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Original Article

Some Inclusion Properties of p-Valent Meromorphic Functions defined by the Wright Generalized Hypergeometric Function

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Abstract: In this paper, using the Wright generalized hypergeometric function we define a new operator and some classes of meromorphic functions associated to it and investigate several inclusion properties of these classes. Some applications involving integral operator are also considered.

Keywords and phrases: p-Valent meromorphic functions, Hadamard product, Wright generalized hyper-geaometric hypergeaometric function, inclusion relationships

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1. INTRODUCTION

Let Σ_p denote the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=1-p}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1,2,\dots\}), \tag{1.1}$$

which are analytic and p-valent in the punctured unit disc $U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = U \setminus \{0\}$. For two analytic functions f and g in U, f is said to be subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ in U, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. If $g(z)$ is univalent function, then $f \prec g$ if and only if (see [8] and [16])

$$f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For functions $f(z) \in \Sigma_p$ given by (1.1) and $g(z) \in \Sigma_p$ defined by

$$g(z) = z^{-p} + \sum_{k=1-p}^{\infty} b_k z^k \quad (p \in \mathbb{N}), \tag{1.2}$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = z^{-p} + \sum_{k=1-p}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.3}$$

For $0 \le \eta, \gamma < p$, let $MS^*(\eta,p)$, $MK(\eta,p)$, $MC(\eta,\gamma,p)$ and $MC^*(\eta,\gamma,p)$, be the subclasses of Σ_p consisting of all meromorphic functions which are, respectively, starlike of order η, convex of order η, close-to-convex of order γ and type η and quasi-convex functions of order γ and type η in U. Let $\mathcal{S}$ be the class of all functions ϕ which are analytic and univalent in U and for which $\phi(U)$ is convex with $\phi(0) = 1$ and $\mathrm{Re}\{\phi(z)\} > 0$ $(z \in U)$. Making use of the principle of subordination between analytic functions, let the subclasses $MS_p^*(\eta;\phi)$, $MK_p(\eta;\phi)$, $MC_p(\eta,\gamma;\phi,\psi)$ and $MC_p^*(\eta,\gamma;\phi,\psi)$ of the class Σ_p for

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$0\le\eta,\ \gamma<p$ and $\phi,\psi\in\mathcal{S}$, be defined as follows:

$$MS_p^*(\eta;\varphi)=\left\{f\in\Sigma_p\ :\ \frac{1}{p-\eta}\left(-\frac{zf'(z)}{f(z)}-\eta\right)\prec\varphi(z)\ (z\in U)\right\},$$

$$MK_p(\eta;\varphi)=\left\{\begin{array}{c} f\in\Sigma_p\ :\ \dfrac{1}{p-\eta} \\ \left(-\left[1+\dfrac{zf''(z)}{f'(z)}\right]-\eta\right)\prec\varphi(z)\ (z\in U)\end{array}\right\},$$

$$MC_p(\eta,\gamma;\varphi,\psi)=\left\{\begin{array}{c} f\in\Sigma_p\ :\exists g\in MS_p^*(\eta;\varphi) \\ \text{s. t. } \dfrac{1}{p-\gamma}\left(-\dfrac{zf'(z)}{g(z)}-\gamma\right)\prec\psi(z)\ (z\in U)\end{array}\right\}$$

and

$$MC_p^*(\eta,\gamma;\varphi,\psi)=\left\{\begin{array}{c} f\in\Sigma_p\ :\exists g\in MS_p^*(\eta;\varphi)\text{s. t.} \\ \dfrac{1}{p-\gamma}\left(-\dfrac{(zf'(z))'}{g(z)}-\gamma\right)\prec\psi(z)\ (z\in U)\end{array}\right\}.$$

From these defnitions, we can obtain some well-known subclasses of Σ_p by special choices of the functions ϕ and ψ as well as special choices of η and γ see ([6], [13] and [22]).

Let $\alpha_1,A_1,\ldots,\alpha_q,A_q$ and $\beta_1,B_1,\ldots,\beta_s,B_s$ $(q,s\in\mathbb{N})$ be positive and real parameters such that

$$1+\sum_{j=1}^{s}B_j-\sum_{j=1}^{q}A_j\ \ge\ 0.$$

The Wright generalized hypergeometric function [23] (see also [25]

$$\begin{aligned}&{}_q\Psi_s\left[(\alpha_1,A_1),\ldots,(\alpha_q,A_q);(\beta_1,B_1),\ldots,(\beta_s,B_s);z\right]\\&={}_q\Psi_s\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,s};z\right],\end{aligned}$$

is defined by

$$\begin{aligned}&{}_q\Psi_s\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,s};z\right]\\&=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{q}\Gamma(\alpha_i+kA_i)}{\prod_{i=1}^{s}\Gamma(\beta_i+kB_i)}\cdot\frac{z^k}{k!}\quad(z\in U).\end{aligned}$$

