Proceedings of the Pakistan Academy of Sciences 49 (1): 1-8 (2012)
Pakistan Academy of Sciences

ISSN: 0377 - 2969

Original Article

On Regions of Variability of Some Differential Operators Implying Starlikeness

Sukhwinder Singh Billing*

Department of Applied Sciences
Baba Banda Singh Bahadur Engineering College
Fatehgarh Sahib-140 407, Punjab, India

Abstract: In this paper, we prove a subordination theorem and use it to extend the regions of variability of some differential operators implying starlikeness of normalized analytic functions. Mathematica 7.0 is used to show the extended regions of the complex plane.

Keywords: Analytic functions, Starlike functions, Differential subordination.

2000 Mathematical Subject Classification: Primary 30C80, Secondary 30C45.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ be the class of functions f, analytic in the open unit disk $\mathrm{E}=\{z:|z|<1\}$ and normalized by the conditions $f(0)=f'(0)-1=0$. Denote by $\mathcal{S}^*(\alpha)$, the class of starlike functions of order α which is analytically defined as follows:

$$\mathcal{S}^*(\alpha)=\left\{f\in\mathcal{A}:\Re\left(\frac{zf'(z)}{f(z)}\right)>\alpha,\, z\in\mathrm{E},\, 0\le\alpha<1\right\}.$$

We write $\mathcal{S}^*=\mathcal{S}^*(0)$, the class of univalent starlike functions w.r.t. the origin. Obtaining different criteria for starlikeness of an analytic function has always been a subject of interest. A number of criteria for starlikeness of analytic functions have been developed. We state below some of them.

Miller et al [4] studied the class of α-convex functions and proved the following result.

Theorem 1.1. If a function $f\in\mathcal{A}$ satisfies the differential inequality

$$\Re\left[(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right]>0, z\in\mathrm{E},$$

where α is any real number, then f is starlike in E.

Later on, Fukui [1] proved the more general result given below for the class of α-convex functions.

Theorem 1.2. Let α, $\alpha\ge 0$ be a given real number. For all $z\in\mathrm{E}$, let a function $f\in\mathcal{A}$ satisfy

$$\Re\left[(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right]>\begin{cases}\lambda-\dfrac{\alpha\lambda}{2(1-\lambda)}, 0\le\lambda<1/2,\\ \lambda-\dfrac{\alpha(1-\lambda)}{2\lambda}, 1/2\le\lambda<1.\end{cases}$$

Then $f\in\mathcal{S}^*(\lambda)$.

Lewandowski et al [2] proved the following result.

Received, February 2011; Accepted, March 2012
*Email: ssbilling@gmail.com

Theorem 1.3. For a function $f \in \mathcal{A}$, the differential inequality

$$\Re\left[\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)\right] > 0, z \in \mathrm{E},$$

ensures the membership for f in the class $\mathcal{S}^*$.

In 2002, Li and Owa [3] proved the following two results:

Theorem 1.4. If $f \in \mathcal{A}$ satisfies

$$\Re\left[\frac{zf'(z)}{f(z)}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right)\right] > -\frac{\alpha}{2}, z \in \mathrm{E},$$

for some $\alpha, \alpha \geq 0$, then $f \in \mathcal{S}^*$.

Theorem 1.5. If $f \in \mathcal{A}$ satisfies

$$\Re\left[\frac{zf'(z)}{f(z)}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right)\right] > -\frac{\alpha^2(1-\alpha)}{4}, z \in \mathrm{E},$$

for some $\alpha, 0 \leq \alpha < 2$, then $f \in \mathcal{S}^*(\alpha/2)$.

Later on Ravichandran et al. [10] proved the following result:

Theorem 1.6. If $f \in \mathcal{A}$ satisfies

$$\Re\left[\frac{zf'(z)}{f(z)}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right)\right] > \alpha\beta\left(\beta-\frac{1}{2}\right)+\beta-\frac{\alpha}{2}, z \in \mathrm{E},$$

for some $\alpha, \beta, 0 \leq \alpha, \beta \leq 1,$ then $f \in \mathcal{S}^*(\beta)$.

