A Study on Subordination Results for Certain Subclasses of Analytic Functions defined by Convolution

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Abstract: In this paper, we drive several interesting subordination results of certain classes of analytic functions defined by convolution.

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1.  INTRODUCTION

Let \( A \) denote the class of functions of the form:

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}
\]

which are analytic in the open unit disc \( \mathbb{U} = \{z \in \mathbb{C}: |z| < 1\} \). Let \( \varphi \in A \) be given by

\[
\varphi(z) = z + \sum_{k=2}^{\infty} c_k z^k. \tag{1.2}
\]

Definition 1. (Hadamard product or convolution). Given two functions \( f \) and \( \varphi \) in the class \( A \), where \( f(z) \) is given by (1.1) and \( \varphi(z) \) is given by (1.2) the Hadamard product (or convolution) \( f \ast \varphi \) of \( f \) and \( \varphi \) is defined (as usual) by

\[
(f \ast \varphi)(z) = z + \sum_{k=2}^{\infty} a_k c_k z^k = (\varphi \ast f)(z). \tag{1.3}
\]

We also denote by \( K \) the class of functions \( f(z) \in A \) that are convex in \( \mathbb{U} \).

Let \( M(\beta) \) be the subclass of \( A \) consisting of functions \( f(z) \) which satisfy the inequality:

\[
Re \left\{ \frac{zf''(z)}{f'(z)} \right\} < \beta \quad (z \in \mathbb{U}), \tag{1.4}
\]

for some \( \beta > 1 \). Also let \( N(\beta) \) denote the subclass of \( A \) consisting of functions \( f(z) \) which satisfy the inequality:

\[
Re \left\{ 1 + \frac{zf'''(z)}{f''(z)} \right\} < \beta \quad (z \in \mathbb{U}), \tag{1.5}
\]

for some \( \beta > 1 \) (see [7], [8], [9] and [10]). For \( 1 < \beta \leq \frac{4}{3} \), the classes \( M(\beta) \) and \( N(\beta) \) were investigated earlier by Uralegaddi et al. [14] (see also [12] and [13]). It follows from (1.4) and (1.5) that

\[
f(z) \in N(\beta) \iff zf'(z) \in M(\beta). \tag{1.6}
\]

For \( 0 \leq \lambda < 1, \beta > 1 \) and for all \( z \in \mathbb{U} \), let \( T(g, \lambda, \beta) \) be the subclass of \( A \) consisting of functions \( f(z) \) of the form (1.1) and functions

\[
g(z) = z + \sum_{k=2}^{\infty} b_k z^k \quad (b_k > 0), \tag{1.7}
\]

which satisfying the analytic criterion:

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\[ \text{Re} \left\{ \frac{z(f \ast g)'(z)}{(1 - \lambda)(f \ast g)(z) + \lambda z(f \ast g)'(z)} \right\} < \beta. \quad (1.8) \]

We note that:

(i) \( T\left( \frac{z}{1-z}, 0, \beta \right) = M(\beta) \) and \( T\left( \frac{z}{(1-z)^2}, 0, \beta \right) = N(\beta) \) (\( \beta > 1 \)) (see [7]);

(ii) \( T(g, 0, \beta) = M(g, \beta)(\beta > 1) \) (see [1]).

Also we note that:

(i) \( T\left( \frac{z}{1-z}, \lambda, \beta \right) = T_M(\lambda, \beta) \)
\[ = \left\{ f \in A : \text{Re} \left\{ \frac{zf'(z)}{(1 - \lambda)f(z) + \lambda zf'(z)} \right\} < \beta \ (0 \leq \lambda < 1, \beta > 1, z \in \mathbb{U} \right\}; \]

(ii) \( T\left( \frac{z}{(1-z)^2}, \lambda, \beta \right) = T_N(\lambda, \beta) \)
\[ = \left\{ f \in A : \text{Re} \left\{ \frac{f'(z) + zf''(z)}{f'(z) + \lambda zf''(z)} \right\} < \beta \ (0 \leq \lambda < 1, \beta > 1, z \in \mathbb{U} \right\}; \]

