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Original Article

Indexable and Strongly Indexable Graphs

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Abstract: Let $G = (V, E)$ be an (n, m) graph. G is said to be strongly indexable if there exists a bijection $f : V \rightarrow \{0,1,2, \ldots \ldots, n-1\}$, such that $f^{+}(E) = \{1,2, \ldots, m\}$, where $f^{+}(uv) = f(u) + f(v)$ for any edge $uv \in E$. G is said to be indexable if f^{+} is injective on E. In this paper we construct classes of indexable graphs, and we give an upper bound for the number of edges of any graph on n vertices to be indexable. Also, we determine all indexable graphs of order ≤ 6.

Keywords: Indexable graph / labeling, Strongly Indexable graph, C++ programming language

1. INTRODUCTION

By a graph G we mean a finite, undirected, connected graph without loops or multiple edges. We denote by n and m the order and size of the graph G. Terms not defined here are used in the sense of Harary [4]. The concepts of indexable and strongly indexable graphs were introduced in 1990 by Acharya and Hegde [1,2]. They call a graph with n vertices and m edges (k, d)-indexable if there is an injective function from V to $\{0, 1, 2, \ldots, n-1\}$ such that the set of edge labels induced by adding the vertex labels is a subset of $\{k, k+d, k+2d, \ldots, k+m(d-1)\}$. When the set of edges is $\{k, k+d, k+2d, \ldots, k+m(d-1)\}$ the graph is said to be strongly (k, d)-indexable. A $(k, 1)$-graph is more simply called k-indexable and strongly 1-indexable graphs are simply called strongly indexable.

An example of a strongly indexed graph is given in Fig. 1.1.

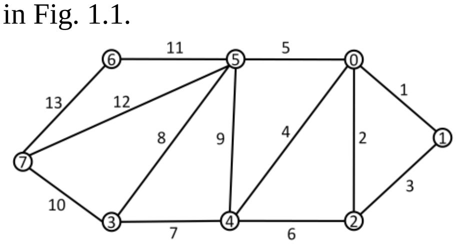

Fig. 1.1

Acharya and Hegde [1] call a graph with n vertices indexable if there is an injective labeling of the vertices with labels from $\{0, 1, 2, \ldots, n-1\}$ such that the edge labels induced by addition of the vertex labels are distinct. They conjecture that all unicyclic graphs are indexable. This conjecture was proved by Arumugam and Germina [3] who also proved that all trees are indexable.

An example of an indexable graph is displayed in Fig. 1.2.

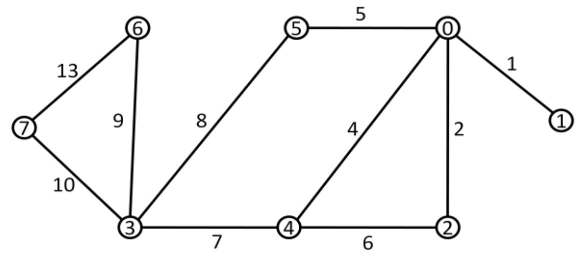

Fig. 1.2

Hegde [5] has shown the following: every graph can be embedded as an induced subgraph of an indexable graph; if a connected graph with n vertices and m edges $(m \geq 2)$ is (k, d)-indexable, then $d \leq 2$; if G is a connected (1,2)-indexable graph, then G is a tree; the minimum degree of any $(k, 1)$-indexable graph with at least two vertices is at most 3; a caterpillar with partite sets of orders a and b is strongly (1,2)-indexable if and only if

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$|a-b| \leq 1$. Hegde and Shetty [6] also prove that if G is strongly k-indexable Eulerian graph with m edges then $m \equiv 0, 3 \ (mod\ 4)$ if k is even and $m \equiv 0, 1 \ (mod\ 4)$ if k is odd.

2. INDEXABLE AND STRONGLY INDEXABLE GRAPHS

Theorem 2.1: The graph obtained from the Fan $F_n = P_n + K_1$ by inserting one vertex between every two consecutive vertices of the Path P_n is indexable.

Proof: This graph has the vertex v_0 and the set of vertices $\{v_1, v_2, v_3, \dots, v_{2n-1}\}$, such that $|V| = 2n$ vertices and $|E| = 3n-2$ edges.