For more such results, we refer the readers to [5, 7, 9]. Recently, Singh et al [11] proved the following more general result for starlikeness which unifies all the above mentioned results.

Theorem 1.7. Let α, $\alpha \geq 0$, $\lambda, 0 \leq \lambda < 1,$ and $\beta, 0 \leq \beta \leq 1$, be given real numbers. Let

$$M(\alpha,\beta,\lambda) = [1-\alpha(1-\beta)]\lambda + \alpha(1-\beta)\lambda^2 - \frac{\alpha(1-\beta)(1-\lambda)}{2} - \frac{\alpha\beta(1-\lambda)}{2\lambda},$$ and

$$N(\alpha,\beta,\lambda) = [1-\alpha(1-\beta)]\lambda + \alpha(1-\beta)\lambda^2 - \frac{\alpha(1-\beta)(1-\lambda)}{2} - \frac{\alpha}{2(1-\lambda)}[2\sqrt{\beta\lambda(1-2\lambda)(1-\beta)(3-2\lambda)} + \beta\lambda - \lambda^2(1-\beta)(3-2\lambda)].$$

(i) For $0 \leq \lambda < 1/2$, let a function $f \in A$, $\frac{f(z)}{z} \neq 0$ in E, satisfy

(a) $$\Re\left[\frac{zf'(z)}{f(z)}\left(1+\frac{\alpha zf''(z)}{f'(z)}\right) + \alpha\beta\left(1-\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}\right)\right] > M(\alpha,\beta,\lambda),$$

whenever

$$\beta(2\lambda-1-3\lambda^3+2\lambda^4)+(3-2\lambda)\lambda^3 \geq 0,$$ and

(b) $$\Re\left[\frac{zf'(z)}{f(z)}\left(1+\frac{\alpha zf''(z)}{f'(z)}\right) + \alpha\beta\left(1-\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}\right)\right] > N(\alpha,\beta,\lambda),$$

whenever

$$\beta(2\lambda-1-3\lambda^3+2\lambda^4)+(3-2\lambda)\lambda^3 \leq 0.$$

Then $f \in \mathcal{S}^*(\lambda)$.

(ii) For $1/2 \leq \lambda < 1$, if a function $f \in A$, $\frac{f(z)}{z} \neq 0$ in E, satisfies

$$\Re\left[\frac{zf'(z)}{f(z)}\left(1+\frac{\alpha zf''(z)}{f'(z)}\right) + \alpha\beta\left(1-\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}\right)\right] > M(\alpha,\beta,\lambda),$$

then $f \in \mathcal{S}^*(\lambda)$.

The main objective of this paper is to extend the region of variability of above mentioned differential operators implying starlikeness. The extended regions are shown pictorially using Mathematica 7.0.

To prove our main results, we use the technique of differential subordination and need the following lemma of Miller and Mocanu [6].

For two analytic functions f and g in the unit disk E, we say that a function f is subordinate to a function g in E and write $f \prec g$ if there exists a Schwarz function w analytic in E with $w(0)=0$ and $|w(z)|<1, z\in \mathrm{E}$ such that $f(z)=g(w(z)), z\in\mathrm{E}.$ In case the function g is univalent, the above subordination is equivalent to $f(0)=g(0)$ and $f(\mathrm{E})\subset g(\mathrm{E})$.

Let $\psi : C\times C\to C$ be an analytic function, p be an analytic function in E, with $\left(p(z), zp'(z)\right)\in C\times C$ for all $z\in \mathrm{E}$ and let h be univalent in E, then the function p is said to satisfy first order differential subordination if

$$\psi(p(z), zp'(z)) \prec h(z), \psi(p(0),0)=h(0). \tag{1}$$

A univalent function q is called a dominant of the differential subordination (1) if $p(0)=q(0)$ and $p\prec q$ for all p satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q}\prec q$ for each dominant q of (1), is said to be the best dominant of (1).

