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Original Article

Some New *s*-Hermite-Hadamard Type Inequalities for Differentiable Functions and Their Applications

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Abstract: In this paper, we establish several inequalities for some differentiable mappings that are connected with the celebrate Hermit - Hadamard integral inequality for *s*-convex functions. Also a parallel development is made base on concavity. Applications to some special means of real numbers are found. Also applications to numerical integration are provided.

Keywords and Phrases: Hermite-Hadamard type inequality, s-Convex function, *p*-logarithmic mean, Hölder's inequality, Trapezoidal formula, special means.

(AMS SUBJECT CLASSFICATION: 26D15, 39A10 and 26A51)

1. INTRODUCTION

Let a function defined as $f: \emptyset \neq I \subseteq \mathrm{R} \to \mathrm{R}$ is said to be convex if the following inequality holds

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

For all $x, y \in I$ and $t \in [0, 1]$. Geometrically, this means that if P, Q and R are three distinct points on graph of *f(x)* with Q between P and R, Then Q is on or below chord PR. There are many result associated with convex function in the area of inequalities, but one of those is the classical Hermite Hadamard inequality.

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \qquad (1.1)$$

for $a, b \in I, with\ a < b$.

Hudzik and Maligranda [3] considered, among others, the class of functions which are *s*-convex in the second sense. This is defining as follows.

A function $f: [0, \infty) \to \mathrm{R}$ is said to be *s*-convex in the second sense if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) \qquad (1.2)$$

holds for all $x, y \in [0, \infty), t \in [0,1]$ and for some fixed $s \in (0,1]$. It may be noted that every 1-convex function is convex. In the same Paper [3] H. Hudzik and L. Maligranda discussed a few result connecting with *s*-convex function in second sense and some new result about Hadamard inequality for *s*-convex function is discussed in [2, 7]. On the hand, there are many important inequalities connecting with 1-convex (Convex) function [2], but one of these is (1.1).

Dragomir et al [7], proved a variant of Hermit-Hadamard inequality for *s*-convex function in second sense.

Theorem 1. Suppose that $f: [0, \infty) \to [0, \infty)$ is *s*-convex function in the second sense. Where $s \in (0,1]$, and let $a, b \in [0, \infty)$, $a < b$. If $f \in L^1[a, b]$, then the following inequality holds.

$$2^{s-1} f\left(\frac{a+b}{2}\right) \frac{1}{b-a}\int_a^b f(x)dx\ \frac{f(a)+f(b)}{s+1} \qquad (1.3)$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.3). The inequality in (1.3) becomes reverse when the function is

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concave. The result in (1.3) was improved by Jagers [4] who gave both the upper and lower bounds for the constant c(s) in the inequality

$$c(s)f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx.$$

He proved that

$$\frac{2^{s+1}-1}{s+2} \leq c(s) \leq 2^{\frac{s-1}{s+1}}\left(\frac{2^s-1}{s}\right)^{\frac{s}{s+1}} \leq \frac{2^{s+1}-2^{s-1}-1}{s+1}$$

Dragomir et al [2] discussed inequality for differentiable and twice differentiable function connecting with the Hermite – Hadamard (H-H) Inequality in the basis of the following Lemmas.

Lemma 1. Let $f: I \subseteq R \to R$ be differentiable function on I^o (interior of I) $a, b \in I$ with $a < b$, If $f' \in L^1[a,b]$, then we have

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx$$
$$= \frac{(b-a)}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)dt. \quad (1.4)$$

Dragomir and Agarwal [1] established the following result connected with the right part of (1.4) as well as to apply them for some elementary inequalities for real numbers and numerical integration.

Lemma 2. Let $f: I^o \subseteq R \to E$ be differentiable function on I^o , $a, b \in I^o$, with $a < b$. If $f' \in L^1[a,b]$, *then*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx$$
$$= \frac{(b-a)}{2}\int_0^1\int_0^1 \begin{pmatrix} f'(ta+(1-t)b) \\ -f'(ua+(1-u)b)\end{pmatrix}(u-t)dtdu. \quad (1.5)$$

This paper is organized as follows: after Introduction, we discuss some new s-Hermite Hadamard type inequalities for differentiable function in section 2, and in section 3 we give some applications of the results from section 2 for some special means of real numbers. In section 4, we give some application, to trapezoidal formula.

