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Original Article

Some Inclusion Properties of Certain Operators

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Abstract: In this paper we introduce several new subclasses of analytic $p-$vhalent functions which are defined by means of a general integral operators $I_{\lambda,\,p}(a,b,c)\;(a,b,c\in\Re\setminus Z_0^-,\;\lambda>-p,\;\;p\in\mathrm{N})$ and investigate various inclusion properties of these subclasses. Many interesting applications involving these and other families of $p-$ valent operators are also considered.

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1. INTRODUCTION

Let $A(p)$ denote the class of functions of the form:

$$f(z)=z^p+\sum_{k=1}^{\infty}a_{k+p}z^{k+p}\quad(p\in\mathrm{N}=\{1,2,...\}),\tag{1.1}$$

which are analytic and $p-$valent in the open unit disc $U=\{z:|z|<1\}$. A function $f(z)\in A(p)$ is said to be in the class $S_p^*(\alpha)$ of $p-$valently starlike of order α, if it satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha\quad(0\le\alpha<p;z\in U).\tag{1.2}$$

We write $S_p^*(0)=S_p^*$, the class of $p-$valently starlike in U. A function $f(z)\in A(p)$ is said to be in the class $K_p(\alpha)$ of $p-$valently convex of order α, if it satisfies

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha\quad(0\le\alpha<p;z\in U).\tag{1.3}$$

It follows form (1.2) and (1.3) that

$$f(z)\in K_p(\alpha)\Leftrightarrow\frac{zf'(z)}{p}\in S_p^*(\alpha)\;\;(0\le\alpha<p).\tag{1.4}$$

The classes $S_p^*(\alpha)$ and $K_p(\alpha)$ were studied by Owa [1] and Patil and Thakare [2].

Furthermore, a function $f(z)\in A(p)$ is said to be $p-$valently close-to-convex functions of order β and type γ in U, if there exists a function $g(z)\in S_p^*(\gamma)$ such that

$$\mathrm{Re}\left\{\frac{zf'(z)}{g(z)}\right\}>\beta\quad(0\le\beta,\gamma<p;z\in U).\tag{1.5}$$

We denote by $B_p(\beta,\gamma)$, the subclass of $A(p)$ consisting of all such functions. The class $B_p(\beta,\gamma)$ was studied by Aouf [3].

Suppose that $f(z)$ and $g(z)$ are analytic in U. Then we say that the function $g(z)$ is subordinate to $f(z)$ if there exists an analytic

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function $w(z)$ in U with $|w(z)| \le |z|$ for all $z \in U$, such that $g(z) = f(w(z))$, denoted $g \prec f$ of $g(z) \prec f(z)$. In case $f(z)$ is univalent in U we have that the subordination $g(z) \prec f(z)$ is equivalent to $g(0) = f(0)$ and $g(U) \subset f(U)$ (see [4]; see also [5],[6, p. 4]).

For the functions $f_j(z)\ (j = 1,2)$ defined by

$$f_j(z) = z^p + \sum_{k=1}^{\infty} a_{k+p,j} z^{k+p} \quad (p \in \mathrm{N}) \tag{1.6}$$

we denote the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p,1} a_{k+p,2} z^{k+p} . \tag{1.7}$$

Let M be the class of analytic functions $\varphi(z)$ in U normalized by $\varphi(0) = 1$, and let S be the subclass of M consisting of those functions $\varphi(z)$ which are univalent in U and for which $\varphi(U)$ is convex and $\mathrm{Re}\{\varphi(z)\} > 0\ (z \in U)$.

Making use of the principle of subordination between analytic functions, we introduce the subclasses $S_p^*(\varphi)$, $K_p(\varphi)$ and $C_p(\varphi,\psi)$ of the class $A(p)$ for $\varphi,\ \psi \in S$, which are defined by

$$S_p^*(\varphi) = \left\{ f \ :\ f \in A(p) \text{ and } \frac{zf'(z)}{pf(z)} \prec \varphi(z) \text{ in } U \right\},$$

$$K_p(\varphi) = \left\{ f \ :\ f \in A(p) \text{ and } \frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \text{ in } U \right\},$$

$$C_p(\phi,\psi) = \left\{ \begin{array}{l} f \ :\ f \in A(p) \text{ and } \exists\, h \in \\ K_p(\phi) \text{ s.t. } \dfrac{f'(z)}{h'(z)} \prec \psi(z) \text{ in } U \end{array} \right\} .$$

We note that for $p = 1$, the classes $S_1^*(\varphi) = S^*(\varphi)$, $K_1(\varphi) = K(\varphi)$ and $C_1(\varphi, \psi = C(\varphi,\psi)$ are investigated by Ma and Minda [7] and Kim et al [8].

