-\beta|b|,$$

that, is that

$$\operatorname{Re}\left\{\frac{zf'(z)-(p-\beta|b|)z'f(z)}{z'f(z)}\right\}$$

$$=\operatorname{Re}\left\{\frac{\beta|b|z^p-\sum_{k=p+1}^{\infty}(k-p+\beta|b|)a_k z^k-\sum_{k=p}^{\infty}(k+p-\beta|b|)b_k\bar{z}^k}{z^p-\sum_{k=p+1}^{\infty}a_k z^k+\sum_{k=p}^{\infty}b_k\bar{z}^k}\right\}>0. \tag{2.5}$$

By choosing the values of z on the positive real axis where $0\le z=r<1$, we have

$$\frac{\beta|b|-\sum_{k=p+1}^{\infty}(k-p+\beta|b|)a_k-\sum_{k=p}^{\infty}(k+p-\beta|b|)b_k}{z^p-\sum_{k=p+1}^{\infty}a_k+\sum_{k=p}^{\infty}b_k}\ge 0. \tag{2.6}$$

If the condition (2.4) does not hold, then the numerator in (2.6) is negative for $r\to 1$. This contradicts (2.6), then the proof of Theorem 2 is completed.

Putting $p=1$ in Theorem 2, we obtain the following corollary:

Corollary 1. Let $f=h+\bar{g}$, where h and g are given by (1.4). Then $f(z)\in\overline{H}S^*(b,\beta)$ if and only if

$$\sum_{k=2}^{\infty}\frac{k-1+\beta|b|}{\beta|b|}|a_k|+\sum_{k=1}^{\infty}\frac{k+1-\beta|b|}{\beta|b|}|b_k|\le 1. \tag{2.7}$$

Putting $b=(p-\alpha)cos\lambda e^{-i\lambda}$ $(0\le\alpha<p,|\lambda|<\frac{\pi}{2})$ and $\beta=1$ in Theorem 2, we obtain the following corollary:

Corollary 2. Let $f=h+\bar{g}$, where h and g are given by (1.4). Then $f(z)\in\bar{S}_H^{\lambda}(p,\alpha)$ if and only if

$$\sum_{k=p+1}^{\infty}\frac{k-p+(p-\alpha)cos\lambda}{(p-\alpha)cos\lambda}|a_k|+\sum_{k=p}^{\infty}\frac{k+p-(p-\alpha)cos\lambda}{(p-\alpha)cos\lambda}|b_k|\le 1. \tag{2.8}$$

3. SOME PROPERTIES FOR THE CLASS $\bar{S}_H(b,p,\beta)$

Distortion bounds for the class $\bar{S}_H(b,p,\beta)$ are given in the following theorem.

Thereom 3. Let the function $f(z)$ given by (1.3) be in the class $\bar{S}_H(b,p,\beta)$. Then for $|z|=r<1$, we have

$$f(z)|\le\left(1+|b_p|\right)r^p+\left(\frac{\beta|b|}{1+\beta|b|}-\frac{2p-\beta|b|}{1+\beta|b|}|b_p|\right)r^{p+1} \tag{3.1}$$

and

$$|f(z)|\ge\left(1-|b_p|\right)r^p-\left(\frac{\beta|b|}{1+\beta|b|}-\frac{2p-\beta|b|}{1+\beta|b|}|b_p|\right)r^{p+1}. \tag{3.2}$$

The equalities in (3.1) and (3.2) are attained for the functions $f(z)$ given by

$$f(z) = (1+b_p)\bar{z}^p + \left(\frac{\beta|b|}{1+\beta|b|} - \frac{2p-\beta|b|}{1+\beta|b|}b_p\right)\bar{z}^{p+1}$$

and

$$f(z) = (1-b_p)\bar{z}^p - \left(\frac{\beta|b|}{1+\beta|b|} - \frac{2p-\beta|b|}{1+\beta|b|}b_p\right)\bar{z}^{p+1}.$$