Lemma 1.1. ([6], p.132, Theorem 3.4 h) Let q be univalent in E and let θ and ϕ be analytic in a domain D containing $q(\mathrm{E})$, with $\phi(w)\neq 0$, |when $w\in q(\mathrm{E})$. Set $Q(z)=zq'(z)\phi[q(z)]$, $h(z)=\theta[q(z)]+Q(z)$ and suppose that either

(i) h is convex, or

(ii) Q is starlike.

In addition, assume that

(iii) $\Re\, \dfrac{zh'(z)}{Q(z)}>0,\ z\in\mathrm{E}$.

If p is analytic in E, with $p(0)=q(0), p(\mathrm{E})\subset \mathrm{D}$ and

$$\theta[p(z)]+zp'(z)\phi[p(z)]\prec\theta[q(z)]+zq'(z)\phi[q(z)],$$

then $p(z)\prec q(z)$ and q is the best dominant.

2. MAIN RESULTS

Theorem 2.1. Let a, b, c and d be complex numbers such that c and d are not simultaneously zero. Let $q, q(\mathrm{E})\subset \mathrm{D}$, be a univalent function in E such that

(*i*) $\Re\left(1+\dfrac{zq''(z)}{q'(z)}-\dfrac{zq'(z)}{q(z)}+\dfrac{czq'(z)}{cq(z)+d}\right)>0,$

and

(*ii*) $\Re\left(1+\dfrac{zq''(z)}{q'(z)}-\dfrac{zq'(z)}{q(z)}+\dfrac{czq'(z)}{cq(z)+d}+\dfrac{(a+2bq(z))q(z)}{cq(z)+d}\right)>0.$

If p, $p(z)\neq 0, z\in\mathrm{E}$, satisfies the differential subordination

$$ap(z)+b(p(z))^2+\left(c+\frac{d}{p(z)}\right)zp'(z)\prec aq(z)+b(q(z))^2+\left(c+\frac{d}{q(z)}\right)zq'(z), \tag{2}$$

then $p(z)\prec q(z)$ and q is the best dominant.

Proof. Let us define the functions θ and ϕ as follows:

$\theta(w)=aw+bw^2$, and $\phi(w)=c+\dfrac{d}{w}.$

Obviously, the functions θ and ϕ are analytic in domain $\mathrm{D}=C-\left\{-\dfrac{d}{c}\right\}$ and $\phi(w)\neq 0$ in D.

Now, define the functions Q and h as follows:

$$Q(z)=zq'(z)\phi(q(z))=\left(c+\frac{d}{q(z)}\right)zq'(z), \text{ and}$$

$$h(z)=aq(z)+b(q(z))^2+\left(c+\frac{d}{q(z)}\right)zq'(z).$$

Now Q is starlike in E in view of condition (i) and the condition (ii) implies that $\Re\,\dfrac{z\,h'(z)}{Q(z)}>0,\ z\in\mathrm{E}.$ Also by (2), we have

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)].$$

Therefore, the proof, now, follows from Lemma 1.1.

Setting $p(z) = \frac{zf'(z)}{f(z)}$ in Theorem 2.1, we have the following result.

Theorem 2.2. Let $q, q(z) \neq 0,$ be a univalent function in E and satisfy the conditions (i) and (ii) of Theorem 2.1. If $f \in \mathcal{A}, \frac{zf'(z)}{f(z)} \neq 0, z \in \mathrm{E}$, satisfies the differential subordination

$$\frac{zf'(z)}{f(z)}\left[\begin{array}{l} a + b\frac{zf'(z)}{f(z)} + \left(c + d\frac{f(z)}{zf'(z)}\right) \\ \left(1 - \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\right) \end{array}\right]$$

$$\prec aq(z) + b(q(z))^2 + \left(c + \frac{d}{q(z)}\right)zq'(z),$$

where a, b, c and d are complex numbers, then $\frac{zf'(z)}{f(z)} \prec q(z)$ and q is the best dominant.