Obviously, for special choices for the functions φ and ψ involved in the above definitions, we have the following relationships:

$$S_p^*\left(\frac{1+z}{1-z}\right) = S_p^* \ ,$$

$$S_p^*\left(\frac{p+(p-2\alpha)z}{1-z}\right) = S_p^*(\alpha)\ (0 \le \alpha < p),$$

$$K_p\left(\frac{1+z}{1-z}\right) = K_p \ ,$$

$$K_p\left(\frac{p+(p-2\alpha)z}{1-z}\right) = K_p(\alpha)\ (0 \le \alpha < p),$$

$$C_p\left(\frac{1+z}{1-z}, \frac{1+z}{1-z}\right) = C_p \ ,$$

$$C_p\left(\frac{p+(p-2\gamma)z}{1-z}, \frac{p+(p-2\alpha)z}{1-z}\right) = C_p(\beta,\gamma)\ (0 \le \beta,\gamma < p).$$

Furthermore, for the function classes $S_p^*[A,B,\alpha]$ and $K_p[A,B,\alpha]$ investigated by Aouf ([9, 10], it is easily seen that

$$S_p^*\left(\frac{1+[B+(A-B)(1-\frac{\alpha}{p})]}{1+Bz}\right) = S_p^*[A,B,\alpha]\ (-1 \le B < A \le 1; 0 \le \alpha < p)$$

(see Aouf [9]),

And

$$K_p\left(\frac{1+[B+(A-B)(1-\frac{\alpha}{p})]}{1+Bz}\right) = K_p[A,B,\alpha]\ (-1 \le B < A \le 1; 0 \le \alpha < p)$$

(see Aouf [10]).

For real or complex number a,b,c other than $0,-1,-2,\ldots$, the hypergeometric series is defined by

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k (1)_k} z^k \ , \tag{1.8}$$

where $(x)_k$ is Pochhammer symbol defined by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = \begin{cases} x(x+1)\ldots(x+k-1) & (k \in N; x \in C), \\ 1 & (k = 0; k \in C \setminus \{0\}). \end{cases}$$

We note that the series (1.8) converges absolutely for all $z \in U$ so that it represents an analytic function in U (see, for details, [11, Chapter 14]).

Now we set

$$f_{\lambda,p}(z) = \frac{z^p}{(1-z)^{\lambda+p}} \quad (\lambda > -p) \tag{1.9}$$

and define $f_{\lambda,p}(z)$ by means of the Hadamard product

$$f_{\lambda,p}(z) * f_{\lambda,p}^{(-1)}(z) = z^p\ {}_2F_1(a,b;c;z) \quad (z \in U), \tag{1.10}$$

This leads us to a family of linear operators

$$I_{\lambda,p}(a,b,c) = f_{\lambda,p}^{(-1)}(z) * f(z) \quad (a,b,c \in R\setminus Z_0^-,\ \lambda > -p,\ p \in N, z \in U). \tag{1.11}$$

After some computations, we obtain

$$I_{\lambda,p}(a,b,c)f(z) = z^p + \sum_{k=1}^{\infty} \frac{(a)_k (b)_k}{(c)_k (\lambda+p)_k} a_{k+p} z^{k+p}. \tag{1.12}$$

From (1.12), we deduce that

$$I_{\lambda,p}(a,\lambda+p,a)f(z) = f(z)\ (\lambda > -p, p \in N)$$

and

$$I_{1,p}(p+1,p+1,p)f(z) = \frac{zf'(z)}{p},$$

$$z(I_{\lambda+1,p}(a,b,c)f(z))' = (\lambda+p)I_{\lambda,p}(a,b,c)f(z) - \lambda I_{\lambda+1,p}(a,b,c)f(z)\ \ (\lambda > -p), \tag{1.13}$$

and

$$z(I_{\lambda,p}(a,b,c)f(z))' = aI_{\lambda,p}(a+1,b,c)f(z) - (a-p)I_{\lambda,p}(a,b,c)f(z). \tag{1.14}$$

We note that;

(i) $I_{n,p}(a,p+1,a)f(z) = I_{n+p-1}\ (n > -p)$, where I_{n+p-1} is the Noor integral operator of $(n+p-1)-th$ order (see Liu and Noor [12] and Patel and Cho [13]);

(ii) $I_{1,p}(p+1,n+p,1)f(z) = D^{n+p-1}f(z)\ (n > -p)$, where $D^{n+p-1}f(z)$ is the $(n+p-1)-th$ order Ruscheweyh derivative of a function $f(z) \in A(p)$ (see Kumar and Shukla [14]);