Proof. Let $f(z) \in \overline{S}_H(b,p,\beta)$. Then we have

$$|f(z)| \le r^p + \sum_{k=p+1}^{\infty} |a_k|\, r^k + \sum_{k=p}^{\infty} |b_k|\, r^k$$

$$\le (1+|b_p|)r^p + r^{p+1} \sum_{k=p+1}^{\infty} (|a_k| + |b_k|)$$

$$= (1+|b_p|)r^p + \frac{\beta|b|}{1+\beta|b|} \sum_{k=p+1}^{\infty} \left(\frac{1+\beta|b|}{\beta|b|}|a_k| + \frac{1+\beta|b|}{\beta|b|}|b_k|\right) r^{p+1}$$

$$\le (1+|b_p|)r^p + \frac{\beta|b|}{1+\beta|b|} \sum_{k=p+1}^{\infty} \left(\frac{k-p+\beta|b|}{\beta|b|}|a_k| + \frac{k+p-\beta|b|}{\beta|b|}|b_k|\right) r^{p+1}$$

$$\le (1+|b_p|)r^p + \frac{\beta|b|}{1+\beta|b|}\left(1 - \frac{2p-\beta|b|}{\beta|b|}|b_p|\right) r^{p+1}$$

$$= (1+|b_p|)r^p + \left(\frac{\beta|b|}{1+\beta|b|} - \frac{2p-\beta|b|}{1+\beta|b|}|b_p|\right) r^{p+1}.$$

Similarly, we can prove the left-hand inequality, where

$$|f(z)| \ge r^p - \sum_{k=p+1}^{\infty} |a_k|\, r^k - \sum_{k=p}^{\infty} |b_k|\, r^k. \qquad (3.3)$$

This completes the proof of Theorem 3.

Putting $p = 1$ in Theorem 3, we obtain the following corollary:

Corollary 3. Let the function $f(z)$ given by (1.3) be in the class $\overline{H}S^*(b,\beta)$. Then for $|z| = r < 1$, we have

$$|f(z)| \le (1+|b_1|)r + \left(\frac{\beta|b|}{1+\beta|b|} - \frac{2-\beta|b|}{1+\beta|b|}|b_1|\right) r^2 \qquad (3.4)$$

and

$$|f(z)| \ge (1-|b_1|)r - \left(\frac{\beta|b|}{1+\beta|b|} - \frac{2-\beta|b|}{1+\beta|b|}|b_1|\right) r^2. \qquad (3.5)$$

The equalities in (3.4) and (3.5) are attained for the functions $f(z)$ given by

$$f(z) = (1+b_1)\bar{z} + \left(\frac{\beta|b|}{1+\beta|b|} - \frac{2-\beta|b|}{1+\beta|b|}b_1\right)\bar{z}^2$$

and

$$f(z) = (1-b_1)\bar{z} - \left(\frac{\beta|b|}{1+\beta|b|} - \frac{2-\beta|b|}{1+\beta|b|}b_1\right)\bar{z}^2.$$

Putting $b = (p-\alpha)cos\lambda e^{-i\lambda}$ $(0 \le \alpha < p, |\lambda| < \frac{\pi}{2})$ and $\beta = 1$ in Theorem 3, we obtain the following corollary:

Corollary 4. Let the function $f(z)$ given by (1.3) be in the class $S_H^{\lambda}(p,\alpha)$. Then for $|z| = r < 1$, we have

$$f(z)| \le (1+|b_p|)r^p + \left(\frac{(p-\alpha)cos\lambda}{1+(p-\alpha)cos\lambda} - \frac{2p-(p-\alpha)cos\lambda}{1+(p-\alpha)cos\lambda}|b_p|\right) r^{p+1} \qquad (3.6)$$

and

$$|f(z)| \ge (1-|b_p|)r^p - \left(\frac{(p-\alpha)cos\lambda}{1+(p-\alpha)cos\lambda} - \frac{2p-(p-\alpha)cos\lambda}{1+(p-\alpha)cos\lambda}|b_p|\right) r^{p+1}. \qquad (3.7)$$