We define the labeling function $f: V \to \{0,1,2, \dots \dots, 2n-1\}$ as follows:

$f(v_0) = 0$

$f(v_{2j-1}) = j; \qquad 1 \leq j \leq n$

$f(v_{2j}) = n + j; \qquad 1 \leq j \leq n-1$.

It is obvious that f is injective. An indexable labeling of the graph defined in Theorem 2.1 with n = 5 is shown in Fig. 2.1.

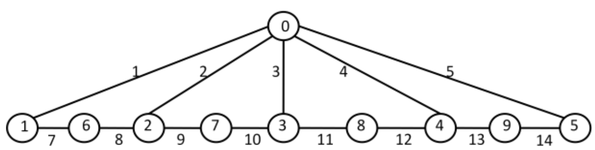


Fig. 2.1

Theorem 2.2: The one point union of m copies of the complete bipartite graph $K_{2,n}^{(m)}$ is indexable.

Proof: This graph has the set of vertices $\{v_0; v_1, v_1^1, v_2^1, \dots, v_n^1; v_2, v_1^2, v_2^2, \dots, v_n^2; \dots; v_m,$

$v_1^m, \dots, v_n^m\}$, such that $|V| = nm + m + 1$ vertices and $|E| = 2nm$ edges.

We define the labeling function $f: V \to \{0,1,2, \dots \dots, nm + m\}$ as follows:

$f(v_0) = 0$

$f(v_j) = mn + j \ ; \qquad 1 \leq j \leq m$,

$f(v_i^j) = (j-1)n + i \ ; \quad 1 \leq i \leq n, \ 1 \leq j \leq m$

It is clear that f is injective. An indexable labeling of $K_{2,n}^{(m)}$ with $n = 4$ and $m = 3$ is shown in Fig. 2.2.

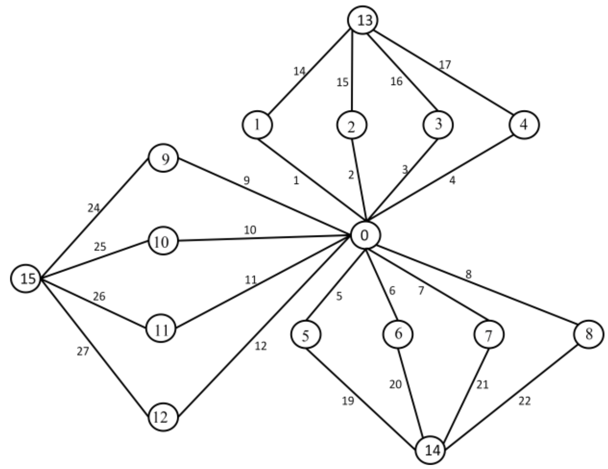


Fig. 2.2

Theorem 2.3: The Cartesian product $P_n \times P_m$ is indexable for $n, m \geq 2$.

Proof: This graph has the set of vertices $\{v_1^1, v_2^1, \dots, v_m^1; v_1^2, v_2^2, \dots, v_m^2; \dots; v_1^n, v_2^n, \dots, v_m^n\}$, such that $|V| = nm$ vertices and $|E| = n(m-1) + m(n-1)$ edges.

We define the labeling function $f: V \to \{0,1,2, \dots \dots, nm-1\}$ as follows:

if $> m$:

$f(v_1^1) = 0$

$f(v_1^{i+1}) = i + f(v_1^i) \ ; \quad 1 \leq i \leq m$

$f(v_1^{i+1}) = m + f(v_1^i) \ ; \qquad 1 + m \leq i \leq n-1$

$f(v_{j+1}^n) = f(v_j^n) + m - j + 1 \ ; 1 \leq j \leq m-1$

$f(v_{j+1}^i) = f(v_j^{i+1}) + 1 \ ;$

$1 \leq i \leq n-1, \ 1 \leq j \leq m-1$

if $= m$:

$f(v_1^1) = 0$

$f(v_1^{i+1}) = i + f(v_1^i) \ ; \quad 1 \leq i \leq n-1$

$f(v_{j+1}^n) = f(v_j^n) + m - j + 1 \ ; \quad 1 \leq j \leq m-1$

$f(v_{j+1}^i) = f(v_j^{i+1}) + 1 \ ;$

$1 \leq i \leq n-1, \ 1 \leq j \leq m-1$

Then one can verify that f is injective. An indexable labelings of $P_4 \times P_4$ and $P_6 \times P_4$ are shown in Fig. 2.3.