(iii) $I_{n,1}(a,2,a)f(z) = I_n f(z)\ (n > -1)$, where I_n is the Noor integral operator of $n-th$ order (see [15]);

(iv) $$I_{1-\lambda,p}(a,p+1,a)f(z) = \Omega_z^{(\lambda,p)}f(z) = z^p + \sum_{k=1}^{\infty} \frac{\Gamma(k+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+p+1-\lambda)} a_{k+p} z^{k+p} = z^p\ {}_2F_1(1,p+1;p+1-\lambda;z) * f(z) \quad (-\infty < \lambda < p+1; z \in U).$$

The operator $\Omega_z^{(\lambda,p)}$ was introduced and studied by Patel and Mishra [16]:

(v) $I_{\lambda,p}(\sigma+p,\lambda+p,\sigma+p+1)f(z) = J_{\sigma,p}f(z)\ (\sigma > -p)$, where $J_{\sigma,p}$ is the generalized Bernardi-Libera-Livingston operator defined by (3.1) (see [17]);

(vi) $I_{\lambda,1}(\mu,b,b)f(z) = I_{\lambda,\mu}f(z)\ (\lambda > -1, \mu > 0,\ f(z) \in A(1) = A)$, where $I_{\lambda,\mu}$ is the Choi-Saigo-Srivastava operator (see [17]).

We also note that:

$$I_{\lambda,p}(\mu,b,b)f(z) = I_{\lambda,\mu}^{p}f(z)\ \ (\lambda > -p, \mu > 0,\ f(z) \in A(p)),$$

where $I_{\lambda,\mu}^{p}$ is the generalized Choi-Saigo-Srivastava operator (see [17]) defined by

$$I_{\lambda,\mu}^{p}f(z) = z^p + \sum_{k=1}^{\infty} \frac{(\mu)_k}{(\lambda+p)_k} a_{k+p} z^{k+p}\ \ (\lambda > -p; \mu > 0; z \in U).$$

Next, by using the general operator $I_{\lambda,p}(a,b,c)$, we introduce the following classes of analytic $p-$valent functions for

$$S_{\lambda,p}^{*}(a,b,c;\phi) = \left\{ \begin{matrix} f\ :\ f \in A(p) \text{ and} \\ I_{\lambda,p}(a,b,c)f(z) \in S_p^{*}(\phi) \end{matrix} \right\},$$

$$K_{\lambda,p}(a,b,c;\phi) = \left\{ \begin{matrix} f\ :\ f \in A(p) \text{ and} \\ I_{\lambda,p}(a,b,c)f(z) \in K_p(\phi) \end{matrix} \right\},$$

And

$$C_{\lambda,p}(a,b,c;\phi,\psi) = \left\{ \begin{matrix} f\ :\ f \in A(p) \text{ and} \\ I_{\lambda,p}(a,b,c)f(z) \in C_p(\phi,\psi) \end{matrix} \right\}.$$

We also note that

$$f(z) \in K_{\lambda,p}(a,b,c;\varphi) \Leftrightarrow \frac{zf'(z)}{p} \in S_{\lambda,p}^{*}(a,b,c;\varphi). \tag{1.15}$$

In particular, we set

$$S_{n,p}^{*}\left(a,p+1,a;\frac{1+z}{1-z}\right) = S_{n+p-1}^{*}\ \ (n > -p),$$

$$S_{\lambda,p}^{*}\left(a,b,c;\frac{1+Az}{1+Bz}\right) = S_{\lambda,p}^{*}[a,b,c;A,B]\ \ (-1 \le B < A \le 1),$$

and

$$K_{\lambda,p}\left(a,b,c;\frac{1+Az}{1+Bz}\right)=K_{\lambda,p}[a,b,c;A,B]\ \ (-1\le B<A\le 1).$$

Inclusion properties was investigated by several authors (e.g. see [18], [19], [20] and [21]). In this paper, we investigate several inclusion properties of the classes $S^*_{\lambda,p}(a,b,c;\varphi)$, $K_{\lambda,p}(a,b,c;\varphi)$ and $C_{\lambda,p}(a,b,c;\varphi,\psi)$ associated with the general integral operator $I_{\lambda,p}(a,b,c)$. Some applications involving these and other families of integral operators also considered.

2 . INCLUSION PROPERTIES INVOLVING $I_{\lambda,p}$

To establish our main results, we shall need the following lemmas.

Lemma 1 [22]. *Let* h *be convex univalent in* U *with* $h(0)=1$ *and*
$\operatorname{Re}\{\beta h(z)+\mu\}>0\quad(\beta,\mu\in C)$.