The equalities in (3.6) and (3.7) are attained for the functions$f(z)$ given by

$$f(z) = (1+b_p)\bar{z}^p + \left(\frac{(p-\alpha)cos\lambda}{1+(p-\alpha)cos\lambda} - \frac{2p-(p-\alpha)cos\lambda}{1+(p-\alpha)cos\lambda}b_p\right)\bar{z}^{p+1}$$

and

$$f(z) = (1-b_p)\bar{z}^p - \left(\frac{(p-\alpha)cos\lambda}{1+(p-\alpha)cos\lambda} - \frac{2p-(p-\alpha)cos\lambda}{1+(p-\alpha)cos\lambda}b_p\right)\bar{z}^{p+1}.$$

Our next theorem is on the extreme points of convex hulls of the class $\overline{S}_H(b,p,\beta)$ denoted by $clco\, \overline{S}_H(b,p,\beta)$.

Thereom 4. Let $f(z)$ be given by (1.3). Then $f \in \overline{S}_H(b,p,\beta)$ if and only if $f(z) = \sum_{k=p}^{\infty}(X_k h_k + Y_k g_k)$, where

$$h_p(z) = z^p, h_k(z) = z^p - \frac{\beta|b|}{k-p+\beta|b|} z^k \ (k = p+1, p+2, \dots), \qquad (3.8)$$

and

$$g_k(z) = z^p + \frac{\beta|b|}{k+p-\beta|b|}\bar{z}^k \; (k = p, p+1, ...)$$

$$\left(X_k \geq 0; Y_k \geq 0; \sum_{k=p}^{\infty}(X_k + Y_k) = 1\right). \qquad (3.9)$$

Proof. Let

$$f(z) = \sum_{k=p}^{\infty}(X_k h_k + Y_k g_k)$$

$$= \sum_{k=p}^{\infty}(X_k + Y_k)z^p - \sum_{k=p+1}^{\infty}\frac{\beta|b|}{k-p+\beta|b|}X_k z^k$$

$$+ \sum_{k=p}^{\infty}\frac{\beta|b|}{k+p-\beta|b|}\overline{Y_k z}^k$$

$$= z^p - \sum_{k=p+1}^{\infty}\frac{\beta|b|}{k-p+\beta|b|}X_k z^k$$

$$+ \sum_{k=p}^{\infty}\frac{\beta|b|}{k+p-\beta|b|}Y_k \bar{z}^k. \qquad (3.10)$$

Using (2.4), we get

$$\sum_{k=p+1}^{\infty}\frac{k-p+\beta|b|}{\beta|b|}|a_k| + \sum_{k=p}^{\infty}\frac{k+p-\beta|b|}{\beta|b|}|b_k|$$

$$= \sum_{k=p+1}^{\infty}\frac{k-p+\beta|b|}{\beta|b|}\cdot\frac{\beta|b|}{k-p+\beta|b|}X_k$$

$$+ \sum_{k=p}^{\infty}\frac{k+p-\beta|b|}{\beta|b|}\cdot\frac{\beta|b|}{k+p-\beta|b|}Y_k$$

$$= \sum_{k=p+1}^{\infty}X_k + \sum_{k=p}^{\infty}Y_k = \sum_{k=p}^{\infty}(X_k + Y_k) - X_p \leq 1,$$

then $f \in \overline{S}_H(b, p, \beta)$.

Conversely, if $f \in \overline{S}_H(b, p, \beta)$, let

$$|a_k| = \frac{\beta|b|}{k-p+\beta|b|}X_k \; (k = p+1, p+2, \dots) \qquad (3.11)$$

and

$$|b_k| = \frac{\beta|b|}{k+p-\beta|b|}Y_k \; (k = p, p+1, \dots), \qquad (3.12)$$

where $\sum_{k=p}^{\infty}(X_k + Y_k) = 1$.. Then, we have

$$f(z) = z^p - \sum_{k=p+1}^{\infty}|a_k|z^k + \sum_{k=p}^{\infty}|b_k|\bar{z}^k;$$

$$= z^p - \sum_{k=p+1}^{\infty}\frac{\beta|b|}{k-p+\beta|b|}X_k z^k + \sum_{k=p}^{\infty}\frac{\beta|b|}{k+p-\beta|b|}Y_k \bar{z}^k$$

$$= z^p + \sum_{k=p+1}^{\infty}(h_k(z) - z^p)X_k + \sum_{k=p}^{\infty}(g_k(z) - z^p)Y_k$$

$$= \sum_{k=p}^{\infty}(X_k h_k + Y_k g_k).$$

This completes the proof of Theorem 4.