2. MAIN RESULTS

Theorem 2. Let $f: I^o \subseteq \mathrm{R} \to \mathrm{R}$ be differentiable function on I^o , $a, b \in I^o$, with $a < b$. If $f' \in L^1[a,b]$, if the mapping $|f'|$ is s-convex on $[a, b]$, then

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq$$

$$\frac{b-a}{2^s}\left[\frac{s.2^s+1}{(s+2)(s+1)}\right](|f'(a)|+|f'(b)|) \quad (2.6)$$

Proof. From Lemma 1,

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq$$

$$\frac{b-a}{2}\int_0^1 |(1-2t)||f'(ta+(1-t)b)|dt$$

$|f'|$ is s-convex on $[a, b]$ for t $\in [0,1]$, *then*

$$|f'(ta+(1-t)b) \leq t^s|f'(a)| + (1-t)^s|f'(b)|$$

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq$$
$$\frac{(b-a)}{2}\int_0^1 |1-2t|[t^s|f'(a)| + (1-t)^s|f'(b)|]dt. \quad (2.7)$$

Where

$$\int_0^1 t^s|1-2t|dt = \int_0^1 (1-t)^s|1-2t|dt = \frac{2^s.\ s+1}{2^s(s+1)(s+2)} \quad (2.8)$$

By (2.8) and (2.7), we get (2.6).

Theorem 3. Let the assumptions of Theorem 2 are satisfied with $p > 1$ such that $q = \frac{p}{p-1}$. If the mapping $|f'|^q$ is concave on $[a, b]$ then,

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{2}(p+1)^{-1/p}\left|f'\left(\frac{a+b}{2}\right)\right|. \quad (2.9)$$

Proof. From Lemma 1,

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq$$

$\frac{(b-a)}{2}\int_0^1|1-2t||f'(ta+(1-t)b)|dt.$ (2.10)

By applying Hölder's inequality on right side of (2.10). We have;

$$\int_0^1|(1-2t)||f'(ta+(1-t)b)|dt$$
$$\leq \left(\int_0^1|1-2t|^p\,dt\right)^{1/p}\left(\int_0^1|f'(ta+(1-t)b)|^q\,dt\right)^{1/q} \quad (2.11)$$

Here

$$\int_0^1|1-2t|^p\,dt=\frac{1}{p+1} \quad (2.12)$$

Since $|f'|^q$ is concave, by applying Jensen's Integral Inequality on the second integral of R.H.S. of (2.11). We have

$$\int_0^1|f'(ta+(1-t)b|^q dt \leq \left(\int_0^1 t^o dt\right)\left|f'\left(\frac{\int_0^1(ta+(1-t)b)dt}{\int_0^1 t^o dt}\right)\right|^q$$
$$=\left|f'\left(\frac{a+b}{2}\right)\right|^q \quad (2.13)$$

By (2.10), (2.12) and (2.13).We get (2.14).

Theorem 4. Let the assumptions of theorem 2 are satisfied with $p>1$ such that $q=\frac{p}{p-1}$. If the mapping $|f'|^q$is s-convex on $[a,b]$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)}{2(p+1)^{1/p}}\left(\frac{|f'(a)|^q+f'(b)|^q}{s+1}\right)^{1/q} \quad (2.14)$$

Proof. From Lemma 1**,**

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)}{2}\int_0^1|1-2t||f'(ta+(1-t)b)|dt \quad (2.15)$$

By applying Hölder's inequality on right side of (2.15). We get

$$\int_0^1|(1-2t)||f'(ta+(1-t)b)|dt$$
$$\leq \left(\int_0^1|1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1|f'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}} \quad (2.16)$$

Since $|f'|^q$ is s-convex on $[a,b]$ for $t\in[0,1]$, then

$|f'(ta+(1-t)b)|^q\leq t^s|f'(a)|^q+(1-t)^s|f'(b)|^q$.

