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Original Article

On Stability of a Class of Fractional Differential Equations

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Abstract: In this paper, we consider the Hyers-Ulam stability for fractional differential equations of the form: $D_z^\alpha f(z) = G(f(z), zf'(z); z), \quad 1 < \alpha \le 2$ in a complex Banach space. Furthermore, applications are illustrated.

Keywords: Analytic function; Unit disk; Hyers-Ulam stability; Admissible functions; Fractional calculus; Fractional differential equation

1. INTRODUCTION

A classical problem in the theory of functional equations is that: If a function f approximately satisfies functional equation E when does there exists an exact solution of E which f approximates. In 1940, Ulam [1] imposed the question of the stability of Cauchy equation and in 1941, D. H. Hyers solved it [2]. In 1978, Rassias [3] provided a generalization of Hyers, theorem by proving the existence of unique linear mappings near approximate additive mappings. The problem has been considered for many different types of spaces [4-6]. Recently, Li and Hua [7] discussed and proved the Hyers-Ulam stability of spacial type of finite polynomial equation, and Bidkham, Mezerji and Gordji [8] introduced the Hyers-Ulam stability of generalized finite polynomial equation. Finally, Rassias [9] imposed a Cauchy type additive functional equation and investigated the generalised Hyers-Ulam "product-sum" stability of this equation.

The class of fractional differential equations of various types plays important roles and tools not only in mathematics but also in physics, control systems, dynamical systems and engineering to create the mathematical modeling of many physical phenomena. Naturally, such equations required to be solved. Many studies on fractional calculus and fractional differential equations, involving different operators such as Riemann-Liouville operators [10], Erdélyi-Kober operators [11], Weyl-Riesz operators [12], Caputo operators [13] and Grünwald-Letnikov operators [14], have appeared during the past three decades. The existence of positive solution and multi-positive solutions for nonlinear fractional differential equation are established and studied [15]. Moreover, by using the concepts of the subordination and superordination of analytic functions, the existence of analytic solutions for fractional differential equations in complex domain are suggested and posed [16-18].

Srivastava and Owa [19] gave definitions for fractional operators (derivative and integral) in the complex z-plane C as follows:

1.1. Definition: The fractional derivative of order α is defined, for a function $f(z)$ by

$$D_z^\alpha f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\alpha} d\zeta,$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane C containing the origin and the multiplicity of

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$(z-\zeta)^{-\alpha}$ is removed by requiring $log(z-\zeta)$ to be real when $(z-\zeta)>0.$

1.2. Definition: The fractional integral of order $\alpha > 0$ is defined, for a function $f(z),$ by

$$I_z^\alpha f(z) := \frac{1}{\Gamma(\alpha)}\int_0^z f(\zeta)(z-\zeta)^{\alpha-1}d\zeta; \alpha > 0,$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane $(\mathbb{C})$ containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $log(z-\zeta)$ to be real when $(z-\zeta)>0.$

1.1. Remark:

$$D_z^\alpha z^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} z^{\mu-\alpha}, \mu > -1$$

and

$$I_z^\alpha z^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} z^{\mu+\alpha}, \mu > -1.$$

In [17], it was shown the relation

$$I_z^\alpha D_z^\alpha f(z) = D_z^\alpha I_z^\alpha f(z) = f(z).$$

Let $U := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $\mathbb{H}$ denote the space of all analytic functions on U. Here we suppose that $\mathbb{H}$ as a topological vector space endowed with the topology of uniform convergence over compact subsets of U. Also for $a \in \mathbb{C}$ and $m \in \mathbb{N}$, let $\mathbb{H}[a,m]$ be the subspace of $\mathbb{H}$ consisting of functions of the form

$$f(z) = a + a_m z^m + a_{m+1}z^{m+1} + \ldots, \quad z \in U.$$

Definition 1.3. Let p be a real number. We say that

$$\sum_{n=0}^{\infty} a_n z^{n+\alpha} = f(z) \qquad (1)$$

has the generalized Hyers-Ulam stability if there exists a constant $K > 0$ with the following property:

for every $\varepsilon > 0, w \in \overline{U} = U \bigcup \partial U,$ if

$$\left|\sum_{n=0}^{\infty} a_n w^{n+\alpha}\right| \le \varepsilon\left(\sum_{n=0}^{\infty}\frac{|a_n|^p}{p(n+1)^2}\right),$$
$$p \in (0,1)$$

then there exists some $z \in \overline{U}$ that satisfies equation (1) such that

$$|z^i - w^i| \le \varepsilon K,$$

$$(z, w \in \overline{U}, \quad i \in \mathbb{N}).$$

In the present paper, we study the generalized Hyers-Ulam stability for holomorphic solutions of the fractional differential equation in complex Banach spaces X and Y

$$D_z^\alpha f(z) = G(f(z), zf'(z); z), \quad 1 < \alpha \le 2, \quad (2)$$

where $G: X^2 \times U \to Y$ and $f: U \to X$ are holomorphic functions such that $f(0) = \Theta$ (Θ is the zero vector in X).