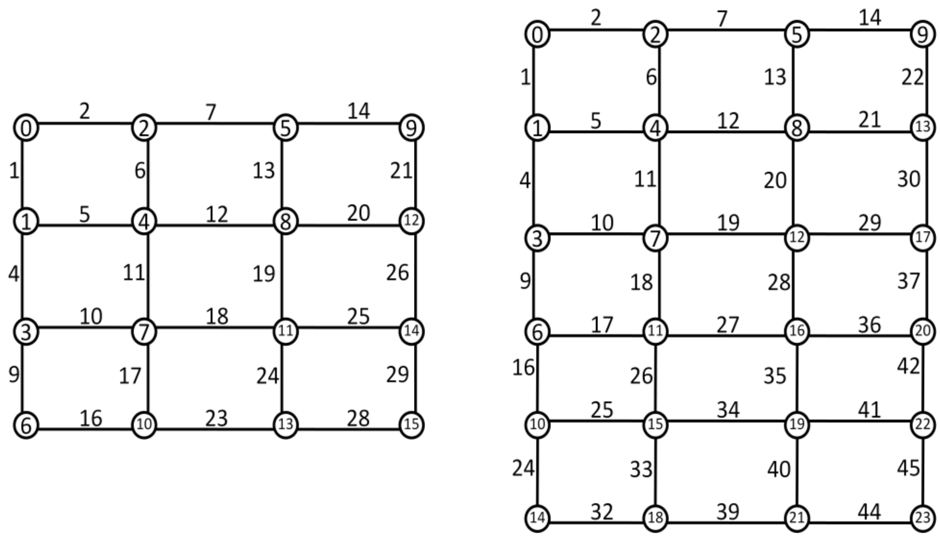


Fig. 2.3

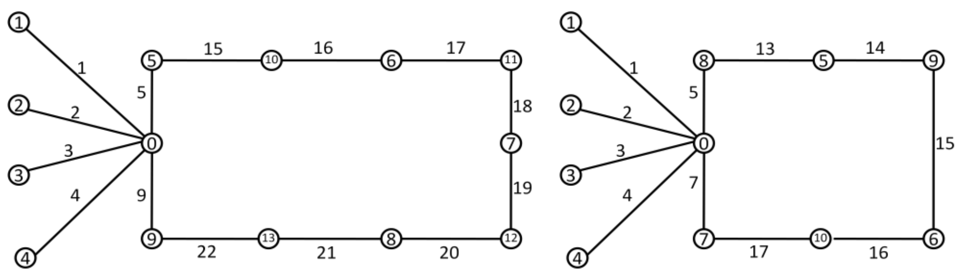


Fig. 2.4

Theorem 2.4: The graph $C_n \circ K_{1,m}$ obtained by identifying a vertex of C_n with the centre of $K_{1,m}$ is indexable.

Proof: This graph has the set of vertices $\{v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_m\}$, such that $|V| = n + m$ vertices and $|E| = n + m$ edges.

We define the labeling function $f : V \to \{0,1,2, \dots \dots, n+m-1\}$ as follows:

$f(v_1) = 0$

$f(u_i) = i\,;$ $\quad 1 \le i \le m$

if n is even:

$f(v_{2j}) = m + j;$ $\quad 1 \le j \le \frac{n}{2},$

$f(v_{2j-1}) = \frac{n}{2} + m + j - 1\,;$ $\quad 2 \le j \le \frac{n}{2}\,,$

if n is odd:

$f(v_{2j-1}) = m + j - 1;$ $\quad 2 \le j \le \frac{n+1}{2},$

$f(v_{2j}) = m + \frac{n-1}{2} + j;$ $\quad 1 \le j \le \frac{n-1}{2}.$

Then one can verify that f is injective. Indexable labelings of $C_{10} \circ K_{1,4}$ and $C_7 \circ K_{1,4}$ are shown in Fig. 2.4.

Theorem 2.5: All Paths P_n are strongly $\left\lceil \frac{n}{2} \right\rceil$-indexable.