And

$$\int_0^1|1-2t|^p\,dt=\frac{1}{p+1};$$

$$\int_0^1|f'(ta+(1-t)b)|^q\,dt=\frac{|f'(a)|^q+|f'(b)|^q}{s+1} \quad (2.17)$$

By (2.16) and (2.17), we get (2.14).

Corollary 5. From theorem 4, the assumptions of theorem 2 are satisfied with $p>1$ such that $q=\frac{p}{p-1}$. If the mapping $|f'|^q$ is s-convex on $[a,b]$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq$$
$$\frac{(b-a)}{2(p+1)^{1/p}}\left(\frac{1}{s+1}\right)^{1/q}(|f'(a)|+|f'(b)|)$$

Proof. The above inequality is obtained by using the fact $\sum_{i=1}^n(\alpha_i+\beta_i)^k\leq\sum_{i=1}^n(\alpha_i)^k+\sum_{i=1}^n(\beta_i)^k$ for $k\in(0,1)$ with $0\leq\frac{p-1}{p}<1,\ for\ p>1$.

Theorem 6. Let the assumptions of theorem 2 are satisfied with $p>1$ such that $q=\frac{p}{p-1}$. If the mapping $|f'|^q$ is s-concave on $[a,b]$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq$$
$$\frac{(b-a)}{2}(p+1)^{-\frac{1}{p}}.2^{\frac{s-1}{q}}\left|f'\left(\frac{a+b}{2}\right)\right|. \quad (2.18)$$

Proof. We proceed similarly as in theorem 4.

By $s-concavity$ of $|f'|^q$, we obtain

$$\int_0^1|f'(ta+(1-t)b)|^q\,dt\leq 2^{s-1}\left|f'\left(\frac{a+b}{2}\right)\right|. \quad (2.19)$$

Now (2.18) immediately follows from theorem 1.

Theorem 7. Let the assumptions of theorem 4 are satisfied, we have another result:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq$$
$$\frac{(b-a)}{2^{\frac{p+1}{p}}}\left[\left(\frac{s.2^s+1}{2^s}\right)\frac{|f'(a)|^q+|f'(b)|^q}{(s+1)(s+2)}\right]^{1/q} \quad (2.20)$$

Proof. From Lemma 1,

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{(b-a)}{2}\int_0^1 |1-2t||f'(ta+(1-t)b)|dt$$

$$= \frac{(b-a)}{2}\int_0^1 |1-2t|^{1/p}|1-2t|^{1/q}\left|\begin{matrix} f'(ta \\ +(1-t)b\end{matrix}\right| dt \tag{2.21}$$

By applying Hölder's inequality on (2.21) for $q > 1$, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\frac{(b-a)}{2}\left(\int_0^1 |1-2t|dt\right)^{\frac{1}{p}}\left(\int_0^1 |1-2t|\left|\begin{matrix} f'(ta \\ +(1-t)b)\end{matrix}\right|^q dt\right)^{1/q} \tag{2.22}$$

By s-convexity of $|f'|^q$ on $[a, b]$ for all $t \in [0,1]$. (2.22) can be written as:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{(b-a)}{2}\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\int_0^1 \begin{matrix}(t^s|1-2t||f'(a)| \\ +(1-t^s)|1-2t||f'(b)|)\end{matrix} dt\right)^{\frac{1}{q}}$$

$$= \frac{(b-a)}{2^{\frac{p-1}{p}}}\left(\begin{matrix}|f'(a)|^q \int_0^1 t^s|1-2t|dt + |f'(b)|^q \\ \int_0^1 (1-t)^s|1-2t|dt\end{matrix}\right)^{1/q} \tag{2.23}$$

Here,

$$\int_0^1 t^s|1-2t|dt = \int_0^1 (1-t)^s|1-2t|dt$$

$$= \frac{2^s.s+1}{2^s(s+1)(s+2)} \tag{2.24}$$

By (2.23) and (2.24) in (2.21), we get (2.20).