(iii) \( T\left( z + \sum_{k=0}^{\infty} \Gamma_k(a_k)z^k, \lambda, \beta \right) = T_{q,s}(\alpha_1, \lambda, \beta) \)
\[ = \left\{ f \in A : \text{Re} \left\{ \frac{z(H_q,s(\alpha_1, \beta_1)f(z))'}{(1 - \lambda)H_q,s(\alpha_1, \beta_1)f(z) + \lambda z(H_q,s(\alpha_1, \beta_1)f(z))'} \right\} < \beta \}, \]

where \( \Gamma_k(\alpha_1) \) is defined by
\[ \Gamma_k(\alpha_1) = \frac{(\alpha_1)_{k-1} \ldots (\alpha_q)_{k-1}}{(\beta_1)_{k-1} \ldots (\beta_s)_{k-1}(1)_{k-1}} \quad (1.9) \]
\[ (\alpha_i > 0, i = 1, \ldots, q; \beta_j > 0, j = 1, \ldots, s; q \leq s + 1, q, s \in \mathbb{N}_0, \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1,2, \ldots\} ) , \]

and the operator \( H_{q,s}(\alpha_1, \beta_1) \) was introduced and studied by Dziok and Srivastava ([4] and [5]), which is a generalization of many other linear operators considered earlier.

(iv) \( T\left( z + \sum_{k=0}^{\infty} c_k(b_0)z^k, \lambda, \beta \right) = T(\mu, \ell, \lambda, \beta) \)
\[ = \left\{ f \in A : \text{Re} \left\{ \frac{z(\Gamma^m(\mu, \ell)f(z))'}{(1 - \lambda)\Gamma^m(\mu, \ell)f(z) + \lambda z(\Gamma^m(\mu, \ell)f(z))'} \right\} < \beta \}, \]

where \( m \in \mathbb{N}_0, \mu, \ell \geq 0, z \in \mathbb{U} \) and the operator \( \Gamma^m(\mu, \ell) \) was defined by Cătaş et al. [3], which is a generalization of many other linear operators considered earlier;

\[ \text{Re} \left\{ z \Gamma^m(\mu, \ell)f(z) \right\}' \]
\[ = \left\{ f \in A : \text{Re} \left\{ \frac{z(\Gamma^m(\mu, \ell)f(z))'}{(1 - \lambda)\Gamma^m(\mu, \ell)f(z) + \lambda z(\Gamma^m(\mu, \ell)f(z))'} \right\} < \beta \} \]
\[ \left\{ f \in A : \text{Re} \left\{ \frac{z(\Gamma^m(\mu, \ell)f(z))'}{(1 - \lambda)\Gamma^m(\mu, \ell)f(z) + \lambda z(\Gamma^m(\mu, \ell)f(z))'} \right\} < \beta \} \]

Where \( C_k(b, \mu) \) is defined by
\[ C_k(b, \mu) = \frac{(1 + b)\mu}{k + b} \quad (\mu \in \mathbb{C}, b \in \mathbb{C} \{\mathbb{Z}_0^+\}, \mathbb{Z}_0^- = \mathbb{Z} \backslash \mathbb{N} ), \quad (1.10) \]

and the operator \( f^\mu_b \) was introduced by Srivastava and Attiya [11], which is a generalization of many other linear operators considered earlier.