Putting $p = 1$ in Theorem 4, we obtain the following corollary:

Corollary 5. Let t $f(z)$ be given by (1.3) with $p = 1$. Then $f \in \overline{\mathrm{H}}\mathrm{S}^*(\mathrm{b}, \beta)$ if and only if $f(z) = \sum_{k=1}^{\infty}(X_k h_k + Y_k g_k)$, where

$$h_1(z) = z, h_k(z) = z - \frac{\beta|b|}{k-1+\beta|b|}z^k \; (k = 2,3, \dots), \qquad (3.13)$$

and

$$g_k(z) = z + \frac{\beta|b|}{k+1-\beta|b|}\bar{z}^k \; (k = 1,2, ...)$$

$$\left(X_k \geq 0; Y_k \geq 0; \sum_{k=1}^{\infty}(X_k + Y_k) = 1\right). \qquad (3.14)$$

Putting $b = (p-\alpha)cos\lambda e^{-i\lambda}$ $(0 \leq \alpha < p, |\lambda| < \frac{\pi}{2})$ and $\beta = 1$ in Theorem 4, we obtain the following corollary:

Corollary 6. Let $f(z)$ be given by (1.3). Then $f \in \overline{\mathrm{S}}_{\mathrm{H}}^{\lambda}(\mathrm{p}, \alpha)$ if and only if $f(z) = \sum_{k=p}^{\infty}(X_k h_k + Y_k g_k)$, where

$h_p(z) = z^p, h_k(z)$

$$= z^p - \frac{(p-\alpha)cos\lambda}{k-p+(p-\alpha)cos\lambda} z^k \ (k = p+1, p+2, \ldots), \quad (3.15)$$

and

$$g_k(z) = z^p + \frac{(p-\alpha)cos\lambda}{k+p-(p-\alpha)cos\lambda} \bar{z}^k \ (k = p, p+1, \ldots)$$

$$\left(X_k \geq 0; Y_k \geq 0; \sum_{k=p}^{\infty} (X_k + Y_k) = 1 \right). \quad (3.16)$$

For harmonic functions of the form:

$$f(z) = z^p - \sum_{k=p+1}^{\infty} |a_k| z^k + \sum_{k=p}^{\infty} |b_k| \bar{z}^k \quad (3.17)$$

and

$$F(z) = z^p - \sum_{k=p+1}^{\infty} |A_k| z^k + \sum_{k=p}^{\infty} |B_k| \bar{z}^k, \quad (3.18)$$

we define the convolution of two harmonic functions f and F by

$$(f * F)(z) = z^p - \sum_{k=p+1}^{\infty} |a_k A_k| z^k + \sum_{k=p}^{\infty} |b_k B_k| \bar{z}^k. \quad (3.19)$$

Thereom 5. For $0 < \gamma \leq \beta \leq 1$, let $f \in \overline{S}_H(b, p, \beta)$ and $F \in \overline{S}_H(b, p, \gamma)$. Then $f * F \in \overline{S}_H(b, p, \beta) \subset \overline{S}_H(b, p, \gamma)$.