Corollary 8. From theorem 7, the assumptions of theorem 4 are satisfied with p > 1 such that $q = \frac{p}{p-1}$. If the mapping $|f'|^q$ is s-convex on $[a, b]$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq$$

$$\frac{(b-a)}{2^{\frac{p+1}{p}}}\left[\frac{s.2^s+1}{2^s(s+1)(s+2)}\right]^{1/q}(|f'(a)|+|f'(b)|)$$

Proof. The proof is similar to that of corollary 5.

Theorem 9. Let the assumptions of theorem 2 are satisfied with $p > 1$ such that $q = \frac{p}{p-1}$. If the mapping $|f'|^q$ is s-concave on $[a, b]$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq$$

$$\frac{(b-a)}{2^{\frac{p-1}{p}}}\left(\frac{s^{2^s}+1}{2(s+2)}\right)^{1/q}\left|f'\left(\frac{a+b}{2}\right)\right|. \tag{2.25}$$

Proof. We proceed similarly as in theorem 6.

By $s-concavity$ of $|f'|^q$ we obtain

$$\int_0^1 |1-2t||f'(ta+(1-t)b)|^q dt \leq.$$

$$\left(\frac{s^{2^s}+1}{2(s+2)}\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q \tag{2.26}$$

Now (2.25) immediately follows from theorem 1.

Theorem 10. Let the assumptions of theorem 2 are satisfied, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq$$

$$\frac{(b-a)}{2}\left[\frac{s^2+3s+4}{(s+1)(s+2)(s+3)}\right](|f'(a)|+|f'(b)|). \tag{2.27}$$

Proof. From Lemma 2.

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{(b-a)}{2}\int_0^1\int_0^1 |f'(ta+(1-t)b) -$$

$f'(ua+(1-u)b)\big||u-t|dtdu.$

$$\leq \frac{(b-a)}{2}\int_0^1\int_0^1\left|f'(ta+(1-t)b)\right||u-t|dtdu$$
$$+\frac{(b-a)}{2}\int_0^1\int_0^1\left|f'(ua+(1-u)b)\right||u-t|dtdu.$$
$$=(b-a)\int_0^1\int_0^1\left|f'(ta+(1-t)b)\right||u-t|dtdu. \quad (2.28)$$

By using s-convexity of $|f'|$ on $[a, b]$ for all $t \in [0,1]$ on right side of (2.28), we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq (b-a)\int_0^1\int_0^1 (t^s|u-t|\ |f'(a)|$$
$$+(1-t)^s|u-t|\ |f'(b)|)dtdu \quad (2.29)$$

But

$$\int_0^1\int_0^1 t^s|u-t|dtdu$$
$$=\int_0^1\int_0^1(1-t)^s|u-t|dtdu$$
$$=\frac{s^2+3s+4}{2(s+1)(s+2)(s+3)} \quad (2.30)$$

By (2.29) and (2.30) we get (2.27).

Theorem 11. Let the assumptions of Theorem 2 are satisfied. Furthermore, if the mapping $|f'|^q$ is concave on $[a, b]$ for $q > 1$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq$$
$$(b-a)\left[\frac{2}{(p+1)(p+2)}\right]^{1/p}\left|f'\left(\frac{a+b}{2}\right)\right|. \quad (2.31)$$

Proof. From Lemma 2, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)}{2}\int_0^1\int_0^1|f'(ta+(1-t)b)-$$
$$f'(ua+(1-u)b)\big||u-t|dtdu.$$
$$\leq \frac{(b-a)}{2}\int_0^1\int_0^1\left|f'(ta+(1-t)b)\right||u-t|dtdu$$
$$+\frac{(b-a)}{2}\int_0^1\int_0^1\left|f'(ua+(1-u)b)\right||u-t|dtdu.$$
$$=(b-a)\int_0^1\int_0^1\left|f'(ta+(1-t)b)\right||u-t|dtdu. \quad (2.32)$$