If $A_i=1(i=1,\ldots,q)$ and $B_i=1(i=1,\ldots,s)$, we have the relationship:

$$\begin{aligned}&\Omega_q\Psi_s\left[(\alpha_i,1)_{1,q};(\beta_i,1)_{1,s};z\right]\\&={}_qF_s\left(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z\right),\end{aligned}$$

where ${}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)$ is the generalized hypergeometric function (see for details [20] and [24]) and

$$\Omega=\frac{\prod_{i=1}^{s}\Gamma(\beta_i)}{\prod_{i=1}^{q}\Gamma(\alpha_i)}. \tag{1.4}$$

Consider the following linear operator due to Dziok and Raina [10] (see also [3] and [11]):

$$\theta_{p,q,s}\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,s}\right]:\Sigma_p\to\Sigma_p,$$

defined by the convolution

$$\begin{aligned}&\theta_{p,q,s}\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,s}\right]f(z)\\&=\Phi_{p,q,s}\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,s};z\right]*f(z),\end{aligned}$$

where $\Phi_{p,q,s}\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,s};z\right]$ was defined by Bansal et al. [7] as follows:

$$\begin{aligned}&\Phi_{p,q,s}\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,s};z\right]\\&=\Omega\, z^{-p}{}_q\Psi_s\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,s};z\right](z\in U^*).\end{aligned} \tag{1.5}$$

We observe that, for a function $f(z)$ of the form (1.1), we have

$$\begin{aligned}&\theta_{p,q,s}\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,s}\right]f(z)\\&=z^{-p}+\sum_{k=1-p}^{\infty}\Omega\sigma_{k,p}(\alpha_1,A_1,B_1)a_kz^k,\end{aligned} \tag{1.6}$$

where Ω is given by (1.4) and $\sigma_{k,p}(\alpha_1,A_1,B_1)$ is defined by

$$\sigma_{k,p}(\alpha_1,A_1,B_1)=\frac{\Gamma(\alpha_1+A_1(k+p))\ldots\Gamma(\alpha_q+A_q(k+p))}{\Gamma(\beta_1+B_1(k+p))\ldots\Gamma(\beta_s+B_s(k+p))(k+p)!}.$$

Corresponding to the function $\Phi_{p,q,s}\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,s};z\right]$ defined by (1.5), we introduce a function

$\Phi^{\lambda}_{p,q,s}\left[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z\right]$ by

$$\Phi_{p,q,s}\left[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z\right] * \Phi^{\lambda}_{p,q,s}\left[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z\right] = \frac{1}{z^p (1-z)^{\lambda}} \ (\lambda > 0).$$

Analogous to $\theta_{p,q,s}\left[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}\right]$ defined by (1.6), we define the linear operator $\theta^{\lambda}_{p,q,s}\left[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}\right] : \Sigma_p \to \Sigma_p$ as follows:

$$\theta^{\lambda}_{p,q,s}\left[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}\right] f(z) = \Phi^{\lambda}_{p,q,s}\left[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z\right] * f(z)$$
$$= z^{-p} + \sum_{k=1-p}^{\infty} \frac{\Gamma(\lambda+k+p)\prod_{i=1}^{s}\Gamma(\beta_i+(k+p)B_i)\prod_{i=1}^{q}\Gamma(\alpha_i)}{\Gamma(\lambda)\prod_{i=1}^{q}\Gamma(\alpha_i+(k+p)A_i)\prod_{i=1}^{s}\Gamma(\beta_i)} a_k z^k \quad (1.8)$$

$(f \in \Sigma_p; \lambda > 0; z \in U^*)$.

For convenience, we write

$$\theta^{\lambda}_{p,q,s}[\alpha_1, A_1, B_1] f(z) = \theta^{\lambda}_{p,q,s}\begin{bmatrix}(\alpha_1, A_1), \dots, (\alpha_q, A_q); \\ (\beta_1, B_1), \dots, (\beta_s, B_s)\end{bmatrix} f(z).$$

One can easily verify from (1.8) that

$$zA_1\left(\theta^{\lambda}_{p,q,s}[\alpha_1+1, A_1, B_1] f(z)\right)' = \alpha_1 \theta^{\lambda}_{p,q,s}[\alpha_1, A_1, B_1] f(z) - (\alpha_1 + pA_1)\theta^{\lambda}_{p,q,s}[\alpha_1+1, A_1, B_1] f(z) \quad (A_1 > 0) \quad (1.9)$$

and

$$z\left(\theta^{\lambda}_{p,q,s}[\alpha_1, A_1, B_1] f(z)\right)' = \lambda\theta^{\lambda+1}_{p,q,s}[\alpha_1, A_1, B_1] f(z) - (\lambda+p)\theta^{\lambda}_{p,q,s}[\alpha_1, A_1, B_1] f(z) \quad (\lambda > 0). \quad (1.10)$$