If we restrict the constants a, b, c and d to real numbers. By selecting $a = 1 - \alpha(1-\beta)$, $b = \alpha(1-\beta), c = \alpha(1-\beta)$ and $d = \alpha\beta$ in Theorem 2.2, we obtain the following result.

Theorem 2.3. Let $q, q(z) \neq 0,$ be a univalent function in E and satisfy the conditions

(i) $$\Re\left(\begin{array}{l} 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \\ \frac{(1-\beta)zq'(z)}{(1-\beta)q(z) + \beta} \end{array}\right) > 0, \text{ and}$$

(ii) $$\Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{(1-\beta)zq'(z)}{(1-\beta)q(z) + \beta} + \frac{[(1-\alpha(1-\beta)) + 2\alpha(1-\beta)q(z)]q(z)}{\alpha(1-\beta)q(z) + \alpha\beta}\right) > 0.$$

If $f \in \mathcal{A}, \frac{zf'(z)}{f(z)} \neq 0, z \in \mathrm{E},$ satisfies the differential subordination

$$\frac{zf'(z)}{f(z)}\left(1 + \alpha\frac{zf''(z)}{f'(z)}\right) + \alpha\beta\left(1 - \frac{zf'(z)}{f(z)}\right)\left(1 + \frac{zf''(z)}{f'(z)}\right)$$

$$\prec [1 - \alpha(1-\beta)]q(z) + \alpha(1-\beta)(q(z))^2 + \alpha\left(1 - \beta + \frac{\beta}{q(z)}\right)zq'(z),$$

where α and β are real numbers, then $\frac{zf'(z)}{f(z)} \prec q(z)$ and q is the best dominant.

3. APPLICATIONS TO STARLIKE FUNCTIONS

Throughout this section, we restrict the constants a, b, c and d to real numbers.

Remark 3.1. When we select $q(z) = \frac{1 + (1-2\lambda)z}{1-z}, 0 \le \lambda < 1$. Then

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} = 1 + \frac{z}{1-z} - \frac{(1-2\lambda)z}{1 + (1-2\lambda)z}.$$

Thus,

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) > 0.$$

Also

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{1}{\alpha}q(z) = 1 + \frac{z}{1-z} - \frac{(1-2\lambda)z}{1 + (1-2\lambda)z} + \frac{1}{\alpha}\frac{1 + (1-2\lambda)z}{1-z}.$$

For $\alpha > 0,$ we have

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{1}{\alpha}q(z)\right) > 0.$$

Therefore, $q(z)$ satisfies the conditions of Theorem 2.2 in case where $a = 1, b = 0, c = 0$ and $d = \alpha, \alpha > 0$ and we get the following result.

Corollary 3.1. Let α be a real number with $\alpha > 0$. If $f \in \mathcal{A}, \frac{zf'(z)}{f(z)} \neq 0, z \in \mathrm{E}$, satisfies the differential subordination

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec$$
$$\frac{1+(1-2\lambda)z}{1-z} + \frac{2\alpha(1-\lambda)z}{(1-z)(1+(1-2\lambda)z)},$$

then $f \in \mathcal{S}^*(\lambda)$.

Remark 3.2. For $\alpha = \frac{1}{2}$ and $\lambda = \frac{3}{4}$, Corollary 3.1 reduces to the following result.

If $f \in \mathcal{A}, \frac{zf'(z)}{f(z)} \neq 0, z \in \mathrm{E}$, satisfies the condition

$$1 + \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} \prec \frac{2-z}{1-z} + \frac{z}{(1-z)(2-z)} = h_1(z),$$

then $f \in \mathcal{S}^*(3/4)$.

