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Original Article

Existence and Uniqueness for Solution of Differential Equation with Mixture of Integer and Fractional Derivative

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Abstract: By employing the Krasnosel'ski˘ıfixed point theorem, we establish the existence of solutions for mixed differential equation (ordinary and fractional). Moreover, we suggest the uniqueness of solution and we examine our abstract results by applications.

Keywords: Fractional calculus, Fractional differential equation, Integral boundary condition, Krasnosel'skiĭ Fixed Point Theorem

1. INTRODUCTION

In recent years, fractional equations have gained considerable interest due to their applications in various fields of the science such as physics, mechanics, chemistry, biology, engineering and computer sciences. Significant development has been made in ordinary and partial differential equations involving fractional derivatives [1, 2].

The class of fractional differential equations of various types plays an important role not only in mathematics but also in physics, control systems, diffusion, dynamical systems and engineering to create the mathematical modeling of many physical phenomena.

The existence of positive solution and multi-positive solutions for nonlinear fractional. Moreover, by using the concepts of the subordination and superordination of analytic functions, the existence of analytic solutions for fractional differential equations in complex domain are posed in [3, 4]. About the development of existence theorems for fractional functional differential equations.

Many papers on fractional differential equations are devoted to existence and uniqueness of solutions such a type of equations (e.g., [5, 6]). In this paper we investigate the existence of solution of differential equation with mixture of integer and fractional derivative. Our result is an application of Krasnosel'skiĭ fixed point theorem. Such differential equation plays a very important rule in applications in sciences and engineering problems [7].

2. PRELIMINARIES

Recall the following basic definitions and results:

Definition 2.1. For a function f given on the interval [a,b], the Caputo fractional order derivative of f is defined by

$$ {}_a^tD^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds $$

where $n = [\alpha] + 1$ and $[\alpha]$ denote the integer part of α.

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Lemma 2.2. Let $\alpha > 0$, then

${}_a^tI^{\alpha}\, {}_a^tI^{-\alpha} y(t) = y(t) + c_0 + c_1 t + \cdots + c_{n-1}t^{n-1}$
for some $c_i \in R$, i=0,1,..,n-1 , $n = [\alpha]+1$

Definition 2.3. Let f be a function which is defined almost everywhere on [a,b] , for $\alpha > 0$, we define

$${}_a^bI^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^b (b-\tau)^{\alpha-1} f(\tau)\, d\tau$$

provided that the integral (Lebesgue) exists.

Theorem 2.4. [8] **(Krasnosel'skiĭ Theorem)**

Let M be a closed convex bounded nonempty subset of a Banach space X. Let A and B be two operators such that:

i)Ax+By=M, whenever x, y ∈ M ;

ii) A is compact and continuous ;

iii) B is a contraction mapping .

Then, there exists $z \in M$ such that z =Az+Bz.

Let R be a Banach space with the norm$\|.\|$. Let $C = ([0,b], R)$, be Banach space of all continuous functions $h: [0,b] \to R$, with supermum norm
$\|h\| = sup\{|h(s)|: s \in [0,b]\}$.

Consider the extraordinary differential equation with initial conditions , which has the form

$$D^m y(t) + \lambda^c D^{\alpha} y(t) = f\big(t, y(t)\big), \qquad (2.1)$$
$\lambda \in R$, $m = 1,2$ and $0 < \alpha < 1$
$$y(0) = k_0 \ ,\ y'(0) = k_1 \qquad (2.2)$$

Where D^{α} is the Caputo fractional derivative and the nonlinear functions $f: [0,b] \times R \to R$ is continuous .