Specializing the parameters $p, q, s, A_i (i = 1, \dots, q)$, $B_i (i = 1, \dots, s)$ and λ in (1.8) we have:

(i) For $A_i = 1 \, (i = 1, \dots, q)$, $B_i = 1 \, (i = 1, \dots, s)$ and $\lambda = \mu + p \ (\mu > -p, \ p \in \mathbb{N})$, we have $\theta^{\mu+p}_{p,q,s}[\alpha_1, 1, 1] f(z) = M^{\mu}_{p,q,s}(\alpha_1) f(z)$, where the operator $M^{\mu}_{p,q,s}(\alpha_1)$ was introduced by Patel and Patil [21] and Mostafa [17];

(ii) For $A_i = 1, \dots, q)$, $B_i = 1(I = 1, \dots, s)$, $q = 2$, $s = 1$, $\alpha_1 = n + p \ (n > -p, \ p \in \mathbb{N})$ and $\alpha_2 = \beta_1 = \lambda \ (\lambda > 0)$, we have $\theta^{\lambda}_{p,2,1}[n+p, \lambda; \lambda] f(z) = I_{n+p-1,\lambda} f(z)$, where the operator $I_{n+p-1,\lambda}$ was introduced by Aouf and Xu [5] which for $p = 1$ reduces to $I_{n,\lambda} \ (n > -1, \lambda > 0)$, where the operator $I_{n,\lambda}$ was introduced by Yuan et al. [28];

(iii) For $A_i = 1 \, (i = 1, \dots, q), B_i = 1 \, (i = 1, \dots, s)$, $\lambda = n + p \ (n > -p, p \in \mathbb{N})$, $q = 2$, $s = 1$ and $\alpha_1 = \alpha_2 = \beta_1 = 1$, we have $\theta^{n+p}_{p,2,1}[1,1;1] f(z) = D^{n+p-1} f(z)$, where the operator D^{n+p-1} was introduced by Yang [26] and Aouf ([1] and [2]);

(iv) For $p = 1$, we have $\theta^{\lambda}_{1,q,s}[\alpha_1, A_1, B_1] f(z) = \theta_{\lambda,q,s}(\alpha_1, A_1, B_1) f(z)$, where the operator $\theta_{\lambda,q,s}(\alpha_1, A_1, B_1)$ was introduced by Aouf et al. [4];

(v) For $A_i = 1 \ (i = 1, \dots, q), B_i = 1 \ (i = 1, \dots, s)$ and $p = 1$, we have $\theta^{\lambda}_{1,q,s}[\alpha_1, 1, 1] f(z) = H_{\lambda,q,s}(\alpha) f(z)$, where the operator $H_{\lambda,q,s}(\alpha)$ was introduced by Cho and Kim [9], Muhamad [18] and Noor and Muhamad [19].

Also, we note that:

(i) For $\lambda = 1$, then the operator $\theta^{1}_{p,q,s}[\alpha_1, A_1, B_1]$ reduces to the operator $\theta_{p,q,s}[\alpha_1, A_1, B_1]$, defined by:

$$\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z) = z^{-p} + \sum_{k=1-p}^{\infty} \frac{\prod_{i=1}^{s}\Gamma(\beta_i+(k+p)B_i)\prod_{i=1}^{q}\Gamma(\alpha_i)}{\prod_{i=1}^{q}\Gamma(\alpha_i+(k+p)A_i)\prod_{i=1}^{s}\Gamma(\beta_i)} a_k z^k,$$

(ii) For $A_i = 1 \ (i = 1, \dots, q), B_i = 1 \ (i = 1, \dots, s)$ and $\lambda = 1$, then the operator $\theta^{1}_{p,q,s}[\alpha_1, 1, 1]$ reduces to the operator $N_{p,q,s}[\alpha_1]$, defined by:

$$N_{p,q,s}[\alpha_1] f(z) = z^{-p} + \sum_{k=1-p}^{\infty} \frac{(\beta_1)_{k+p} \dots (\beta_s)_{k+p}}{(\alpha_1)_{k+p} \dots (\alpha_q)_{k+p}} a_k z^k$$

$(\alpha_i \notin Z_0^- = \{0,-1,-2,...\};$
$(i = 1,2,...,q);\ p \in \mathbb{N}).$