By applying Hölder Inequality in (2.32), we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq (b-a)\left(\int_0^1\int_0^1|f'(ta+(1-t)b)|^q\right.$$
$$\left.dtdu\right)^{\frac{1}{q}}\left(\int_0^1\int_0^1|u-t|^p dtdu\right)^{\frac{1}{p}} \quad (2.33)$$

But

$$\int_0^1\int_0^1|u-t|^p\ dtdu$$
$$=\int_0^1\left\{\int_0^u(u-t)^p dt+\int_u^1(t-u)^p dt\right\}du$$
$$=\frac{2}{(p+1)(p+2)}. \quad (2.34)$$

Since $|f'|^q$ is concave on $[a, b]$ so by using Jensen's Integral Inequality on first integral in R.H.S., we have

$$\int_0^1\int_0^1|f'(ta+(1-t)b)|^q\ dtdu$$
$$\leq \int_0^1\left[\left(\int_0^1 t^o dt\right)\left|f'\left(\frac{\int_0^1(ta+(1-t)b)dt}{\int_0^1 t^o dt}\right)\right|^q\right]du$$
$$=\int_0^1\left|f'\left(\frac{a+b}{2}\right)\right|^q du=\left|f'\left(\frac{a+b}{2}\right)\right|^q \quad (2.35)$$

Hence (2.33), (2.34) and (2.35) together imply (2.31).

Theorem 12. Let the assumptions of Theorem 2 are satisfied. Furthermore, if the mapping $|f'|^q$ is s-convex on $[a, b]$ for $q > 1$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq (b-a)\left(\frac{2}{(p+1)(p+2)}\right)^{1/p}\left(\frac{|f'(a)|^q+|f'(b)|^q}{(s+1)}\right)^{1/q}. \quad (2.36)$$

Proof. From Lemma 2, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{(b-a)}{2}\int_0^1\int_0^1 \left|f'(ta+(1-t)b)-f'(ua+(1-u)b)\right||u-t|dtdu.$$

$$\leq \frac{(b-a)}{2}\int_0^1\int_0^1\left|f'(ta+(1-t)b)\right||u-t|dtdu$$

$$+\frac{(b-a)}{2}\int_0^1\int_0^1\left|f'(ua+(1-u)b)\right||u-t|dtdu.$$

$$=(b-a)\int_0^1\int_0^1\left|f'(ta+(1-t)b)\right||u-t|dtdu. \quad (2.37)$$

By applying Hölder Inequality, (2.37) becomes

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq (b-a)\left(\int_0^1\int_0^1|f'(ta+(1-t)b)|^q\,dtdu\right)^{1/q}\left(\int_0^1\int_0^1|u-t|dtdu\right)^{1/p} \quad (2.38)$$

By s-convexity of $|f'|^q$ on $[a,b]$, for $t\in[0,1]$, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq (b-a)\left(\int_0^1\int_0^1\left(t^s|f'(a)|^q+(1-t)^s|f'(b)|^q\right)dtdu\right)^{1/q} \quad (2.39)$$

But

$$\int_0^1\int_0^1\left|u-t\right|^p dtdu$$

$$=\int_0^1\left\{\int_0^u(u-t)^p dt+\int_u^1(t-u)^p dt\right\}du$$

$$=\frac{2}{(p+1)(p+2)} \quad (2.40)$$

And

$$\int_0^1\int_0^1 t^s dtdu=\int_0^1\int_0^1(1-t)^s dtdu=\frac{1}{s+1}. \quad (2.41)$$

By (2.39), (2.40) and (2.41), we have (2.36).

Corollary 13. From theorem 12, Let the assumptions of Theorem 2 are satisfied. Furthermore, if the mapping $|f'|^q$ is s-convex on $[a,b]$ for $q>1$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq (b-a)$$

$$\left(\frac{2}{(p+1)(p+2)}\right)^{1/p}\left(\frac{1}{(s+1)}\right)^{1/q}(|f'(a)|+|f'(a)|).$$

Proof. The proof is similar to that of corollary 5.