Substituting the same values of α and λ in the result of Fukui [1] stated in Theorem 1.2, we obtain the following result:

If $f \in \mathcal{A}$, satisfies the condition

$$\Re\left(1 + \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\right) > \frac{4}{3}, z \in \mathrm{E},$$

then $f \in \mathcal{S}^*(3/4)$.

To compare both the results, we plot $h_1(\mathbb{E})$ and the line $\Re(z) = \frac{4}{3}$ in Fig. 1.

We see that according to the result of Fukui [1], for the starlikeness of order $3/4$ of $f(z)$, the differential operator $1 + \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}$ can vary in the complex plane on the right side of the line $\Re(z) = \frac{4}{3}$ shown with dashes in Fig. 1 whereas according to our result, the same operator can vary over the portion of the plane right to the curve $h_1(z)$ for the same conclusion. Thus our result extends the region of variability of this operator for the same implication and the region bounded by the dashing line and the curve is the claimed extension as shown in Fig. 1.

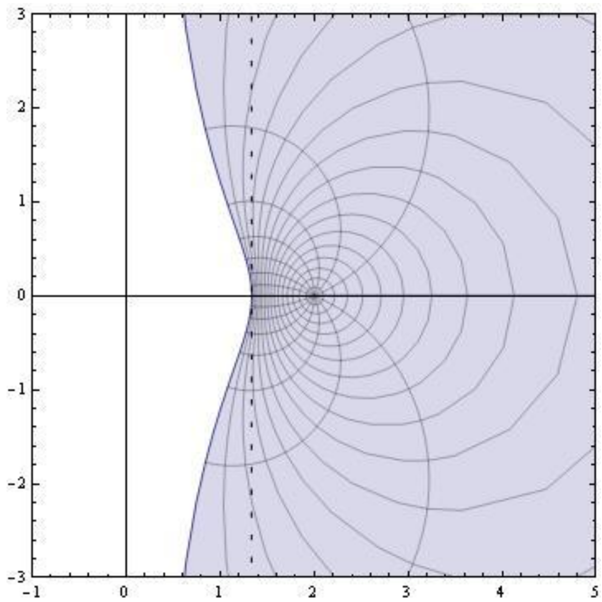

Fig. 1.

Remark 3.3. When we select $q(z) = \frac{1+(1-2\lambda)z}{1-z}, 0 \leq \lambda < 1$. Then

$$1 + \frac{zq''(z)}{q'(z)} = \frac{1+z}{1-z}, \text{i.e.} \Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > 0,$$

and

$$1 + \frac{zq''(z)}{q'(z)} + 2q(z) + \frac{1-\alpha}{\alpha} = \frac{1+z}{1-z} +$$
$$2\frac{1+(1-2\lambda)z}{1-z} + \frac{1-\alpha}{\alpha}.$$

Therefore for $0 < \alpha \leq 1$, we have

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} + 2q(z) + \frac{1-\alpha}{\alpha}\right) > 0.$$

Therefore, $q(z)$ satisfies the conditions of Theorem 2.2 for $a=1-\alpha, b=\alpha, c=\alpha$ and $d=0$ and we obtain the following result.

Corollary 3.2. Let α be a real number $0<\alpha\le 1$. If $f\in\mathcal{A}, \dfrac{zf'(z)}{f(z)}\neq 0, z\in \mathrm{E}$, satisfies the differential subordination

$$\frac{zf'(z)}{f(z)}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right)\prec$$
$$\frac{1+(1-2\lambda)z}{1-z}\left(1-\alpha+\alpha\frac{1+(1-2\lambda)z}{1-z}\right.$$
$$\left.+\frac{2\alpha(1-\lambda)z}{(1-z)[1+(1-2\lambda)z]}\right),$$

then $f\in\mathcal{S}^*(\lambda)$.