Next, by using the operator $\theta_{p,q,s}^{\lambda}[\alpha_1, A_1, B_1]$, we introduce the following classes of meromorphic functions for $0 \le \eta,\ \gamma < p,\ \lambda > 0$ and $\phi,\psi \in \mathcal{S}$:

$$MS_{p,q,s}^{*\lambda}(\alpha_1, A_1, B_1;\eta;\varphi) = \left\{f \in \Sigma_p\ :\ \theta_{p,q,s}^{\lambda}[\alpha_1, A_1, B_1] f \in MS_p^*(\eta;\varphi)\right\},$$

$$MK_{p,q,s}^{\lambda}(\alpha_1, A_1, B_1;\eta;\varphi) = \left\{f \in \Sigma_p\ :\ \theta_{p,q,s}^{\lambda}[\alpha_1, A_1, B_1] f \in MK_p(\eta;\varphi)\right\},$$

$$MC_{p,q,s}^{\lambda}(\alpha_1, A_1, B_1;\eta,\gamma;\varphi,\psi) = \left\{f \in \Sigma_p\ :\ \theta_{p,q,s}^{\lambda}[\alpha_1, A_1, B_1] f \in MC_p(\eta,\gamma;\varphi,\psi)\right\}$$

and

$$MC_{p,q,s}^{*\lambda}(\alpha_1, A_1, B_1;\eta,\gamma;\varphi,\psi) = \left\{f \in \Sigma_p\ :\ \theta_{p,q,s}^{\lambda}[\alpha_1, A_1, B_1] f \in MC_p^*(\eta,\gamma;\varphi,\psi)\right\}.$$

We can easily see that:

$$f(z) \in MK_{p,q,s}^{\lambda}(\alpha_1, A_1, B_1;\eta;\phi) \Leftrightarrow -\frac{zf'(z)}{p} \in MS_{p,q,s}^{*\lambda}(\alpha_1, A_1, B_1;\eta;\phi) \tag{1.11}$$

and

$$f(z) \in MC_{p,q,s}^{*\lambda}(\alpha_1, A_1, B_1;\eta,\gamma;\phi,\psi) \Leftrightarrow -\frac{zf'(z)}{p} \in MC_{p,q,s}^{\lambda}(\alpha_1, A_1, B_1;\eta,\gamma;\phi,\psi). \tag{1.12}$$

In particular, for $-1 < B < A \le 1$, we set

$$MS_{p,q,s}^{*\lambda}(\alpha_1, A_1, B_1;\eta;\frac{1+Az}{1+Bz}) = MS_{p,q,s}^{*\lambda}(\alpha_1, A_1, B_1;\eta;A,B)$$

and

$$MK_{p,q,s}^{\lambda}(\alpha_1, A_1, B_1;\eta;\frac{1+Az}{1+Bz}) = MK_{p,q,s}^{\lambda}(\alpha_1, A_1, B_1;\eta;A,B).$$

In this paper we investigate several properties of the classes $MS_{p,q,s}^{*\lambda}(\alpha_1, A_1, B_1, \eta;\phi)$, $MK_{p,q,s}^{\lambda}(\alpha_1, A_1, B_1;\eta;\phi)$, $MC_{p,q,s}^{\lambda}(\alpha_1, A_1, B_1;\eta,\gamma;\phi,\psi)$ and $MC_{p,q,s}^{*\lambda}(\alpha_1, A_1, B_1;\eta,\gamma;\phi,\psi)$ associated with the operator $\theta_{p,q,s}^{\lambda}[\alpha_1, A_1, B_1]$. Some applications involving integral operator are also considered.

2. INCLUSION PROPERTIES INVOLVING THE OPERATOR $\theta_{p,q,s}^{\lambda}[\alpha_1, A_1, B_1]$

In order to prove our results, we need the following lemmas.

Lemma 1 [12]. *Let* ϕ *be convex univalent in* U *with* $\phi(0) = 1$ *and* $\operatorname{Re}\{l\phi(z) + \upsilon\} > 0$ $(l,\upsilon \in \mathbb{C})$. *If* q *is analytic in* U *with* $q(0) = 1$, *then*

$$q(z) + \frac{zq'(z)}{lq(z) + \upsilon} \prec \phi(z),$$

implies

$$q(z) \prec \phi(z).$$

Lemma 2 [15]. *Let* ϕ *be convex univalent in* U *and* ω *be analytic in* U *with* $\operatorname{Re}\{\omega(z)\} \ge 0$. *If* q *is analytic in U and* $q(0) = \phi(0)$, *then*

$$q(z) + \omega(z)zq'(z) \prec \phi(z),$$

implies

$$q(z) \prec \phi(z).$$

Theorem 1. *Let* $\phi \in \mathcal{S}$ *with*

$$\max_{z\in U}\operatorname{Re}\{\varphi(z)\} < \min\left\{\frac{\lambda+p-\eta}{p-\eta},\ \frac{\frac{\alpha_1}{A_1}+p-\eta}{p-\eta}\right\}$$

$$(\lambda, \frac{\alpha_1}{A_1} > 0, 0 \le \eta < p).$$

Then

$$MS_{p,q,s}^{*\lambda+1}(\alpha_1, A_1, B_1;\eta;\varphi) \subset MS_{p,q,s}^{*\lambda}(\alpha_1, A_1, B_1;\eta;\varphi) \subset MS_{p,q,s}^{*\lambda}(\alpha_1+1, A_1, B_1;\eta;\varphi).$$