Lemma 2.5. Let $0 < \alpha < 1$ and $f: [0,b] \times R \to R$ be a continuous function, then the solution of fractional differential equation (2.1) with the initial condition (2.2) is :

$$y(t) = {}_0^tI^{\gamma} f(t,y) + {}_0^tI^{\beta} g(t,y) \ ,$$

$$\beta = m - \alpha, m = 1,2 \quad 0 < \alpha < 1$$

Proof. we reduce the problem (2.1) to an equivalent integral equation

$${}_0^tI^{-m} y(t) + \lambda^c\, {}_0^tI^{-\alpha} y(t) = f(t,y) \qquad (2.3)$$

Operate both side of equation (2.3) by the operator ${}_0^tI^{\alpha}$, we get

$${}_0^tI^{\alpha}\, {}_0^tI^{-m} y(t) + \lambda^c\, {}_0^tI^{\alpha}\, {}_0^tI^{-\alpha} y(t) = {}_0^tI^{\alpha} f(t,y)$$

In view of the relations $I^{-\alpha} I^{\alpha} y(t) = y(t)$,

$I^{\alpha} I^{\beta} y(t) = I^{\alpha+\beta} y(t)$, for$\alpha, \beta > 0$, and by Lemma (2.2), we obtain

$${}_0^tI^{\alpha-m} y(t) + \lambda^c\, [y(t) + c_0 + c_1 t] = {}_0^tI^{\alpha} f(t,y)$$

$$y(t) = \frac{1}{\lambda^c}\, {}_0^tI^{\alpha} f(t,y) - \frac{1}{\lambda^c}\, {}_0^tI^{\alpha-m} y(t) - c_0 - c_1 t \qquad (2.4)$$

By applying the condition (2.2), we get

$$y(0) = c_0 = k_0$$

and

$$\acute{y}(t) = \frac{1}{\lambda^c}\, {}_0^tI^{\alpha-1} f(t,y) - \frac{1}{\lambda^c}\, {}_0^tI^{\alpha-m-1} y(t) + c_1$$

$$\acute{y}(0) = c_1 = k_1$$

Then by substitute c_0 and c_1 in equation (2.4) , we get

$$y(t) = \frac{1}{\lambda^c}\, {}_0^tI^{\alpha} f(t,y)$$

$$-\frac{1}{\lambda^c}\, {}_0^tI^{\alpha-m} y(t) - k_0 - k_1 t \qquad (2.5)$$

The equation (2.5) will become of the form

$$y(t) = {}_0^tI^{\gamma} f(t,y) + {}_0^tI^{\beta} g(t,y),$$

$$\beta = m - \alpha, m = 1,2 \quad 0 < \alpha < 1$$

$$g(t,y) = (D + Ct) - \lambda^c y(t), D = -\lambda^c k_0 \ ,$$

$$C = -\lambda^c k_1 \ ,\ \gamma = \alpha + \beta$$

which completes the proof.

To prove the main results, we need the following assumptions:

(H1) There exists $G_1 > 0$ such that

$\|f(t,x) - f(t,y)\| \leq G_1\|x - y\|$

for $t \in [0,b]$ and $x,y \in R$.

(H2) There exists constants $\omega > 0$, such that

$\|g(t,x) - g(t,y)\| \leq \omega\|x - y\|, \forall\, x,y \in R$.

(H3) $\|f(t,y)\| \leq \mu(t)$, for all

$(t,y) \in [0,b] \times R$, and $\mu \in L^1([0,b],R^+)$.

For convenience, let us set

$$\Lambda = \left(\frac{G_1 b^\gamma}{\Gamma(\gamma+1)} + \frac{|\lambda^c| b^\beta}{\Gamma(\beta+1)}\right) \qquad (2.6)$$

3. MAIN RESULTS

Theorem 3.1. Assume that $f: [0,b] \times R \to R$ is a continuous function and satisfies the assumption (H1). Then the boundary value problem (2.1) has a unique solution.

Proof. Consider the operator $T: C \to C$ by

$Ty(t) = {}_0^tI^\gamma f(t,y) + {}_0^tI^\beta g(t,y)$

Setting $\sup_{t\in[0,b]}|f(t,y)| = M$, we show that

$TBr \subset Br$, where $Br = \{y \in C: \|y\| \leq r\}$.