Recently, the authors studied the ulam stability for different types of fractional differential equations [20-22].

2. RESULTS

In this section we present extensions of the generalized Hyers-Ulam stability to holomorphic vector-valued functions. Let X, Y represent complex Banach space. The class of admissible functions $\mathbb{G}(X,Y),$ consists of those functions $g: X^2 \times U \to Y$ that satisfy the admissibility conditions:

$$\begin{aligned}&\| \; g(r, ks; z) \; \| \ge 1,\\ &when \; \| \; r \; \| = 1, \; \| \; s \; \| = 1,\end{aligned} \qquad (3)$$

$$(\quad z \in U, k \ge 1).$$

We need the following results:

2.1. Lemma: [23] Let $g \in \mathbb{G}(X,Y).$ If $f: U \to X$ is the holomorphic vector-valued functions defined in the unit disk U with $f(0) = \Theta,$ then

$$\| \ g(f(z), zf'(z); z) \ \| < 1 \\ \Rightarrow \| \ f(z) \ \| < 1. \tag{4}$$

2.1. Theorem: In Eq. (2), if $G \in \mathsf{G}(X,Y)$ is the holomorphic vector-valued function defined in the unit disk U then

$$\| \ G(f(z), zf'(z); z) \ \| < 1 \\ \Rightarrow \| \ I_z^\alpha G(f(z), zf'(z); z) < 1 \|. \tag{5}$$

Proof. By continuity of G, the fractional differential equation (2) has at least one holomorphic solution f. According to Remark 1.1, the solution $f(z)$ of the problem (2) takes the form

$$f(z) = I_z^\alpha G(f(z), zf'(z); z).$$

Therefore, in virtue of Lemma 2.1, we obtain the assertion (5).

2.2. Theorem: Let $G \in \mathsf{G}(X,Y)$ be holomorphic vector-valued functions defined in the unit disk U then the equation (2) has the generalized Hyers-Ulam stability for $z \to \partial U$.

Proof. Assume that

$$G(z) := \sum_{n=0}^{\infty} \varphi_n z^n, \quad z \in U$$

therefore, by Remark 1.1, we have

$$I_z^\alpha G(z) = \sum_{n=0}^{\infty} a_n z^{n+\alpha} = f(z).$$

Also, $z \to \partial U$ and thus $|z| \to 1$. According to Theorem 2.1, we have

$$\| \ f(z) \ \| < 1 = |z|.$$

Let $\varepsilon > 0$ and $w \in \overline{U}$ be such that

$$|\sum_{n=1}^{\infty} a_n w^{n+\alpha}| \le \varepsilon \left(\sum_{n=1}^{\infty} \frac{|a_n|^p}{p(n+1)^2}\right).$$

We will show that there exists a constant K independent of ε such that

$$|w^i - u^i| \le \varepsilon K, \quad w \in \overline{U}, u \in U$$

and satisfies (1). We put the function

$$f(w) = \frac{-1}{\lambda a_i} \sum_{n=1, n\ne i}^{\infty} a_n w^{n+\alpha}, \\ a_i \ne 0, 0 < \lambda < 1, \tag{6}$$

thus, for $w \in \partial U$, we obtain

$$\begin{aligned} |w^i - u^i| &\le \frac{1}{|a_i|(1-\lambda)} |\sum_{n=1}^{\infty} a_n w^{n+\alpha}| \\ &\le \frac{\varepsilon}{|a_i|(1-\lambda)} \left(\sum_{n=0}^{\infty} \frac{|a_n|^p}{p(n+1)^2}\right) \\ &\le \frac{\varepsilon |a_i|^{p-1}}{p(1-\lambda)} \left(\sum_{n=0}^{\infty} \frac{1}{(n+1)^2}\right) \\ &\le \frac{\pi^2 \varepsilon |a_i|^{p-1}}{6(1-\lambda)} \\ &:= K\varepsilon. \end{aligned}$$