If $q(z)$ is analytic in U with $q(0)=1$, then

$$q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}\prec h(z)\quad(z\in U)$$

implies that $q(z)\prec h(z)\ (z\in U)$.

Lemma 2 [23]. *Let* h *be convex in* U *with* $h(0)=1$. *Suppose also that* $Q(z)$ *is analytic in* U *with* $\operatorname{Re}\{Q(z)\}>0\,(z\in U)$. *If* $q(z)$ *is analytic in* U *with* $q(0)=1$, *then*

$$q(z)+Q(z)zq'(z)\prec h(z)\quad(z\in U)$$

implies that $q(z)\prec h(z)\ (z\in U)$.

Theorem 1. *Let* $\lambda>-p,\ a\ge p$ *and* $p\in\mathrm{N}$. *Then*

$$S^*_{\lambda,p}(a+1,b,c;\phi)\subset S^*_{\lambda,p}(a,b,c;\phi)$$
$$\subset S^*_{\lambda+1,p}(a,b,c;\phi)\ \ (\phi\in S).$$

Proof. First of all, we show that

$$S^*_{\lambda,p}(a+1,b,c;\phi)$$
$$\subset S^*_{\lambda,p}(a,b,c;\phi)\ \ (\phi\in S;\lambda>-p;a\ge p;p\in N).$$

Let $f(z)\in S^*_{\lambda,p}(a+1,b,c;\varphi)$ and set

$$\frac{z\left(I_{\lambda,p}(a,b,c)f(z)\right)'}{pI_{\lambda,p}(a,b,c)f(z)}=q(z)\,, \tag{2.1}$$

where $q(z)=1+q_1z+q_2z^2+\ldots$ is analytic in U and $q(z)\ne 0$ for all $z\in U$. Using the identity (1.14) in (2.1), we obtain

$$a\frac{I_{\lambda,p}(a+1,b,c)f(z)}{I_{\lambda,p}(a,b,c)f(z)}=pq(z)+a-p\,. \tag{2.2}$$

Differentiating (2.2) logarithmically with respect to z, we have

$$\frac{z\left(I_{\lambda,p}(a+1,b,c)f(z)\right)'}{I_{\lambda,p}(a+1,b,c)f(z)}=\frac{z\left(I_{\lambda,p}(a,b,c)f(z)\right)'}{I_{\lambda,p}(a,b,c)f(z)}+\frac{zq'(z)}{pq(z)+a-p} \tag{2.3}$$
$$=q(z)+\frac{zq'(z)}{pq(z)+a-p}.$$

Since $a\ge p,\ \varphi(z)\in S$, and $f(z)\in S^*_{\lambda,p}(a+1,b,c;\varphi)$, from (2.3) we see that

$$\operatorname{Re}\{p\varphi(z)+a-p\}>0\quad(z\in U)$$

and

$$q(z)+\frac{zq'(z)}{pq(z)+a-p}\prec\varphi(z)\quad(z\in U)$$

Thus, by using Lemma 1 and (2.1), we observe that
$q(z)\prec\varphi(z)\quad(z\in U)$,

so that
$f(z)\in S^*_{\lambda,p}(a,b,c;\varphi)$.

This implies that
$S^*_{\lambda,p}(a+1,b,c;\varphi)\subset S^*_{\lambda,p}(a,b,c;\varphi)$.

To prove the second part, let $f(z)\in S^*_{\lambda,p}(a,b,c;\phi)\quad(\lambda>-p;\ a\ge p;\ p\in\mathrm{N})$ and put

$$\frac{z\left(I_{\lambda+1,p}(a,b,c)f(z)\right)'}{pI_{\lambda+1,p}(a,b,c)f(z)}=g(z)\,,$$

where $g(z)=1+d_1z+d_2z^2+\ldots$ is analytic in U

and $g(z)\neq 0$ for all $z\in U$. Then, by using arguments similar to those detailed above with the identity (1.13), it follows that

$$g(z)\prec\varphi(z)\quad (z\in U),$$

which implies that $f(z)\in S^*_{\lambda+1,p}(a,b,c;\varphi)$. Hence we conclude that

$$S^*_{\lambda,p}(a+1,b,c;\varphi)\subset S^*_{\lambda,p}(a,b,c;\varphi)\subset S^*_{\lambda+1,p}(a,b,c;\varphi),$$

which completes the proof of Theorem 1.

Putting $\lambda=n,\ c=a,\ b=p+1$ and $\varphi(z)=\frac{1+z}{1-z}\ (z\in U)$ in Theorem 1, we obtain the following corollary.