Proof: This graph has the set of vertices $\{v_0, v_1, v_2, \dots, v_{n-1}\}$, such that $|V| = n$ vertices and $|E| = n - 1$ edges. We define the labeling function $f : V \to \{0,1,2, \dots \dots, n-1\}$ as follows:

$f(v_{2i}) = i;$ $\quad 0 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor\,,$

$f(v_{2i-1}) = \left\lfloor \frac{n-1}{2} \right\rfloor + i;$ $\quad 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor\,.$

And this will induce an edge labeling function $f^+(E) = \{\left\lceil \frac{n}{2} \right\rceil, \left\lceil \frac{n}{2} \right\rceil + 1, \left\lceil \frac{n}{2} \right\rceil + 2, \dots \dots, \left\lceil \frac{n}{2} \right\rceil + (n-2)\}$ since:

$f^+(v_{2i}v_{2i+1}) = f(v_{2i}) + f(v_{2i+1}) = 2i + \left\lfloor \frac{n-1}{2} \right\rfloor + 1\,;$ $\quad 0 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor\,.$

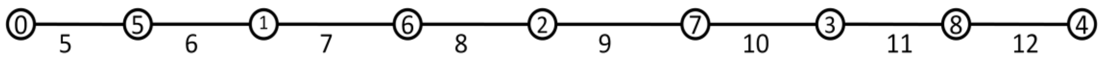


Fig. 2.5

$f^{+}(v_{2i-1}v_{2i}) = f(v_{2i-1}) + f(v_{2i}) = 2i + \left\lfloor\frac{n-1}{2}\right\rfloor$; $\quad 1 \le i \le \left\lfloor\frac{n-1}{2}\right\rfloor$

A strongly 5-indexable labeling of P_9 with $n = 9$ and $m = 8$ using the last theorem is shown in Fig. 2.5.

Theorem 2.6: All odd circuits C_n are strongly $\left\lfloor\frac{n}{2}\right\rfloor$-indexable.

Proof: This graph has the set of vertices $\{v_0, v_1, v_2, \dots, v_{n-1}\}$, such that $|V| = n$ vertices and $|E| = n$ edges. We define the labeling function $f : V \to \{0,1,2, \dots\dots, n-1\}$ as follows:

$f(v_{2i}) = i; \qquad 0 \le i \le \left\lfloor\frac{n}{2}\right\rfloor$,

$f(v_{2i-1}) = \left\lfloor\frac{n}{2}\right\rfloor + i; \qquad 1 \le i \le \left\lfloor\frac{n}{2}\right\rfloor$.

And this will induce an edge labeling function $f^{+}(E) = \{\left\lfloor\frac{n}{2}\right\rfloor, \left\lfloor\frac{n}{2}\right\rfloor + 1, \left\lfloor\frac{n}{2}\right\rfloor + 2, \dots\dots, \left\lfloor\frac{n}{2}\right\rfloor + (n-1)\}$ as follows:

$f^{+}(v_{n-1}v_0) = f(v_{n-1}) + f(v_0) = \left\lfloor\frac{n}{2}\right\rfloor$,

$f^{+}(v_{2i}v_{2i+1}) = f(v_{2i}) + f(v_{2i+1}) = 2i + \left\lfloor\frac{n}{2}\right\rfloor + 1$; $\quad 0 \le i \le \left\lfloor\frac{n-3}{2}\right\rfloor$,

$f^{+}(v_{2i-1}v_{2i}) = f(v_{2i-1}) + f(v_{2i}) = 2i + \left\lfloor\frac{n}{2}\right\rfloor$; $\quad 1 \le i \le \left\lfloor\frac{n}{2}\right\rfloor$.

A strongly 6-indexable labeling of C_{13} with n = 1`3, using the last theorem, is shown in Fig. 2.6.

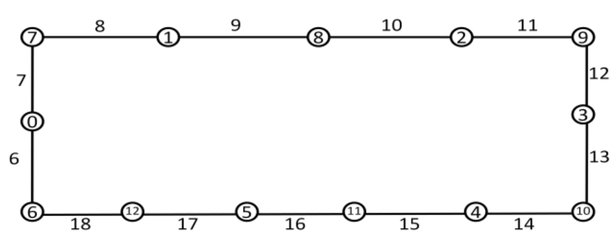


Fig. 2.6

We show that C_4 is not strongly k-indexable, hence even circuits are not necessarily strongly k-indexable for all k.