For $y \in Br$, we have

$$\|(Ty)(t)\| = \left\|{}_0^tI^\gamma f(t,y) + {}_0^tI^\beta g(t,y)\right\|$$

$$\leq {}_0^tI^\gamma \|f(t,y)\| + {}_0^tI^\beta \|g(t,y)\|$$

$$\leq {}_0^tI^\gamma \|f(t,y) - f(t,0) + f(t,0)\|$$

$$+ {}_0^tI^\beta \|g(t,y)\|$$

$$\leq {}_0^tI^\gamma \|f(t,y) - f(t,0)\| + {}_0^tI^\gamma \|f(t,0)\|$$

$$+ {}_0^tI^\beta \|g(t,y)\|$$

$$\leq \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} \|f(s,y) - f(s,0)\| ds$$

$$+ \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} \|f(s,0)\| ds$$

$$+ \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} \|g(s,y)\| ds$$

$$\leq \frac{G_1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} \|y\| ds + \frac{M}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} ds$$

$$+ \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} \|(D + Cs) + \lambda^c y\| ds$$

$$\leq \frac{G_1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} \|y\| ds + \frac{M}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} ds$$

$$+ D\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} ds + \frac{C}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} s ds$$

$$+ \frac{|\lambda^c|}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} \|y\| ds$$

$$\leq \frac{(G_1 r + M)t^\gamma}{\Gamma(\gamma+1)} + \frac{Dt^\beta}{\Gamma(\beta+1)}$$

$$+ \frac{Ct^{\beta+1}}{(\beta+1)\Gamma(\beta+1)} + \frac{r|\lambda^c|t^\beta}{\Gamma(\beta+1)}$$

Where $t^\beta < b^\beta$ and $t^\gamma < b^\gamma$, we obtain:

$$\|(Ty)(t)\| \leq \frac{(G_1 r + M)b^\gamma}{\Gamma(\gamma+1)}$$

$$+ \frac{((r|\lambda^c| + D)(\beta+1) + Cb)b^\beta}{(\beta+1)\Gamma(\beta+1)}$$

Now, for $x,y \in C$ and for each $t \in [0,b]$, we obtain:

$$\|(Tx)(t) - (Ty)(t)\| \leq {}_0^tI^\gamma \|f(t,x) - f(t,y)\|$$

$$+ {}_0^tI^\beta \|g(t,x) - g(t,y)\|$$

$$\leq G_1\, {}_0^tI^\gamma \|x - y\| + |\lambda^c|\, {}_0^tI^\beta \|x - y\|$$

$$\leq \frac{G_1\|x-y\|}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} ds$$

$$+ \frac{|\lambda^c|\|x-y\|}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} ds$$

$$\leq \frac{G_1\|x-y\|b^\gamma}{\Gamma(\gamma+1)} + \frac{|\lambda^c|\|x-y\|b^\beta}{\Gamma(\beta+1)}$$

$$\|(Ty)(t)\| \leq \left(\frac{G_1 b^\gamma}{\Gamma(\gamma+1)} + \frac{|\lambda^c|b^\beta}{\Gamma(\beta+1)}\right)\|x-y\| = \Lambda\|x-y\|$$

As $\Lambda < 1$, therefore T is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem).

Theorem 3.2. Let $f : [0,b] \times R \to R$ be a continuous function mapping bounded subsets of [0,b] × R into relatively compact subsets of R, and the assumptions (H2) and (H3) hold. Then the boundary value problem (2.1) has at least one solution on [0,b].

Proof. Letting $\sup_{t\in[0,b]}|\mu(t)| = \|\mu\|$ and consider $Br = \{y \in C: \|y\| \leq r\}$. We define the operators $\mathcal{P}$and $\mathcal{Q}$as

$$(\mathcal{P}x)(t) = {}_0^tI^\gamma f(t,x)$$

$$= \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} f(s,x(s))ds\,,$$

$$(\mathcal{Q}x)(t) = {}_0^tI^\beta g(t,x)$$

$$= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}(D+Cs+\lambda^c x(s))ds$$

for $x, y \in Br$, we find that

$$\|\mathcal{P}\mathrm{x} + \mathcal{Q}\mathrm{y}\| \leq \frac{1}{\Gamma(\gamma)}\int_0^{\mathrm{t}} (\mathrm{t}-\mathrm{s})^{\gamma-1}\left\|\mathrm{f}(\mathrm{s},\mathrm{x}(\mathrm{s}))\right\|\mathrm{ds}$$