**Definition 2.** (Subordination principle). For two functions \( f \) and \( \varphi \), analytic in \( \mathbb{U} \), we say that the function \( f(z) \) is subordinate to \( \varphi(z) \) in \( \mathbb{U} \), written \( f(z) \prec \varphi(z) \), if there exists a Schwarz function \( w(z) \), which (by definition) is analytic in \( \mathbb{U} \) with \( w(0) = 0 \) and \( |w(z)| < 1 \), such that \( f(z) = \varphi(w(z)) \). Indeed it is known that
\[ f(z) \prec \varphi(z) \Rightarrow f(0) = \varphi(0) \text{ and } f(\mathbb{U}) \subset \varphi(\mathbb{U}) . \]

Furthermore, if the function \( \varphi \) is univalent in \( \mathbb{U} \), then we have the following equivalence (see [2] and [6]):
\[ f(z) \prec \varphi(z) \Leftrightarrow f(0) = \varphi(0) \text{ and } f(\mathbb{U}) \subset \varphi(\mathbb{U}) . \]

**Definition 3.** (Subordinating factor sequence). [15]. A sequence \( \{d_k\}_{k=1}^{\infty} \) of complex numbers is said to be a subordinating factor sequence if, whenever \( f \) of the form (1.1) is analytic, univalent
and convex in $\mathbb{U}$, we have

$$\sum_{k=2}^{\infty} d_k a_k z^k < f(z) \quad (a_1 = 1; z \in \mathbb{U}).$$

**2. MAIN RESULTS**

Unless otherwise mentioned, we assume throughout this paper that $0 \leq \lambda < 1, \beta > 1, z \in \mathbb{U}$ and $g(z)$ is given by (1.7) with $b_{k+1} \geq b_k$ ($k \geq 2$).

To prove our main result we need the following lemmas.

**Lemma 1.** [15]. The sequence $\{d_k\}_{k=1}^{\infty}$ is a subordinating factor sequence if and only if

$$\text{Re} \left\{ 1 + 2 \sum_{k=1}^{\infty} d_k z^k \right\} > 0. \tag{2.1}$$

Now, we prove the following lemma which gives a sufficient condition for functions belonging to the class $T(g; \lambda, \beta)$:

**Lemma 2.** A function $f(z)$ of the form (1.1) is said to be in the class $T(g; \lambda, \beta)$ if

$$\sum_{k=2}^{\infty} \left( \frac{(1-\lambda)(k-1)}{[1+\lambda(k-1)]} \right) b_k |a_k| \leq 2(\beta-1). \tag{2.2}$$

**Proof.** Assume that the inequality (2.2) holds true. Then it suffices to show that

$$\frac{z(f \ast g)'(z)}{(1-\lambda)(f \ast g)(z) + \lambda z(f \ast g)'(z) - (2\beta-1)} < 1.$$

We have

$$\frac{z(f \ast g)'(z)}{(1-\lambda)(f \ast g)(z) + \lambda z(f \ast g)'(z) - (2\beta-1)} \leq \frac{\sum_{k=2}^{\infty} (1-\lambda)(k-1) b_k |a_k| z^{k-1}}{2(\beta-1) - \sum_{k=2}^{\infty} |k - (2\beta-1)(1+\lambda(k-1))| b_k |a_k| z^{k-1}} < 1.$$  

This completes the proof of Lemma 2.

**Corollary 1.** Let the function $f(z)$ defined by (1.1) be in the class $T(g; \lambda, \beta)$, then

$$|a_k| \leq \frac{2(\beta-1)}{[(1-\lambda)(k-1) + |k - (2\beta-1)(1+\lambda(k-1))]|b_k} \tag{2.3}$$

The result is sharp for the function

$$f(z) = z + \frac{2(\beta-1)}{[(1-\lambda)(k-1) + |k - (2\beta-1)(1+\lambda(k-1))]|b_k} \tag{2.4}$$

Let $T^*(g; \lambda, \beta)$ denote the subclass of functions $f(z) \in A$ whose coefficients satisfy the condition (2.2). We note that $T^*(g; \lambda, \beta) \subseteq T(g; \lambda, \beta)$.