To label the vertices of C_4 with the labels $\{0,1,2,3\}$: The vertex labeled 0 is either adjacent to the vertex labeled 3, and consequently we have two edges labeled 3, or the vertex labeled 0 is not adjacent to the vertex labeled 3. In the later case we have the edges labeled $\{1,2,4,5\}$ and C_4 is not strongly k-indexable for all k.

Theorem 2.7: Let G be a simple graph on n vertices. Then the upper bound of the number of edges of G to be indexable is $2n - 3$.

Proof: By using induction on the number of vertices of G.

If $n = 2$, then the upper bound for the number of edges is $2 \times 2 - 3 = 1$, which is true.

Let the upper bound of the number of edges of G on n vertices be $2n - 3$. Then by adding one vertex to G which will be labeled n, this vertex could be joined to the vertices labeled $n - 1$ and $n - 2$, but not to the vertex labeled $n - 3$, since the vertices labeled $n - 1$ and $n - 2$ give the edge label $2n - 3$. Hence the upper bound of the number of edges will be $= 2n - 3 + 2 = 2(n + 1) - 3$.

Corollary 2.8: All complete bipartite graphs $K_{n,m}$, where $n, m > 2$, are not indexable. If $n \le 2$, *or* $m \le 2$, they are indexable.

Proof: Let the number of vertices be k. Then by dividing the vertices into two sets with orders $k - 2$ and 2, the number of edges will be $(k - 2)2 = 2k - 4 < 2k - 3$, then by the last theorem, the number of edges is less than the upper bound and $K_{n,m}$ is indexable by the following labeling function:

$f(v_0) = 0$,

$f(v_1) = k - 1$,

$f(u_i) = i$; $\qquad 1 \le i \le k - 2$.

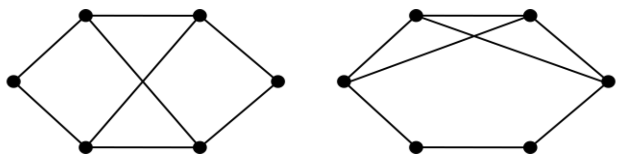
But when dividing the vertices into two sets with orders n and $(n \ge 3, m > 3 \text{ or } n > 3, m \ge 3)$, the number of edges will be greater than the upper bound, which makes the graph not indexable.

In the case $n, m = 3$, (i.e. $K_{3,3}$), using the algorithm in section 3, it is clear that $K_{3,3}$ is not indexable.

Theorem 2.9:

(1) All graphs with $n \leq 5$ are indexable except those which have $m > 2n - 3$,

(2) All graphs with $n = 6$ and $m \leq 8$ are indexable except the following two graphs:



(3) For $n = 6$ and $m = 9$, only the following graphs are indexable (They are strongly indexable):

(4) All graphs having $n = 6$ and $m > 9$ are not indexable.

Proof: Using the Algorithm in section 3 and Theorem 2.7, one can prove (1),

using the Algorithm in section 3, one can prove (2), (3).

And (4) is proved using Theorem 2.7.