Proof. Let the convolution $f * F$ be of the form (3.19), then we want to prove that the coefficient of $f * F$ satisfy the condition of Theorem 2. Since $F \in \overline{S}_H(b, p, \gamma)$ we note that $|A_k| \leq 1$ and $|B_k| \leq 1$. Then we have

$$\sum_{k=p+1}^{\infty} \frac{k-p+\gamma|b|}{\gamma|b|} |a_k A_k| + \sum_{k=p}^{\infty} \frac{k+p-\gamma|b|}{\gamma|b|} |b_k B_k|$$

$$\leq \sum_{k=p+1}^{\infty} \frac{k-p+\gamma|b|}{\gamma|b|} |a_k| + \sum_{k=p}^{\infty} \frac{k+p-\gamma|b|}{\gamma|b|} |b_k|$$

$$\leq \sum_{k=p+1}^{\infty} \frac{k-p+\beta|b|}{\beta|b|} |a_k| + \sum_{k=p}^{\infty} \frac{k+p-\beta|b|}{\beta|b|} |b_k| \leq 1,$$

since $0 < \gamma \leq \beta \leq 1$ and $f \in \overline{S}_H(b, p, \beta)$. Therefore $f * F \in \overline{S}_H(b, p, \beta) \subset \overline{S}_H(b, p, \gamma)$, which completes the proof of Theorem 5.

Now we want to prove that the class $\bar{S}_H(b, p, \beta)$ is closed under convex combinations.

Thereom 6. Let $0 \leq c_i \leq 1$ for $i = 1, 2, \ldots$ and $\sum_{i=1}^{\infty} c_i = 1$. If the functions $f_i(z)$ defined by

$$f_i(z) = z^p - \sum_{k=p+1}^{\infty} |a_{k,i}| z^k$$

$$+ \sum_{k=p}^{\infty} |b_{k,i}| \bar{z}^k \ (z \in U; i = 1, 2, 3, \ldots), \quad (3.20)$$

are in the class $\bar{S}_H(b, p, \beta)$ for every $i = 1, 2, 3, \ldots$, then $\sum_{i=1}^{\infty} c_i f_i(z)$ of the form

$$\sum_{i=1}^{\infty} c_i f_i(z) = z^p - \sum_{k=p+1}^{\infty} \left(\sum_{i=1}^{\infty} c_i |a_{k,i}| \right) z^k$$

$$+ \sum_{k=p}^{\infty} \left(\sum_{i=1}^{\infty} c_i |b_{k,i}| \right) \bar{z}^k \quad (3.21)$$

is in the class $\bar{S}_H(b, p, \beta)$.

Proof. Since $f_i(z) \in \overline{S}_H(b, p, \beta)$, it follows from Theorem 2 that

$$\sum_{k=p+1}^{\infty} \frac{k-p+\beta|b|}{\beta|b|} |a_{k,i}|$$

$$+ \sum_{k=p}^{\infty} \frac{k+p-\beta|b|}{\beta|b|} |b_{k,i}| \leq 1, \quad (3.22)$$

for every $i = 1, 2, 3, \ldots$. Hence

$$\sum_{k=p+1}^{\infty} \left(\frac{k-p+\beta|b|}{\beta|b|} \sum_{i=1}^{\infty} c_i |a_{k,i}| \right)$$

$$+ \sum_{k=p}^{\infty} \left(\frac{k+p-\beta|b|}{\beta|b|} \sum_{i=1}^{\infty} c_i |b_{k,i}| \right)$$

$$= \sum_{i=1}^{\infty} c_i \left(\sum_{k=p+1}^{\infty} \frac{k-p+\beta|b|}{\beta|b|} |a_{k,i}| \right.$$

$$\left. + \sum_{k=p}^{\infty} \frac{k+p-\beta|b|}{\beta|b|} |b_{k,i}| \right) \leq \sum_{i=1}^{\infty} c_i \leq 1.$$

By Theorem 2, it follows that $\sum_{i=1}^{\infty} c_i f_i(z) \in \bar{S}_H(b,p,\beta)$. This proves that $\bar{S}_H(b,p,\beta)$ is closed under convex combinations.

Remarks. (i) The results in Corollaries 1, 3 and 5, respectively, correct the results obtained by Janteng [4, Theorem 2.1, 2.2 and 2.3, respectively];

(ii) Putting $b = 1$ in the above results, we obtain the corresponding results for the class $\bar{S}_H(p, \beta)$.

4. REFERENCES

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