**Thereom 1.** Let $f(z) \in T^*(g; \lambda, \beta)$. Then

$$\text{Re}(f(z)) > \frac{[1-\lambda + |3-2\beta - \lambda(2\beta-1)|]b_2}{[1-\lambda + |3-2\beta - \lambda(2\beta-1)|]b_2} \tag{2.5}$$

for every function $h \in K$, and

$$\text{Re}(f(z)) > \frac{[1-\lambda + |3-2\beta - \lambda(2\beta-1)|]b_2}{[1-\lambda + |3-2\beta - \lambda(2\beta-1)|]b_2} \tag{2.6}$$

The constant factor $\frac{[1-\lambda + |3-2\beta - \lambda(2\beta-1)|]b_2}{[1-\lambda + |3-2\beta - \lambda(2\beta-1)|]b_2}$ in the subordination result (2.5) is the best estimate.

**Proof.** Let $f(z) \in T^*(g; \lambda, \beta)$ and suppose that $h(z) = z + \sum_{k=2}^{\infty} b_k z^k \in K$, then

$$\frac{[1-\lambda + |3-2\beta - \lambda(2\beta-1)|]b_2}{[1-\lambda + |3-2\beta - \lambda(2\beta-1)|]b_2} \left( f \ast h \right)(z)$$

$$= \frac{[1-\lambda + |3-2\beta - \lambda(2\beta-1)|]b_2}{[1-\lambda + |3-2\beta - \lambda(2\beta-1)|]b_2} \left( z + \sum_{k=2}^{\infty} b_k a_k z^k \right) \tag{2.7}$$

Thus, by using Definition 3, the subordination result holds true if

$$\left\{ \frac{[1-\lambda + |3-2\beta - \lambda(2\beta-1)|]b_2}{[1-\lambda + |3-2\beta - \lambda(2\beta-1)|]b_2} \right\}_{k=1}^{\infty}.$$
Subordination Results for Certain Subclasses of Analytic Functions

is a subordinating factor sequence, with \(a_1 = 1\). In view of Lemma 1, this is equivalent to the following inequality:

\[
\Re \left\{ 1 + \sum_{k=1}^{\infty} \frac{[1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|]}{[2(\beta - 1) + [1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|]} a_k z^k \right\} > 0.
\] (2.8)

Now, since

\[
\Psi(k) = \{(1 - \lambda)(k - 1) + |k - (2\beta - 1)(1 + \lambda(k - 1))\} b_k
\]
is an increasing function of \(k \geq 2\), we have

\[
\Re \left\{ 1 + \sum_{k=1}^{\infty} \frac{[1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|]}{[2(\beta - 1) + [1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|]} a_k z^k \right\} = \Re \left\{ 1 + \frac{[1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|]}{[2(\beta - 1) + [1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|]} b_k z^k \right\} \geq 1 - \frac{[1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|]}{[2(\beta - 1) + [1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|]} \sum_{k=1}^{\infty} \left[ \frac{[1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|]}{[2(\beta - 1) + [1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|]} a_k \right] v^{k}
\]

where we have also made use of assertion (2.2) of Lemma 2. Thus (2.8) holds true in \(\mathbb{U}\). This proves the inequality (2.5). The inequality (2.6) follows from (2.5) by taking the convex function

\[
h(z) = \frac{z}{1 - z} = z + \sum_{k=2}^{\infty} z^k \in K.
\] (2.9)

To prove the sharpness of the constant

\[
\frac{[1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|]}{[2(\beta - 1) + [1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|]} b_k z^2.
\]

we consider the function \(f_0(z) \in T^*(g; \lambda, \beta)\) given by

\[
f_0(z) = z - \frac{[1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|]}{[2(\beta - 1) + [1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|]} f_0(z) < \frac{z}{1 - z}
\]

It is easily verified that

\[
\min_{|z| < 1} \Re \left\{ \frac{[1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|]}{[2(\beta - 1) + [1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|]} f_0(z) \right\} = \frac{1}{2}
\]

This show that the constant

\[
\frac{[1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|]}{[2(\beta - 1) + [1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|]} b_k z^2
\]

is the best possible. This completes the proof of Theorem 1.