Corollary 1. *Let* $n>-p$ *and* $p\in\mathrm{N}$. *Then* $S^*_{n+p-1}\subset S^*_{n+p}$.

Remark 1. *Putting* $p=1$ *in Corollary 1, we obtain the result obtained by Noor [15].*

Theorem 2. *Let* $\lambda>-p,\ a\geq p$ *and* $p\in\mathrm{N}$. *Then*
$$C_{\lambda,p}(a+1,b,c;\phi)\subset C_{\lambda,p}(a,b,c;\phi)$$
$$\subset C_{\lambda+1,p}(a,b,c;\phi)\ \ (\phi\in S).$$

Proof. Applying (1.15) and Theorem 1, we observe that

$$f(z)\in C_{\lambda,p}(a+1,b,c;\phi)$$
$$\Leftrightarrow I_{\lambda,p}(a+1,b,c)f(z)\in K_p(\phi)$$
$$\Leftrightarrow \tfrac{z}{p}(I_{\lambda,p}(a+1,b,c)f(z))'\in S^*_p(\phi)$$
$$\Leftrightarrow I_{\lambda,p}(a+1,b,c)\left(\frac{zf'(z)}{p}\right)\in S^*_p(\phi)$$

$$\Leftrightarrow \frac{zf'(z)}{p}\in S^*_{\lambda,p}(a+1,b,c;\phi)$$
$$\Rightarrow \frac{zf'(z)}{p}\in S^*_{\lambda,p}(a,b,c;\phi)$$
$$\Leftrightarrow I_{\lambda,p}(a,b,c)\left(\frac{zf'(z)}{p}\right)\in S^*_p(\phi)$$
$$\Leftrightarrow \tfrac{z}{p}\Big(I_{\lambda,p}(a,b,c)f(z)\Big)'\in S^*_p(\phi)$$
$$\Leftrightarrow I_{\lambda,p}(a,b,c)f(z)\in K_p(\phi)$$
$$\Leftrightarrow f(z)\in C_{\lambda,p}(a,b,c;\phi)$$

and

$$f(z)\in K_{\lambda,p}(a,b,c;\phi)$$
$$\Leftrightarrow \frac{zf'(z)}{p}\in S^*_{\lambda,p}(a,b,c;\phi)$$
$$\Rightarrow \frac{zf'(z)}{p}\in S^*_{\lambda+1,p}(a,b,c;\phi)$$
$$\Leftrightarrow \tfrac{z}{p}\Big(I_{\lambda+1,p}(a,b,c)f(z)\Big)'\in S^*_p(\phi)$$
$$\Leftrightarrow I_{\lambda+1,p}(a,b,c)f(z)\in K_p(\phi)$$
$$\Leftrightarrow f(z)\in K_{\lambda+1,p}(a,b,c;\phi),$$

which evidently proves Theorem 2.
Taking

$$\varphi(z)=\frac{1+Az}{1+Bz}\ \ (-1\leq B<A\leq 1;z\in U)$$

in Theorem 1 and 2, we have

Corollary 2. *Let* $\lambda>-p,\ a\geq p,\ p\in\mathrm{N}$ *and* $-1\leq B<A\leq 1$.
Then
$$S^*_{\lambda,p}[a+1,b,c;A,B]\subset S^*_{\lambda,p}[a,b,c;A,B]$$
$$\subset S^*_{\lambda+1,p}[a,b,c;A,B]$$

and
$$K_{\lambda,p}[a+1,b,c;A,B]\subset K_{\lambda,p}[a,b,c;A,B]$$
$$\subset K_{\lambda+1,p}[a,b,c;A,B].$$

Theorem 3. *Let* $\lambda>-p,\ a\geq p$ and $p\in\mathrm{N}$. Then
$$C_{\lambda,p}(a+1,b,c;\phi,\psi)\subset C_{\lambda,p}(a,b,c;\phi,\psi)$$
$$\subset C_{\lambda+1,p}(a,b,c;\phi,\psi)\ (\phi,\psi\in S).$$

Proof. We begin by proving that

$$C_{\lambda,p}(a+1,b,c;\phi,\psi)\subset C_{\lambda,p}(a,b,c;\phi,\psi)$$
$$(\lambda>-p;\ a\geq p;\ p\in\mathrm{N};\phi,\psi\in S).$$