Theorem 14. Let the assumptions of Theorem 2 are satisfied. Furthermore, if the mapping $|f'|^q$ is s-concave on $[a,b]$ for $q>1$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq$$

$$(b-a)\left[\frac{2}{(p+1)(p+2)}\right]^{1/p}.2^{\frac{s-1}{q}}\left|f'\left(\frac{a+b}{2}\right)\right| \quad (2.42)$$

Proof. We proceed in a similar way as in theorem 10.

By s-*concavity of* $|f'|^q$ *we obtain*

$$\int_0^1\int_0^1|f'(ta+(1-t)b)|^q\,dtdu$$

$$\leq 2^{s-1}\left|f'\left(\frac{a+b}{2}\right)\right|^q \quad (2.43)$$

Now (2.42) immediately follows from Theorem 1.

Theorem 15. Let $f: I^o\subseteq\mathbb{R}\to\mathbb{R}$ be differentiable function of I^o, a, b, $\in I^o$ with $a<b$, and $f'\in L^1[a,b]$. if the mapping $|f'|^q$ is s-convex on $[a,b] for\ q>1$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{(b-a)}{3^{\frac{1}{p}}}\left(\frac{s^2+3s+4}{2(s+1)(s+2)(s+3)}\right)^{\frac{1}{q}}$$

$$(|f'(a)|^q + |f'(b)|^q)^{1/q} . \quad (2.44)$$

Proof. From Lemma 2, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)}{2}\int_0^1\int_0^1 \left|f'(ta+(1-t)b) - f'(ua+(1-u)b)\right||u-t|dtdu$$

$$\leq \frac{(b-a)}{2}\int_0^1\int_0^1 \left|f'(ta+(1-t)b)\right||u-t|dtdu$$
$$+\frac{(b-a)}{2}\int_0^1\int_0^1 \left|f'(ua+(1-u)b)\right||u-t|dtdu.$$
$$= (b-a)\int_0^1\int_0^1 \left|f'(ta+(1-t)b)\right||u-t|dtdu. \quad (2.45)$$

By applying Hölder inequality on (2.45), we follow as

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq (b-a)\left(\int_0^1\int_0^1 |u-t||f'(ta+(1-t)b)|dtdu\right)^{\frac{1}{q}} \left(\int_0^1\int_0^1 |u-t|dtdu\right)^{1/p} \quad (2.46)$$

Here

$$\int_0^1\int_0^1 |u-t|dtdu = \frac{1}{3} \quad (2.47)$$

And

$$\int_0^1\int_0^1 |u-t||f'(ta+(1-t)b)|^q\, dtdu$$

$$\leq \int_0^1\int_0^1 |u-t|(t^s|f'(a)|^q + (1-t)^s|f'(b)|^q\, dtdu. \quad (2.48)$$

Since $|f'|^q$ $is\ s-convex\ of\ \ on\ [a,b]\ \ for\ t\ \in [0,1]$

By solving (2.48), we have

$$\int_0^1\int_0^1 |u-t||f'(ta+(1-t)b)|^q\, dtdu$$

$$\leq \left[\frac{s^2+3s+4}{2(s+1)(s+2)(s+3)}\right]\left[|f'(a)|^q + |f'(b)|^q\right]. \quad (2.49)$$

Relations (2.46), (2.47), and (2.49) together imply (2.44).

Corollary 16. From theorem 15, Let $f: I^o \subseteq R \to R$ be differentiable function of I^o , $a, b, \in I^o$ with $a < b$, and $f' \in L^1[a,b]$. if the mapping $|f'|^q$ is s-convex on $[a,b]$ for $q > 1$, then

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{3^{\frac{1}{p}}}\left(\frac{s^2+3s+4}{2(s+1)(s+2)(s+3)}\right)^{\frac{1}{q}}$$

$$(|f'(a)| + |f'(b)|)$$

Proof. The proof is similar to that of corollary 5.