**Remark.** (i) Taking \(g(z) = \frac{z}{1 - z}\) and \(\lambda = 0\) in Lemma 2 and Theorem 1, we obtain the result obtained by Srivastava and Attiya [10, Corollary 2] and Nishiwaki and Owa [7, Theorem 2.1];

(ii) Taking \(g(z) = \frac{z}{(1 - z)^2}\) and \(\lambda = 0\) in Lemma 2 and Theorem 1, we obtain the result obtained by Srivastava and Attiya [10, Corollary 4] and Nishiwaki and Owa [7, Corollary 2.2].

Also, we establish subordination results for the associated subclasses, \(M^*(g, \beta)\), \(T_M^*(\lambda, \beta)\), \(T_N^*(\lambda, \beta)\), \(T_{q, s}^*(\alpha_1, \lambda, \beta)\), \(T^*(m, \mu, \ell, \lambda, \beta)\) and \(T^*(\mu, b, \lambda, \beta)\), whose coefficients satisfy the condition (2.2) in the special cases as mentioned in the introduction.

By taking \(\lambda = 0\) in Lemma 2 and Theorem 1, we obtain the following corollary:

**Corollary 2.** Let the function \(f(z)\) defined by (1.1) be in the class \(M^*(g, \beta)\) and satisfy the condition

\[
\sum_{k=2}^{\infty} [k - |k - (2\beta - 1)|] b_k |a_k| \leq 2(\beta - 1).
\] (2.11)

Then for every function \(h \in K\), we have:
\[ \frac{(1 + |3 - 2\beta|)b_2}{2(2\beta - 1) + (1 + |3 - 2\beta|)b_2} (f \ast h)(z) < h(z) \]  
(2.12)

and

\[ \text{Re}(f(z)) > -\frac{2(\beta - 1) + (1 + |3 - 2\beta|)b_2}{1 + |3 - 2\beta|b_2}. \]  
(2.13)

The constant factor \( \frac{1 + |3 - 2\beta|b_2}{2(2\beta - 1) + (1 + |3 - 2\beta|)b_2} \) in the subordination result (2.12) cannot be replaced by a larger one and the function

\[ f_0(z) = z - \frac{2(\beta - 1)}{1 + |3 - 2\beta|b_2} z^2 \]  
(2.14)

gives the sharpness.

By taking \( g(z) = \frac{z}{(1-z)^2} \) in Lemma 2 and Theorem 1, we obtain the following corollary:

**Corollary 3.** Let the function \( f(z) \) defined by (1.1) be in the class \( T^*_N(\lambda, \beta) \) and satisfy the condition

\[ \sum_{k=2}^{\infty} \left\{ \frac{(1 - \lambda)(k - 1) + (1 + \lambda(k - 1))}{(k - (2\beta - 1)[1 + \lambda(k - 1)])} \right\} |a_k| \leq 2(\beta - 1). \]  
(2.15)

Then for every function \( h \in K \), we have:

\[ \frac{[1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|]}{2[2\beta - 1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|]} (f \ast h)(z) < h(z) \]  
(2.16)

and

\[ \text{Re}(f(z)) > -\frac{[2\beta - 1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|]}{[1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|]}. \]  
(2.17)

The constant factor \( \frac{[1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|]}{2[2\beta - 1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|]} \) in the subordination result (2.16) cannot be replaced by a larger one and the function

\[ f_0(z) = z - \frac{2(\beta - 1)}{1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|} z^2 \]  
(2.18)

gives the sharpness.