Let $f(z)\in C_{\lambda,p}(a+1,b,c;\varphi,\psi)$. Then, in view of (1.7), there exists a function $h(z)\in S^*_p(\varphi)$ such that

$$\frac{z(I_{\lambda,p}(a+1,b,c)f(z))'}{ph(z)}\prec\psi(z)\quad (z\in U).$$

Choose the function $g(z)$ such that

$I_{\lambda,p}(a+1,b,c)g(z)=h(z)$. Then $g(z)\in S^*_{\lambda,p}(a+1,b,c;\varphi)$ and

$$\frac{z(I_{\lambda,p}(a+1,b,c)f(z))'}{pI_{\lambda,p}(a+1,b,c)g(z)}\prec\psi(z)\quad(z\in U). \tag{2.4}$$

Now let

$$=\frac{z\left(I_{\lambda,p}(a,b,c)\left(\frac{zf'(z)}{p}\right)\right)'+(a-p)I_{\lambda,p}(a,b,c)\left(\frac{zf'(z)}{p}\right)}{z\left(I_{\lambda,p}(a,b,c)g(z)\right)'+(a-p)I_{\lambda,p}(a,b,c)g(z)}$$

$$\frac{z(I_{\lambda,p}(a+1,b,c)f(z))'}{pI_{\lambda,p}(a+1,b,c)g(z)}=q(z), \tag{2.5}$$

where $q(z)=1+q_1z+a_2z^2+\ldots$ is analytic in U and $q(z)\neq 0$ for all $z\in U$. Thus by using the identity (1.14), we have

$$\frac{z(I_{\lambda,p}(a+1,b,c)f(z))'}{pI_{\lambda,p}(a+1,b,c)g(z)}=\frac{I_{\lambda,p}(a+1,b,c)\left(\frac{zf'(z)}{p}\right)}{I_{\lambda,p}(a+1,b,c)g(z)}$$

$$=\frac{\dfrac{z\left(I_{\lambda,p}(a,b,c)\left(\frac{zf'(z)}{p}\right)\right)'}{I_{\lambda,p}(a,b,c)g(z)}+(a-p)\dfrac{I_{\lambda,p}(a,b,c)\left(\frac{zf'(z)}{p}\right)}{I_{\lambda,p}(a,b,c)g(z)}}{\dfrac{z(I_{\lambda,p}(a,b,c)g(z))'}{I_{\lambda,p}(a,b,c)g(z)}+(a-p)}. \tag{2.6}$$

Since $g(z)\in S^*_{\lambda,p}(a+1,b,c;\varphi)\subset S^*_{\lambda,p}(a,b,c;\varphi)\ (\varphi\in S)$, by Theorem 1, we set

$$\frac{z(I_{\lambda,p}(a,b,c)g(z))'}{pI_{\lambda,p}(a,b,c)g(z)}=G(z),$$

where $G(z)\prec\varphi(z)\ (z\in U)$ for $\varphi\in S$. Then, by virture of (2.5) and (2.6), we observe that

$$I_{\lambda,p}(a,b,c)\left(\frac{zf'(z)}{p}\right)=q(z)I_{\lambda,p}(a,b,c)g(z) \tag{2.7}$$

and

$$\frac{z(I_{\lambda,p}(a+1,b,c)f(z))'}{pI_{\lambda,p}(a+1,b,c)g(z)}=\frac{\dfrac{z\left(I_{\lambda,p}(a,b,c)\left(\frac{zf'(z)}{p}\right)\right)'}{I_{\lambda,p}(a,b,c)g(z)}+(a-p)q(z)}{pG(z)+a-p}. \tag{2.8}$$

Differentiating both sides of (2.7) with respect to z, we obtain

$$\frac{z\left(I_{\lambda,p}(a,b,c)\left(\frac{zf'(z)}{p}\right)\right)'}{I_{\lambda,p}(a,b,c)g(z)}=pG(z)q(z)+zq'(z). \tag{2.9}$$

Making use of (2.4), (2.8) and (2.9), we get

$$\frac{z(I_{\lambda,p}(a+1,b,c)f(z))'}{pI_{\lambda,p}(a+1,b,c)g(z)}=\frac{pG(z)q(z)+zq'(z)+(a-p)q(z)}{pG(z)+a-p}$$

$$=q(z)+\frac{zq'(z)}{pG(z)+a-p}\prec\psi(z)\quad(z\in U). \tag{2.10}$$

Since $a\geq p,\ p\in\mathrm{N}$ and $G(z)\prec\varphi(z)\ (z\in U)$,

$\mathrm{Re}\{pG(z)+a-p\}>0\quad(z\in U)$.

Hence, by taking

$$Q(z)=\frac{1}{pG(z)+a-p}$$

in (2.10), and applying Lemma 2, we can show that

$p(z)\prec\psi(z)\quad(z\in U)$,

so that

$f(z)\in C_{\lambda,p}(a,b,c;\varphi,\psi)\quad(\varphi,\psi\in S)$.