Proof. To prove the first part, let $f \in MS^{*\lambda+1}_{p,q,s}(\alpha_1, A_1, B_1; \eta; \phi)$ and set

$$q(z) = \frac{1}{p-\eta}\left(-\frac{z\left(\theta^{\lambda}_{p,q,s}[\alpha_1, A_1, B_1] f(z)\right)'}{\theta^{\lambda}_{p,q,s}[\alpha_1, A_1, B_1] f(z)} - \eta\right) \ (z \in U), \qquad (2.1)$$

where q is analytic in U with $q(0) = 1$. Applying (1.10) in (2.1), we obtain

$$\frac{1}{p-\eta}\left(-\frac{z\left(\theta^{\lambda+1}_{p,q,s}[\alpha_1, A_1, B_1] f(z)\right)'}{\theta^{\lambda+1}_{p,q,s}[\alpha_1, A_1, B_1] f(z)} - \eta\right) \qquad (2.2)$$

$$= q(z) + \frac{zq'(z)}{-(p-\eta)q(z) + \lambda + p - \eta} \ (z \in U).$$

Since $\max_{z\in U} \mathrm{Re}\{\phi(z)\} < \frac{\lambda+p-\eta}{p-\eta}$, we see that

$$\mathrm{Re}\{-(p-\eta)q(z) + \lambda + p - \eta\} > 0 \ (z \in U).$$

Applying Lemma 1 to (2.2), it follows that $q(z) \prec \phi(z)$, that is $f(z) \in MS^{*\lambda}_{p,q,s}(\alpha_1, A_1, B_1, \eta; \phi)$. Moreover, by using the arguments similar to those detailed above with (1.9), we can prove the second part. Therefore the proof is completed.

Theorem 2. *Let* $\phi \in \mathcal{S}$ *with*

$$\max_{z\in U} \mathrm{Re}\{\varphi(z)\} < \min\left\{\frac{\lambda+p-\eta}{p-\eta}, \frac{\frac{\alpha_1}{A_1}+p-\eta}{p-\eta}\right\} \ (\lambda, \tfrac{\alpha_1}{A_1} > 0, 0 \le \eta < p).$$

Then

$$MK^{\lambda+1}_{p,q,s}(\alpha_1, A_1, B_1; \eta; \varphi)$$
$$\subset MK^{\lambda}_{p,q,s}(\alpha_1, A_1, B_1; \eta; \varphi)$$
$$\subset MK^{\lambda}_{p,q,s}(\alpha_1 + 1, A_1, B_1; \eta; \varphi).$$

Proof. Applying (1.11) and using Theorem 1, we observe that

$$f(z) \in MK^{\lambda+1}_{p,q,s}(\alpha_1, A_1, B_1; \eta; \varphi)$$
$$\Leftrightarrow -\frac{zf'(z)}{p} \in MS^{*\lambda+1}_{p,q,s}(\alpha_1, A_1, B_1; \eta; \varphi)$$
$$\Rightarrow -\frac{zf'(z)}{p} \in MS^{*\lambda}_{p,q,s}(\alpha_1, A_1, B_1; \eta; \varphi)$$
$$\Leftrightarrow f(z) \in MK^{\lambda}_{p,q,s}(\alpha_1, A_1, B_1; \eta; \varphi).$$

Also

$$f(z) \in MK^{\lambda}_{p,q,s}(\alpha_1, A_1, B_1; \eta; \varphi)$$
$$\Leftrightarrow -\frac{zf'(z)}{p} \in MS^{*\lambda}_{p,q,s}(\alpha_1, A_1, B_1; \eta; \varphi)$$
$$\Rightarrow -\frac{zf'(z)}{p} \in MS^{*\lambda}_{p,q,s}(\alpha_1 + 1, A_1, B_1; \eta; \varphi)$$
$$\Leftrightarrow f(z) \in MK^{\lambda}_{p,q,s}(\alpha_1 + 1, A_1, B_1; \eta; \varphi),$$

which evidently proves Theorem 2.