By taking \( g(z) = \frac{z}{(1-z)^2} \) in Lemma 2 and Theorem 1, we obtain the following corollary:

**Corollary 4.** Let the function \( f(z) \) defined by (1.1) be in the class \( T^*_M(\lambda, \beta) \) and satisfy the condition

\[ \sum_{k=2}^{\infty} \left\{ \frac{(1 - \lambda)(k - 1) + (1 + \lambda(k - 1))}{(k - (2\beta - 1)[1 + \lambda(k - 1)])} \right\} |a_k| \leq 2(\beta - 1). \]  
(2.19)

Then for every function \( h \in K \), we have:

\[ \frac{1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|}{2[2\beta - 1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|]} (f \ast h)(z) < h(z) \]  
(2.20)

and

\[ \text{Re}(f(z)) > -\frac{\beta - 1}{1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|}. \]  
(2.21)

The constant factor \( \frac{1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|}{2[2\beta - 1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|]} \) in the subordination result (2.20) cannot be replaced by a larger one and the function

\[ f_0(z) = z - \frac{\beta - 1}{1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|} z^2 \]  
(2.22)

gives the sharpness.

By taking \( b_k = \Gamma_k(\alpha_1) \), where \( \Gamma_k(\alpha_1) \) defined by (1.9), in Lemma 2 and Theorem 1, we obtain the following corollary:

**Corollary 5.** Let the function \( f(z) \) defined by (1.1) be in the class \( T^*_q, s(\alpha_1, \lambda, \beta) \) and satisfy the condition

\[ \sum_{k=2}^{\infty} \left\{ \frac{(1 - \lambda)(k - 1) + (1 + \lambda(k - 1))}{[k - (2\beta - 1)[1 + \lambda(k - 1)]]} \right\} |\Gamma_k(\alpha_1)| a_k | \leq 2(\beta - 1). \]  
(2.23)

Then for every function \( h \in K \), we have:

\[ \frac{1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|}{2[2\beta - 1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|]} (f \ast h)(z) < h(z) \]  
(2.24)
and
\[
\text{Re}(f(z)) > -\left\{ 2(\beta - 1) \right\} \times \frac{2(\beta - 1)}{[1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|][\ell + 1]} \Gamma_2(\alpha_1)
\]
\[
\text{Corollary 6. Let the function } f(z) \text{ defined by (1.1) be in the class } T^\ast(\mu, \lambda, \beta) \text{ and satisfy the condition}
\]
\[
\sum_{k=2}^{\infty} \left\{ \frac{(1 - \lambda)(k - 1)}{[1 + \lambda(k - 1)]} \right\} \left[ \frac{\ell + 1 + \mu(k - 1)}{\ell + 1} \right] |a_k| \leq 2(\beta - 1).
\]
\[
\text{Then for every function } h \in K, \text{ we have:}
\]
\[
\frac{1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|}{{\ell + 1 + \mu}} \left( f \ast h \right)(z) < h(z)
\]
\[
\text{and}
\]
\[
\text{Re}(f(z)) > -\left\{ 2(\beta - 1) \right\} \times \frac{2(\beta - 1)}{[1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|][\ell + 1]} \Gamma_2(\alpha_1)
\]
\[
\text{gives the sharpness.}
\]
\[
\text{By taking } b_k = C_k(b, \mu), \text{ where } C_k(b, \mu) \text{ defined by (1.10), in Lemma 2 and Theorem 1, we obtain the following corollary:}
\]
\[
\text{Corollary 7. Let the function } f(z) \text{ defined by (1.1) be in the class } T^\ast(\mu, b, \lambda, \beta) \text{ and satisfy the condition}
\]
\[
\sum_{k=2}^{\infty} \left\{ \frac{(1 - \lambda)(k - 1)}{[1 + \lambda(k - 1)]} \right\} C_k(b, \mu)|a_k| \leq 2(\beta - 1).
\]
\[
\text{Then for every function } h \in K, \text{ we have:}
\]
\[
\frac{1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|}{{\ell + 1 + \mu}} \left( f \ast h \right)(z) < h(z)
\]
\[
\text{and}
\]
\[
\text{Re}(f(z)) > -\left\{ 2(\beta - 1) \right\} \times \frac{2(\beta - 1)}{[1 - \lambda + |3 - 2\beta - \lambda(2\beta - 1)|][\ell + 1]} \Gamma_2(\alpha_1)
\]
\[
\text{gives the sharpness.}
\]
\[
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\]

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4. REFERENCES