For the second part, by using arguments similar to those detailed above with the identity (1.13), we obtain:

$C_{\lambda,p}(a,b,c;\varphi,\psi)\subset C_{\lambda+1,p}(a,b,c;\varphi,\psi)\quad(\varphi,\psi\in S)$.

The proof of Theorem 3 is thus completed.

3. INCLUSION PROPERTIES INVOLVING $J_{\sigma,p}$

In this section, we consider the generalized Bernardi-Libera-Livingston integral operator $J_{\sigma,p}(\sigma>-p)$ defined by (see [24],[25],and [26]).

$$J_{\sigma,p}(f)(z)=\frac{\sigma+p}{z^\sigma}\int_0^z t^{\sigma-1}f(t)dt\quad(f\in A(p);\sigma>-p). \tag{3.1}$$

Theorem 4. Let $\sigma>-p,\ \lambda>-p,\ a>p$ and $p\in\mathrm{N}$. If $f(z)\in S^*_{\lambda,p}(a,b,c;\varphi)\ (\varphi\in S)$, then

$J_{\sigma,p}(f)(z)\in S^*_{\lambda,p}(a,b,c;\varphi)\quad(\varphi\in S)$.

Proof . Let $f(z)\in S^*_{\lambda,p}(a,b,c;\varphi)$ for $\varphi\in S$, and set

$$\frac{z(I_{\lambda,p}(a,b,c)J_{\sigma,p}(f)(z))'}{pI_{\lambda,p}(a,b,c)J_{\sigma,p}(f)(z)}=q(z), \tag{3.2}$$

where $q(z)=1+q_1z+q_2z^2+\ldots$ is analytic in U and $q(z)\neq 0$ for all $z\in U$. From (3.1), we obtain

$$z(I_{\lambda,p}(a,b,c)J_{\sigma,p}(f)(z))'=(\sigma+p)I_{\lambda,p}(a,b,c)f(z) -\sigma I_{\lambda,p}(a,b,c)J_{\sigma,p}(f)(z)\quad(z\in U). \tag{3.3}$$

By applying (3.2) and (3.3), we obtain

$$(\sigma+p)\frac{I_{\lambda,p}(a,b,c)f(z)}{I_{\lambda,p}(a,b,c)J_{\sigma,p}(f)(z)}=pq(z)+\sigma. \tag{3.4}$$

Differentiating (3.4) logarithmically with respect to z, we obtain

$$\frac{z(I_{\lambda,p}(a,b,c)f(z))'}{I_{\lambda,p}(a,b,c)f(z)}=q(z)+\frac{zq'(z)}{pq(z)+\sigma}. \tag{3.5}$$

Since $\sigma>-p,\ \varphi(z)\in S$, and $f(z)\in S^*_{\lambda,p}(\varphi)$, from (3.5), we have

$$\mathrm{Re}\{p\varphi(z)+\sigma\}>0\quad\text{and}\quad q(z)+\frac{zq'(z)}{pq(z)+\sigma}\prec\varphi(z)\ \ (z\in U).$$

Hence, by virbure of Lemma 1, we conclude that $q(z)\prec\varphi(z)\quad(z\in U)$,

which implies that

$J_{\sigma,p}(f)(z)\in S^*_{\lambda,p}(a,b,c;\varphi)\quad(\varphi\in S)$.

Next, we derive an inclusion property involving $J_{\sigma,p}$, which is given by

Theorem 5. Let $\sigma>-p,\ \lambda>-p,\ a>p$ and $p\in\mathrm{N}$. If $f(z)\in K_{\lambda,p}(a,b,c;\varphi)\,(\varphi\in S)$, then $J_{\delta,p}(f)(z)\in K_{\lambda,p}(a,b,c;\varphi)\quad(\varphi\in S)$.

Proof . By applying Theorem 4, it follows that

$$f(z)\in K_{\lambda,p}(a,b,c;\varphi)\Leftrightarrow\frac{zf'(z)}{p}\in S^*_{\lambda,p}(a,b,c;\varphi)$$

$$\Rightarrow J_{\sigma,p}\left(\frac{zf'(z)}{p}\right)\in S^*_{\lambda,p}(a,b,c;\phi)$$

$$\Leftrightarrow\frac{z}{p}\left(J_{\sigma,p}(f)(z)\right)'\in S^*_{\lambda,p}(a,b,c;\phi)$$

$$\Leftrightarrow J_{\sigma,p}(f)(z)\in K_{\lambda,p}(a,b,c;\varphi)\ \ (\varphi\in S),$$

which proves Theorem 5.