Taking

$$\varphi(z) = \frac{1+Az}{1+Bz} \ (-1 < B < A \le 1),$$

in Theorem 1 and Theorem 2, we have

Corollary 1. *Let*

$$\frac{1+A}{1+B} < \min\left\{\frac{\lambda+p-\eta}{p-\eta}, \frac{\frac{\alpha_1}{A_1}+p-\eta}{p-\eta}\right\}$$

$$(\lambda, \tfrac{\alpha_1}{A_1} > 0, 0 \le \eta < p, -1 < B < A \le 1).$$

Then

$$MS^{*\lambda+1}_{p,q,s}(\alpha_1, A_1, B_1; \eta; A, B)$$
$$\subset MS^{*\lambda}_{p,q,s}(\alpha_1, A_1, B_1; \eta; A, B)$$
$$\subset MS^{*\lambda}_{p,q,s}(\alpha_1 + 1, A_1, B_1; \eta; A, B),$$

and

$$MK^{\lambda+1}_{p,q,s}(\alpha_1, A_1, B_1; \eta; A, B)$$
$$\subset MK^{\lambda}_{p,q,s}(\alpha_1, A_1, B_1; \eta; A, B)$$
$$\subset MK^{\lambda}_{p,q,s}(\alpha_1 + 1, A_1, B_1; \eta; A, B).$$

Next, by using Lemma 2, we obtain the following inclusion relations for the class $MC^{\lambda}_{p,q,s}(\alpha_1, A_1, B_1; \eta, \gamma; \phi, \psi)$.

Theorem 3. *Let* $\phi, \psi \in \mathcal{S}$ *with* $\max_{z\in U} \mathrm{Re}\{\phi(z)\} < \min\left\{\frac{\lambda+p-\eta}{p-\eta}, \frac{\frac{\alpha_1}{A_1}+p-\eta}{p-\eta}\right\}$ $(\lambda, \tfrac{\alpha_1}{A_1} > 0,\ 0 \le \eta, \gamma < p)$. *Then*

$$MC^{\lambda+1}_{p,q,s}(\alpha_1, A_1, B_1; \eta, \gamma; \varphi, \psi)$$
$$\subset MC^{\lambda}_{p,q,s}(\alpha_1, A_1, B_1; \eta, \gamma; \varphi, \psi)$$
$$\subset MC^{\lambda}_{p,q,s}(\alpha_1 + 1, A_1, B_1; \eta, \gamma; \varphi, \psi).$$

Proof. To prove the first inclusion, let $f(z) \in MC_{p,q,s}^{\lambda+1}(\alpha_1, A_1, B_1; \eta, \gamma; \phi, \psi)$. Then, from the definition of $MC_{p,q,s}^{\lambda+1}(\alpha_1, A_1, B_1; \eta, \gamma; \phi, \psi)$, there exists a function $g(z) \in MS_{p,q,s}^{*\lambda+1}(\alpha_1, A_1, B_1; \eta; \phi)$ such that

$$\frac{1}{p-\gamma}\left(-\frac{z(\theta_{p,q,s}^{\lambda+1}[\alpha_1, A_1, B_1]f(z))'}{\theta_{p,q,s}^{\lambda+1}[\alpha_1, A_1, B_1]g(z)} - \gamma\right) \prec \psi(z).$$

Let

$$q(z) = \frac{1}{p-\gamma}\left(-\frac{z(\theta_{p,q,s}^{\lambda}[\alpha_1, A_1, B_1]f(z))'}{\theta_{p,q,s}^{\lambda}[\alpha_1, A_1, B_1]g(z)} - \gamma\right) \quad (z \in U), \tag{2.3}$$

where $q(z)$ is analytic function in U with $q(0)=1$. Using (1.10), we have

$$[-(p-\gamma)q(z)-\gamma]\theta_{p,q,s}^{\lambda}[\alpha_1, A_1, B_1]g(z)$$
$$+(\lambda+p)\theta_{p,q,s}^{\lambda}[\alpha_1, A_1, B_1]f(z)$$
$$= \lambda\theta_{p,q,s}^{\lambda+1}[\alpha_1, A_1, B_1]f(z). \tag{2.4}$$

Differentiating (2.4) with respect to z and multiplying by z, we obtain

$$-(p-\gamma)zq'(z)\theta_{p,q,s}^{\lambda}[\alpha_1, A_1, B_1]g(z)$$
$$+[-(p-\gamma)q(z)-\gamma]z(\theta_{p,q,s}^{\lambda}[\alpha_1, A_1, B_1]g(z))'$$
$$= \lambda z(\theta_{p,q,s}^{\lambda+1}[\alpha_1, A_1, B_1]f(z))'$$
$$-(\lambda+p)z(\theta_{p,q,s}^{\lambda}[\alpha_1, A_1, B_1]f(z))'.$$