Theorem 17. Let $f: I^o \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable function on I^o, a , $b \in I^o$ with $a < b$, and $f' \in L^1[a,b]$. If the mapping $|f'|^q$ is s-concave on $[a,b]$ for $q > 1$, then

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq$$

$$\frac{(b-a)}{3^{1/p}}\left(\frac{s^2+3s+4}{(s+2)(s+3)}\right)^q . 2^{\frac{s-2}{q}}\left|f'\left(\frac{a+b}{2}\right)\right|. \quad (2.50)$$

Proof. We proceed in a similar way as in theorem 12.

By $s-concavity\ of\ |f'|^q$, we obtain

$$\int_0^1\int_0^1 |u-t||f'(ta+(1-t)b)|^q\, dtdu$$

$$= \frac{s^2+3s+4}{(s+2)(s+3)}\left|f'\left(\frac{a+b}{2}\right)\right|. \quad (2.51)$$

Now (2.50) immediately follows from theorem 1.

3. APPLICATION TO SOME SPECIAL MEANS

Let us recall the following means for any two positive numbers a and b.

(1) *The Arithmetic mean*
$A \equiv A(a,b) = \frac{a+b}{2}$

(2) *The Harmonic mean*

$H \equiv H(a,b) = \frac{2ab}{a+b}$

(3) *The p- Logarithmic mean*

$$L_p = L_p(a,b) = \begin{cases} a, & if\ a = b; \\ \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, & if\ a \neq b, \end{cases}$$

(4). *The Identric mean*

$$I = I(a,b) = \begin{cases} a, & if\ a = b; \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & if\ a \neq b, \end{cases}$$

(5). *The Logarithmic mean*

$$L = L(a,b) = \begin{cases} a, & if\ a = b; \\ \frac{b-a}{Inb - lna}, & if\ a \neq b, \end{cases}$$

The following inequality is well known in the literature in [3]:

$H \leq G \leq L \leq I \leq A.$

It is also known that L_p monotonically increasing over $p \in \mathbb{R}, denoting\ L_0 = I\ and\ L_{-1} = L.$

Now here we find some new applications for special means of real numbers by using the results of Section 2.

Proposition 1. Let $p > 1$, $0 < a < b$ and $q = \frac{p}{p-1}$. Then one has the inequality.

$$\left|A(a,b) - L(a,b)\right| \leq \frac{\ln b - \ln a}{3} A(|a|,|b|). \quad (3.52)$$

Proof. By theorem 10 applied for the mapping $f(x) = e^x$ for $s = 1$ we have the above inequality (3.52).

Proposition 2. Let $p > 1$, $0 < a < b$ and $q = \frac{p}{p-1}$, then

$$\left|\frac{\mathrm{I}(1-a,1-b)}{\mathrm{G}(1-a,1-b)}\right| \leq exp\left(\frac{b-a}{3} H^{-1}(|1-a|,|1-b|)\right)$$

Proof. Following by Theorem 12, setting $f(x) = -\mathrm{In}(1-x)$ for $s = 1$.

Another result which is connected with p-Logarithmic mean $L_p(a,b)$ is the following one.

Proposition 3. Let $p > 1$, $0 < a < b$ and $= \frac{p}{p-1}$, then

$$\left|A\left[(1-a)^n,(1-b)^n\right] - L_n^p\left[(1-a)^n,(1-b)^n\right]\right|. \leq |n|(b-a)\left(\frac{2}{(p+1)(p-2)}\right)^{\frac{1}{p}} \left[A\left(|1-a|^{\frac{q}{n-1}}, |1-b|^{\frac{q}{n-1}}\right)\right]^{1/q}$$

Proof. Following by Theorem 15, setting $f(x) = (1-x)^n, |n| \geq 2$ and $n \in \mathbb{Z}$ for $s = 1$.