Finally, we prove

Theorem 6. Let $\sigma>-p,\ \lambda>-p,\ a>p$ and $p\in\mathrm{N}$. If $f(z)\in C_{\lambda,p}(a,b,c;\varphi,\psi)\,(\varphi,\psi\in S)$, then $J_{\delta,p}(f)(z)\in C_{\lambda,p}(a,b,c;\varphi,\psi)\quad(\varphi,\psi\in S)$.

Proof. Let $f(z)\in C_{\lambda,p}(a,b,c;\varphi,\psi)$ for $\varphi,\psi\in S$. Then, in view of (1.7), there exists a function $g(z)\in S^*_{\lambda,p}(a,b,c;\varphi)$ such that

$$\frac{z(I_{\lambda,p}(a,b,c)f(z))'}{pI_{\lambda,p}(a,b,c)g(z)}\prec\psi(z)\quad(z\in U). \tag{3.6}$$

Thus we set

$$\frac{z(I_{\lambda,p}(a,b,c)J_{\sigma,p}(f)(z))'}{pI_{\lambda,p}(a,b,c)J_{\sigma,p}(f)(z)}=q(z),$$

where $q(z)=1+q_1z+q_2z^2+\ldots$ is analytic in U and $q(z)\neq 0$ for all $z\in U$. Applying (3.3), we get

$$\frac{z(I_{\lambda,p}(a,b,c)f(z))'}{pI_{\lambda,p}(a,b,c)g(z)}=\frac{I_{\lambda,p}(a,b,c)\left(\frac{zf'(z)}{p}\right)}{I_{\lambda,p}(a,b,c)g(z)}$$

$$=\frac{z\left(I_{\lambda,p}(a,b,c)J_{\sigma,p}\left(\frac{zf'(z)}{p}\right)\right)'+\sigma I_{\lambda,p}(a,b,c)J_{\sigma,p}\left(\frac{zf'(z)}{p}\right)}{z\left(I_{\lambda,p}(a,b,c)J_{\sigma,p}(g)(z)\right)'+\sigma I_{\lambda,p}(a,b,c)J_{\sigma,p}(g)(z)}$$

$$=\frac{\dfrac{z\left(I_{\lambda,p}(a,b,c)J_{\sigma,p}\left(\frac{zf'(z)}{p}\right)\right)'}{I_{\lambda,p}(a,b,c)g(z)}+\sigma\dfrac{I_{\lambda,p}(a,b,c)J_{\sigma,p}\left(\frac{zf'(z)}{p}\right)}{I_{\lambda,p}(a,b,c)J_{\sigma,p}g(z)}}{\dfrac{z\left(I_{\lambda,p}(a,b,c)J_{\sigma,p}(g)(z)\right)'}{I_{\lambda,p}(a,b,c)J_{\sigma,p}g(z)}+\sigma}. \tag{3.7}$$

Since $g(z)\in S^*_{\lambda,p}(a,b,c;\varphi)\,(\varphi\in S)$, by virtue of Theorem 4, we have $J_{\sigma,p}(g)(z)\in S^*_{\lambda,p}(a,b,c;\varphi)$.

Let us now put

$$\frac{z\left(I_{\lambda,p}(a,b,c)J_{\sigma,p}(g)(z)\right)'}{pI_{\lambda,p}(a,b,c)J_{\sigma,p}(g)(z)} = H(z),$$

where $H(z) \prec \varphi(z)\ (z \in U)$ for $\varphi \in S$. Then, by using the same techniques as in the proof of Theorem 3, we conclude from (3.6) and (3.7) that

$$\frac{z(I_{\lambda,p}(a,b,c)f(z))'}{pI_{\lambda,p}(a,b,c)g(z)} = q(z) + \frac{zq'(z)}{pH(z)+\sigma} \prec \psi(z)\ (z \in U). \quad (3.8)$$

Hence, upon setting

$$Q(z) = \frac{1}{pH(z)+\sigma} \quad (z \in U)$$

in (3.8), if we apply Lemma 2, we obtain

$$q(z) \prec \psi(z) \quad (z \in U),$$

which yields

$$J_{\sigma,p}(f)(z) \in C_{\lambda,p}(a,b,c;\varphi,\psi) \quad (\varphi,\psi \in S).$$

The proof of Theorem 6 is thus completed.

Remark 2.

(i) Putting $a = \mu > 0$ and $b = c$ in the above results we obtain the corresponding results, for the operator $I^{p}_{\lambda,\mu}$;

(ii) Putting $b = p+1,\ a = c$ and replacing λ by $1-\lambda,\ \infty < \lambda < p+1$ in the above results, we obtain the corresponding results for the operator $\Omega^{(\lambda,p)}_{z}$.

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