Without loss of generality, we consider $|a_i| = \max_{n\ge1}(|a_n|)$ yielding

$$\begin{aligned} |w^i - u^i| &= |w^i - \lambda f(w) + \lambda f(w) - u^i| \\ &\le |w^i - \lambda f(w)| + \lambda |f(w) - u^i| \\ &< |w^i - \lambda f(w)| + \lambda |w^i - u^i| \\ &= |w^i + \frac{1}{a_i} \sum_{n=1, n\ne i}^{\infty} a_n w^{n+\alpha}| + \lambda |w^i - u^i| \\ &= \frac{1}{|a_i|} |\sum_{n=1}^{\infty} a_n w^{n+\alpha}| + \lambda |w^i - u^i|. \end{aligned}$$

This completes the proof.

3. APPLICATIONS

In this section, we introduce some applications of functions to achieve the generalized Hyers-Ulam stability.

3.1. Example: Consider the function $G : X^2 \times U \to \mathsf{R}$ by

$$G(r, s; z) = a(\| \ r \ \| + \| \ s \ \|)^n + b|z|^2, \\ n \in R_+$$

with $a \ge 0.5$, $b \ge 0$ and $G(\Theta, \Theta, 0) = 0$. Our aim is to apply Theorem 2.1. this follows since

$$\| G(r,ks;z) \| = a(\| r \| + k\| s \|)^n + b|z|^2 = a(1+k)^n + b|z|^2 \geq 1,$$

when $\| r \| = \| s \| = 1, z \in U$. Hence by Theorem 2.1, we have : If $a \geq 0.5$, $b \geq 0$ and $f: U \to X$ is a holomorphic vector-valued function defined in U, with $f(0) = \Theta$, then

$$a(\| f(z) \| + \| zf'(z) \|)^n + b|z|^2 < 1 \Rightarrow \| f(z) \| < 1.$$

Consequently, $\| I_z^{\alpha} G(f(z), zf'(z); z) \| < 1$, thus in view of Theorem 2.2, f has the generalized Hyers-Ulam stability.

3.2. Example: Assume the function $G: X^2 \to X$ by

$$G(r,s;z) = G(r,s) = re^{\| s \|^m - 1}, \quad m \geq 1$$

with $G(\Theta, \Theta) = \Theta$. By applying Theorem 2.1, we need to show that $G \in \mathcal{G}(X, X)$. Since

$$\| G(r,ks) \| = \| re^{\| ks \|^m - 1} \| = e^{k^m - 1} \geq 1,$$

when $\| r \| = \| s \| = 1, k \geq 1$. Hence, by Theorem 2.1, we have : For $f: U \to X$ is a holomorphic vector-valued function defined in U, with $f(0) = \Theta$, then,

$$\| f(z)e^{\| zf'(z) \|^m - 1} \| < 1 \Rightarrow \| f(z) \| < 1.$$

Consequently, $\| I_z^{\alpha} G(f(z), zf'(z); z) \| < 1$; thus in view of Theorem 2.2, f has the generalized Hyers-Ulam stability.

3.3. Example: Let $a, b: U \to \mathbb{C}$ satisfy

$$|a(z) + \mu b(z)| \geq 1,$$

for every $\mu \geq 1, \nu > 1$ and $z \in U$. Consider the function $G: X^2 \to Y$ by

$$G(r,s;z) = a(z)r + \mu b(z)s,$$

with $G(\Theta, \Theta) = \Theta$. Now for $\| r \| = \| s \| = 1$, we have

$$\| G(r, \mu s; z) \| = |a(z) + \mu b(z)| \geq 1$$

and thus $G \in \mathcal{G}(X, Y)$. If $f: U \to X$ is a holomorphic vector-valued function defined in U, with $f(0) = \Theta$, then

$$\| a(z)f(z) + zb(z)f'(z) \| < 1 \Rightarrow \| f(z) \| < 1.$$

Hence according to Theorem 2.2, f has the generalized Hyers-Ulam stability.