$$+\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}(D+Cs+|\lambda^c|\|y\|)ds$$

$$\|\mathcal{P}x + \mathcal{Q}y\| \leq \frac{\|\mu\|b^\gamma}{\Gamma(\gamma+1)}$$

$$+\frac{((r|\lambda^c|+D)(\beta+1)+Cb)b^\beta}{(\beta+1)\Gamma(\beta+1)}$$

Now prove that $\mathcal{Q}$ is contraction mapping

$$\|\mathcal{Q}x - \mathcal{Q}y\| \leq \omega\|x-y\|$$

where$\omega = \frac{|\lambda^c|b^\beta}{\Gamma(\beta+1)} < 1$.

It is clear that $\mathcal{Q}$ is contraction mapping, Continuity of f implies that the operator $\mathcal{P}$ is continuous. Also, $\mathcal{P}$ is uniformly bounded on Br as

$$\|\mathcal{P}x\| \leq \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\left\|f(s,x(s))\right\|ds$$

$$\|\mathcal{P}x\| \leq \frac{\|\mu\|b^\gamma}{\Gamma(\gamma+1)}$$

Now we prove the compactness of the operator $\mathcal{P}$. We define$\sup_{(t,x)\in[0,b]\times R}|f(t,y)| = N$,

And consequently we have

$$\|\mathcal{P}x(t_1) - \mathcal{P}x(t_2)\| \leq$$

$$\left\|\int_0^{t_1}\frac{[(t_2-s)^{\gamma-1}-(t_1-s)^{\gamma-1}]}{\Gamma(\gamma)}f(s,x(s))ds + \frac{1}{\Gamma(\gamma)}\int_{t_1}^{t_2}(t_2-s)^{\gamma-1}f(s,x(s))ds\right\|$$

$$\leq \frac{N}{\Gamma(\gamma+1)}\left|2(t_2-t_1)^\gamma + t_1^\gamma + t_2^\gamma\right|$$

Which is independent of x. Thus, $\mathcal{P}$ is equicontinuous. Using the fact that f maps bounded subset into relatively compact subsets, so $\mathcal{P}$ is relatively compact on Br. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{P}$ is compact on Br. Thus all the assumptions of Theorem 3.2 aresatisfied. So the conclusion of Theorem 3.2 implies that the initial value problem (2.1) has at least one solution on [0, b].

4. REFERENCES

1. Kilbas, A.A., H.M. Srivastava & J.J. Trujillo. *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies Vol. 204. Elsevier Science B.V., Amsterdam, The Netherlands, p. 347-463 (2006).
2. Podlubny, I. *Fractional Differential Equations*, Vol. 198 of Mathematics in Science and Engineering. Academic Press, San Diego, CA, USA, p. 261-307 (1999).

3. Ibrahim, R.W. & M. Darus. Subordinati-on and superordination for analytic functions involving fractional integral operator. *Complex Variables and Elliptic Equations* 53: 1021-1031 (2008).
4. Ibrahim, R.W. & M. Darus. Subordi-nation and superordination for univalent solutions for fractional differential equations. *J. Math. Anal. Appl*. 345: 871-879 (2008).
5. Hadid S.B. Local and global existence theorems on differential equations of non-integer order. *J. Fractional Calculus* 7: 101-105 (1995).
6. Murad, S.M. & H.J. Zekri & S. Hadid. Existence and uniqueness theorem of fractional mixed Volterra-Fredholm integro-differential equation with integral boundary conditions. *International Journal of Differential Equations* 2011: 1-15 (2011).
7. Cieielski, M. & J.S. Leszczynski. Numerical solution to boundary value problem for anomalous diffusion equation with Riesz-Feller fractional operator. *J. Theo. and Appl. Mech*. 44: 393-403 (2006).
8. Krasnosel'skiĭ, M.A. Two remarks on the method of successive Approximati-ons. *Uspekhi Matematicheskikh Nauk* 63: 123–127 (1955).