Since $g(z) \in MS_{p,q,s}^{*\lambda+1}(\alpha_1, A_1, B_1; \eta; \phi) \subset MS_{p,q,s}^{*\lambda}(\alpha_1, A_1, B_1; \eta; \phi)$, by Theorem 1, we set

$$\chi(z) = \frac{1}{p-\eta}\left(-\frac{z(\theta_{p,q,s}^{\lambda}[\alpha_1, A_1, B_1]g(z))'}{\theta_{p,q,s}^{\lambda}[\alpha_1, A_1, B_1]g(z)} - \eta\right), \tag{2.5}$$

where $\chi(z) \prec \phi(z)$ in U with the assumption $\phi \in \mathcal{S}$. Then, by using (2.3), (2.4) and (2.5), we have

$$\frac{1}{p-\gamma}\left(-\frac{z(\theta_{p,q,s}^{\lambda+1}[\alpha_1, A_1, B_1]f(z))'}{\theta_{p,q,s}^{\lambda+1}[\alpha_1, A_1, B_1]g(z)} - \gamma\right) \tag{2.6}$$
$$= q(z) + \frac{zq'(z)}{-(p-\eta)\chi(z)+\lambda+p-\eta} \prec \psi(z).$$

Since $\lambda > 0$ and $\chi(z) \prec \phi(z)$ in U with $\max_{z\in U}\mathrm{Re}\{\phi(z)\} < \frac{\lambda+p-\eta}{p-\eta}$, then

$$\mathrm{Re}\{-(p-\eta)\chi(z)+\lambda+p-\eta\} > 0 \quad (z \in U).$$

Hence, by taking

$$\omega(z) = \frac{1}{-(p-\eta)\chi(z)+\lambda+p-\eta},$$

in (2.6) and applying Lemma 2, we have $q(z) \prec \psi(z)$ in U, so that $f(z) \in MC_{p,q,s}^{\lambda}(\alpha_1, A_1, B_1; \eta, \gamma; \phi, \psi)$. The second inclusion can be proved by using arguments similar to those detailed above with (1.9). This compelets the proof of Theorem 3.

Theorem 4. *Let* $\phi, \psi \in \mathcal{S}$ *with*

$$\max_{z\in U}\mathrm{Re}\{\phi(z)\} < \min\left\{\frac{\lambda+p-\eta}{p-\eta}, \frac{\frac{\alpha_1}{A_1}+p-\eta}{p-\eta}\right\}$$

$$(\lambda, \tfrac{\alpha_1}{A_1} > 0,\ 0 \le \eta, \gamma < p).$$

Then

$$MC_{p,q,s}^{*\lambda+1}(\alpha_1, A_1, B_1; \eta, \gamma; \varphi, \psi)$$
$$\subset MC_{p,q,s}^{*\lambda}(\alpha_1, A_1, B_1; \eta, \gamma; \varphi, \psi)$$
$$\subset MC_{p,q,s}^{*\lambda}(\alpha_1+1, A_1, B_1; \eta, \gamma; \varphi, \psi).$$

Proof. Just as we derived Theorem 2 as a consequence of Theorem 1 by using the equivalence (1.11), we can also prove Theorem 4 by using Theorem 3 in conjunction with the equivalence (1.12).

3. PROPERTIES FOR THE INTEGRAL OPERATOR $F_{\mu,p}$

Let $F_{\mu,p}$ be the integral operator defined by (see [14] and [27]):

$$F_{\mu,p}(f)(z) = \frac{\mu}{z^{\mu+p}}\int_0^z t^{\mu+p-1}f(t)dt$$
$$= (z^{-p} + \sum_{k=1-p}^{\infty}\frac{\mu}{\mu+k+p}z^k) * f(z) \tag{3.1}$$

$(f \in \Sigma_p; \mu > 0; z \in U^*)$.

From (3.1), we observe that

$$z(\theta^{\lambda}_{p,q,s}[\alpha_1, A_1, B_1]F_{\mu,p}(f)(z))' = \mu\theta^{\lambda}_{p,q,s}[\alpha_1, A_1, B_1]f(z) - (\mu + p)\theta^{\lambda}_{p,q,s}[\alpha_1, A_1, B_1]F_{\mu,p}(f)(z) \ (\mu > 0).$$

The proof of Theorem 5 below, is much akin to that of Theorem 1, so, we omit it.

Theorem 5. *Let* $\phi \in \mathcal{S}$ *with* $\max_{z\in U}\mathrm{Re}\{\phi(z)\} < \frac{\mu+p-\eta}{p-\eta}$ $(\mu > 0,\ 0 \le \eta < p)$. *If* $f(z) \in MS^{*\lambda}_{p,q,s}(\alpha_1, A_1, B_1; \eta; \phi)$, *then* $F_{\mu,p}(f)(z) \in MS^{*\lambda}_{p,q,s}(\alpha_1, A_1, B_1; \eta; \phi)$.