4. APPLICATION TO QUADRATURE FORMULAE

Let D be a division $a = x_0 < x_1 < x_2 \ldots < x_{n-1} < x_n = b$ of the interval $[a, b]$ and consider the quadrature formula

$$\int_a^b f(x)dx = S(f,D) + R(f,D) \quad (4.53)$$

where, for the trapezoidal version $S(f,D)$ is

$$S(f,D) = \sum_{k=0}^{n-1} \frac{f(x_k) + f(x_{k+1})}{2}(x_{k+1} - x_k)$$

and the connected error term $R(f,D)$ for the trapezoidal version

Proposition 4. Let $f: I \subseteq R \to R$ be differentiable function on I^o such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$ and $|f'|^q$ s-convex on $[a, b]$, for every division D of $[a, b]$, the trapezoidal error estimate satisfies

$$|R(f,D)| \leq \frac{1}{2^{\frac{p+1}{p}}}\left(\frac{s.2^s + 1}{2^s(s+1)(s+2)}\right)^{\frac{1}{q}} \sum_{k=0}^{n-1} (x_{k+1} - x_k)^2 (|f'(x_k)| + |f'(x_{k+1})|) \quad (4.54)$$

Where $p > 1$.

Proof. On applying Corollary 8 on the subinterval $[x_k,\ x_{k+1}]$ of the division D of $[a,b]$ for $k = 0, 1, 2, \dots, n-1$, we have

$$\left|\frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}} f(x)dx - \frac{f(x_k)+f(x_{k+1})}{2}\right|$$

$$\leq \frac{(x_{k+1}-x_k)}{2^{\frac{p+1}{p}}}\left(\frac{s.2^s+1}{2^s(s+1)(s+2)}\right)^{1/q}(|f'(x_k)| + |f'(x_{k+1})|) \tag{4.55}$$

Taking sum over k from $0\ to\ n-1$.. And using s- convexity of $|f'|^q$, we get,

$$\left|\int_a^b f(x)dx - S(f,D)\right|$$

$$= \left|\sum_{k=0}^{n-1}\left(\int_{x_k}^{x_{k+1}} f(x)dx - (x_{k+1}-x_k)\frac{f(x_{k+1})+f(x_k)}{2}\right)\right|$$

$$\leq \sum_{k=0}^{n-1}\left|\int_{x_k}^{x_{k+1}} f(x)dx - (x_{k+1}-x_k)\frac{f(x_{k+1})+f(x_k)}{2}\right|$$

$$\sum_{k=0}^{n-1}\left|\int_{x_k}^{x_{k+1}} f(x)dx - (x_{k+1}-x_k)\frac{f(x_{k+1})+f(x_k)}{2}\right|$$

$$|R(f,D)| \leq \sum_{k=0}^{n-1}(x_{k+1}-x_k)\left|\frac{1}{(x_{k+1}-x_k)}\int_{x_k}^{x_{k+1}} f(x)dx - \frac{f(x_{k+1})+f(x_k)}{2}\right| \tag{4.56}$$

Using (4.55) and (4.56), we get (4.54).

Proposition 5. Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable function on I^o such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b\ and\ |f'|^q$ s- convex on $[a,b]$, for every division D of $[a, b]$, the trapezoidal error estimate satisfies

$$|R(f,D)| \leq \frac{1}{3^{\frac{1}{p}}}\left(\frac{s^2+3s+4}{2(s+1)(s+2)(s+3)}\right)^{\frac{1}{q}}$$

$$\sum_{k=0}^{n-1}(x_{k+1}-x_k)^2(|f'(x_k)| + |f'(x_{k+1})|)$$

Where $p > 1$.

Proof. The proof is similar to that of Proposition 4 and using Corollary 16.

5. CONCLUSIONS

By selecting some other convex function, and applying the results given in section 2, we can find out some new relations connecting to some special means. For example, choosing different convex function like $f(x) = -Inx$, $f(x) = \frac{1}{x}$ and $f(x) = -In(1-x)$ for different values of s from (0, 1] in s-convexity (concavity), we get new relation relating to some special means.

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