3.4. Example: Let $\lambda: U \to \mathbb{C}$ be a function such that

$$\Re\left(\frac{1}{\lambda(z)}\right) > 0,$$

for every $z \in U$. Consider the function $G: X^2 \to Y$ by

$$G(r,s;z) = r + \frac{s}{\lambda(z)},$$

with $G(\Theta, \Theta) = \Theta$. Now for $\| r \| = \| s \| = 1$, we have

$$\| G(r,ks;z) \| = \left|1 + \frac{k}{\lambda(z)}\right| \geq 1, \quad k \geq 1$$

and thus $G \in \mathcal{G}(X, Y)$. If $f: U \to X$ is a holomorphic vector-valued function defined in U, with $f(0) = \Theta$, then

$$\left\| f(z) + \frac{zf'(z)}{\lambda(z)} \right\| < 1 \Rightarrow \| f(z) \| < 1.$$

Hence, according to Theorem 2.2, f has the generalized Hyers-Ulam stability.

4. REFERENCES

1. Ulam, S.M. *A Collection of Mathematical Problems. Interscience Publ. New York, 1961. Problems in Modern Mathematics*. Wiley, New York (1964).
2. Hyers, D.H. On the stability of linear functional equation. *Proc. Nat. Acad. Sci.* 27: 222-224 (1941).
3. Rassias, Th.M. On the stability of the linear mapping in Banach space. *Proc. Amer. Math. Soc.* 72: 297-300 (1978).
4. Hyers, D.H. The stability of homomorphisms and related topics, in Global Analysis-Analysis on Manifolds. *Teubner-Texte Math.* 75: 140-153 (1983).
5. Hyers, D.H. & Th.M. Rassias. Approximate homomorphisms. *Aequationes Math.* 44: 125-153 (1992).
6. Hyers, D.H., G. I. Isac & Th.M. Rassias. *Stability of Functional Equations in Several Variables*. Birkhauser, Basel (1998).
7. Li, Y. & L. Hua. Hyers-Ulam stability of a polynomial equation. *Banach J. Math. Anal.* 3: 86-90 (2009).
8. Bidkham, M. & H.A. Mezerji & M.E. Gordji. Hyers-Ulam stability of polynomial equations. *Abstract and Applied Analysis* doi:10.1155/2010/754120 (2010).
9. Rassias, M.J. Generalised Hyers-Ulam "product-sum" stability of a Cauchy type additive functional equation. *European J. Pure and Appl. Math.* 4: 50-58 (2011).
10. Diethelm, K. & N. Ford. Analysis of fractional differential equations. *J. Math. Anal. Appl*. 265: 229-248 (2002).
11. Ibrahim, R.W. & S. Momani. On the existence and uniqueness of solutions of a class of fractional differential equations. *J. Math. Anal. Appl.* 334: 1-10 (2007).
12. Momani, S.M. & R.W. Ibrahim. On a fractional integral equation of periodic functions involving Weyl-Riesz operator in Banach algebras. *J. Math. Anal. Appl.* 339: 1210-1219 (2008).
13. Bonilla, B., M. Rivero & J.J. Trujillo. On systems of linear fractional differential equations with constant coefficients. *App. Math. Comp*. 187: 68-78 (2007).
14. Podlubny, I. *Fractional Differential Equations*. Academic Press, London, (1999).
15. Zhang, S. The existence of a positive solution for a nonlinear fractional differential equation. *J. Math. Anal. Appl.* 252: 804-812 (2000).
16. Ibrahim, R.W. & M. Darus. Subordination and superordination for analytic functions involving fractional integral operator. *Complex Variables and Elliptic Equations* 53:1021-1031 (2008).
17. Ibrahim, R.W. & M. Darus. Subordination and superordination for univalent solutions for fractional differential equations. *J. Math. Anal. Appl.* 345: 871-879 (2008).
18 Ibrahim, R.W. Existence and uniqueness of holomorphic solutions for fractional Cauchy problem. *J. Math. Anal. Appl.* 380: 232-240 (2011).
19 Srivastava, H.M. & S. Owa. Univalent Functions, *Fractional Calculus, and Their Applications*. Halsted Press, John Wiley and Sons, New York (1989).
20. Ibrahim, R.W. Generalized Ulam–Hyers stability for fractional differential equations. *International Journal of Mathematics* 23: 1-9 (2012).
21. Ibrahim, R.W. On generalized Hyers-Ulam stability of admissible functions. *Abstract and Applied Analysis* (in press).
22. Ibrahim, R.W. Approximate solutions for fractional differential equation in the unit disk. *Electronic Journal of Qualitative Theory of Differential Equations* 64: 1-11 (2011).
23 Miller, S.S. & P.T. Mocanu. *Differential Subordinantions: Theory and Applications.* Pure and Applied Mathematics No. 225. Dekker, New York (2000).