Next, we derive an inclusion property involving $F_{\mu,p}$, which is obtained by applying (1.11) and Theorem 1.

Theorem 6. *Let* $\phi \in \mathcal{S}$ *with* $\max_{z\in U}\mathrm{Re}\{\phi(z)\} < \frac{\mu+p-\eta}{p-\eta}$ $(\mu > 0,\ 0 \le \eta < p)$. *If* $f(z) \in MK^{\lambda}_{p,q,s}(\alpha_1, A_1, B_1; \eta; \phi)$, *then* $F_{\mu,p}(f)(z) \in MK^{\lambda}_{p,q,s}(\alpha_1, A_1, B_1; \eta; \phi)$.

Taking $\phi(z) = \frac{1+A}{1+B}$ $(-1 < B < A \le 1)$ and from Theorems 5 and 6, we have

Corollary 2. *Let* $\frac{1+A}{1+B} < \frac{\lambda+p-\eta}{p-\eta}$ $(\mu > 0,\ 0 \le \eta < p,\ -1 < B < A \le 1)$. *Then if* $f(z) \in MS^{*\lambda}_{p,q,s}(\alpha_1, A_1, B_1; \eta; A, B)$ *(or* $MK^{\lambda}_{p,q,s}(\alpha_1, A_1, B_1; \eta; A, B)$*), then*

$F_{\mu,p}(f)(z) \in MS^{*\lambda}_{p,q,s}(\alpha_1, A_1, B_1; \eta; A, B)$ *(or* $MK^{\lambda}_{p,q,s}(\alpha_1, A_1, B_1; \eta; A, B)$*).*

Finally, we obtain Theorems 7 and 8 below by using the same techniques as in the proof of Theorems 3 and 4.

Theorem 7. *Let* $\phi, \psi \in \mathcal{S}$ *with* $\max_{z\in U}\mathrm{Re}\{\phi(z)\} < \frac{\mu+p-\eta}{p-\eta}$ $(\mu > 0,\ 0 \le \eta, \gamma < p)$. *If* $f(z) \in MC^{\lambda}_{p,q,s}(\alpha_1, A_1, B_1; \eta, \gamma; \varphi, \psi)$, *then* $F_{\mu,p}(f)(z) \in MC^{\lambda}_{p,q,s}(\alpha_1, A_1, B_1; \eta, \gamma; \phi, \psi)$.

Theorem 8. *Let* $\phi, \psi \in \mathcal{S}$ *with* $\max_{z\in U}\mathrm{Re}\{\phi(z)\} < \frac{\mu+p-\eta}{p-\eta}$ $(\mu > 0,\ 0 \le \eta, \gamma < p)$. *If* $f(z) \in MC^{*\lambda}_{p,q,s}(\alpha_1, A_1, B_1; \eta, \gamma; \varphi, \psi)$, *then* $F_{\mu,p}(f)(z) \in MC^{*\lambda}_{p,q,s}(\alpha_1, A_1, B_1; \eta, \gamma; \varphi, \psi)$.

Remark 1. *(i) If we take* $p = 1$, $A_n = 1$ $(n = 1, \ldots, q)$ *and* $B_n = 1$ $(n = 1, \ldots, s)$ *in the above results of this paper, we obtain the results obtained by Cho and Kim [9];*

(ii) If we take $p = 1$, $A_i = 1\ (i = 1, \ldots, q), B_i = 1\ (i = 1, \ldots, s)$, $q = 2$, $s = 1$, $\alpha_1 = n + 1\ (n > -1)$ and $\alpha_2 = \beta_1 = \lambda\ (\lambda > 0)$ *in the above results of this paper, we obtain the results obtained by Yuan et al. [28];*

(iii) If we take $p = 1$ *in the above results of this paper, we obtain the results obtained by Aouf et al. [4].*

Remark 2. *Specializing the parameters* $p, q, s, A_i\ (i = 1, \ldots, q)$, $B_i\ (i = 1, \ldots, s)$ *and* λ *in the above results of this paper, we obtain the results for the corresponding operators* $M^{\mu}_{p,q,s}(\alpha_1), I_{n+p-1,\lambda}$ *and* D^{n+p-1} *which are defined in the introduction.*

4. CONCLUSIONS

In this paper, using the Wright generalized hypergeometric function we define a new operator which contains many other operators as special cases of it. Also, we define some classes of meromorphic functions associated to this operator by using the principle of subordination and investigate several inclusion properties of these classes. Some applications involving integral operator are also considered. Our results generalize many previous results.

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