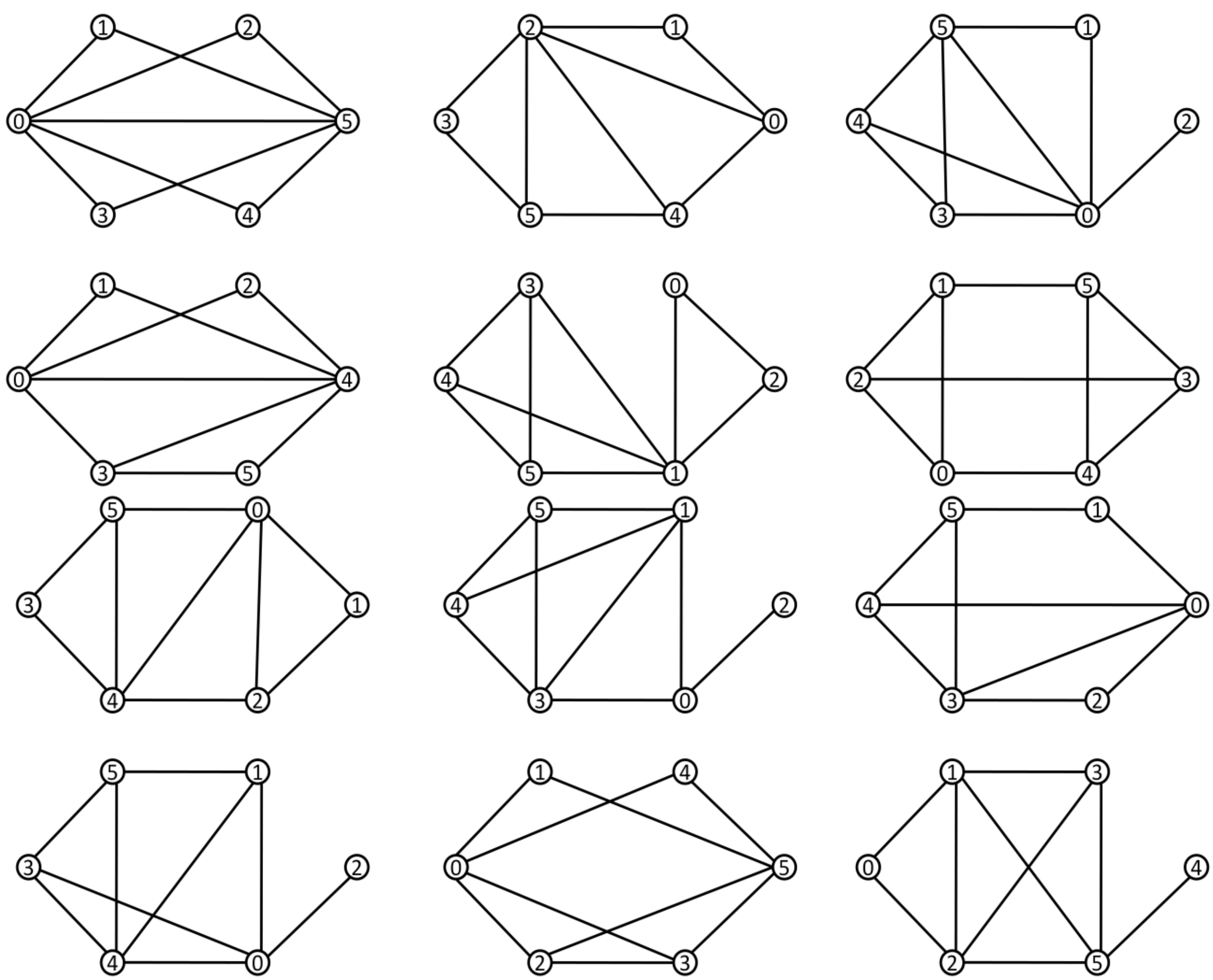
3. CHECK INDEXABLE GRAPH ALGORITHM

We make an algorithm to test any (n, m) graph whether it is indexable or not, and also give all possible indexable labelings for this graph using the labeling function $f : V(G) \rightarrow \{0,1,2, \ldots \ldots, n - 1\}$ as follows:

Given the number of vertices N, the number of edges M and the vertices adjacent to each edge:

INPUT The number of vertices N, the number of edges M of the graph.

OUTPUT State whether the graph is indexable or not and display the vertex labelings if it is indexable.



Step 1 Set *v;* (array with length N stores the labels of the vertices)
v1; (array with length M stores the labels of the first vertex adjacent to each edge)
v2; (array with length M stores the labels of the second vertex adjacent to each edge)
E ; (array with length M stores the calculated labels of all edges)
Step 2: Enter the adjacent vertices to each edge (*v1* and *v2*);
Step 3: Initialize v = [0 1 2 … N-1];
x = 0; (used to count the number of indexable labelings of all the permutations of the vector v)
Step 4: Initialize *FLAG = 0;* (used to decide whether to display *v* or not)
Step 5: Calculate the label of each edge : *E = v[v1] + v[v2]*;
Step 6: FOR *index i = 1:M* (check whether the graph is indexable or not)
Step 7: FOR index j = 1:i
Step 8: If E[i] = E[j] then
Set FLAG = 1;
permute *v*; (make another permutation of the vector v)
Go to step *4*
Step 9: If FLAG = 0, then

OUTPUT (v); (display the labels of the vertices)
x = x+ 1;
Step 10: permute *v*; (make another permutation of the vector v)
Step 11: Go to step 4
Step 12: If x = 0 then

OUTPUT (NO INDEXABLE LABLING FOR THIS GRAPH);

Step 13: STOP;

We implement this algorithm using C++ programming language.

4. ACKNOWLEDGMENTS

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