Note that for $\alpha=1$ and $\lambda=0$ in above corollary, we get Theorem 1 of Nunokawa et al [8].

Remark 3.4. For $\alpha=1$ and $\lambda=\dfrac{1}{2}$, Corollary 3.2 reduces to the following result.

$f\in\mathcal{A}, \dfrac{zf'(z)}{f(z)}\neq 0, z\in\mathrm{E}$, satisfying the condition

$$\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)\prec$$
$$\frac{1+z}{(1-z)^2}=h_2(z),\ \Rightarrow f\in\mathcal{S}^*(1/2).$$

Substituting the same values of α and λ in the result of Ravichandran [10] stated in Theorem 1.6, we obtain the following result.

If $f\in\mathcal{A}$, satisfies the condition

$$\Re\left[\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)\right]>0, z\in\mathrm{E},$$

then $f\in\mathcal{S}^*(1/2)$.

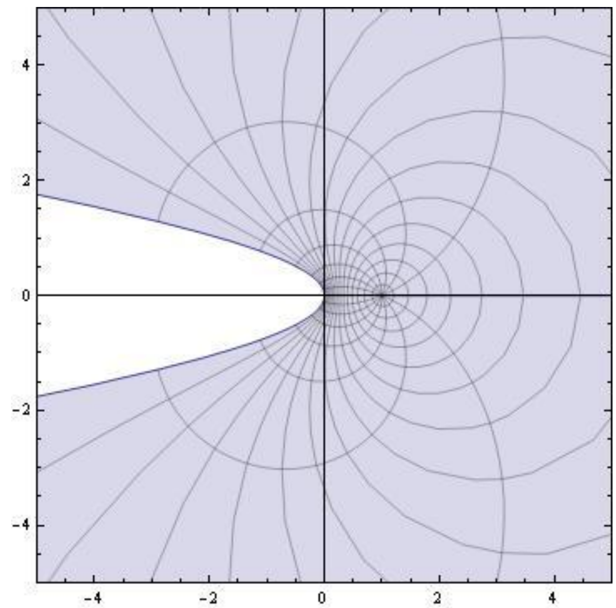

Fig 2.

To compare both the results, we plot $h_2(\mathrm{E})$ in Fig. 2 and we see that according to the result of Ravichandran, for the starlikeness of order $1/2$ of $f(z)$, the differential operator $\dfrac{zf'(z)}{f(z)}\left(1+\dfrac{zf''(z)}{f'(z)}\right)$ can vary in the right half complex plane whereas according to our result, the same operator can vary over the portion of the plane bounded by the curve $h_2(z)$ (entire shaded regiom) for the same conclusion. Thus shaded portion in the left half plane as shown in Fig. 2, is the extension of the region of variability of this operator for the same implication.

Remark 3.5. When we select $q(z)=\dfrac{1}{1-z}$. Then

$$\Re\left(1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{zq'(z)}{q(z)+1}\right)=$$
$$\Re\left(\frac{2-z^2}{(1-z)(2-z)}\right)>0,$$

and

$$\Re\left(1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{zq'(z)}{q(z)+1}+\frac{(1+2q(z))q(z)}{q(z)+1}\right)=$$
$$\Re\left(\frac{5-z-z^2}{(1-z)(2-z)}\right)>0.$$

Therefore, $q(z)$ satisfies the conditions of Theorem 2.3 for $\alpha=1$ and $\beta=\frac{1}{2}$ and we obtain the following result:

Corollary 3.3. If $f\in\mathcal{A}, \frac{zf'(z)}{f(z)}\neq 0, z\in\mathrm{E}$, satisfies the differential subordination

$$\left(1+\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}\right)\prec\frac{2+z-z^2}{(1-z)^2}=h_3(z),$$

then $\frac{zf'(z)}{f(z)}\prec\frac{1}{1-z}, z\in\mathrm{E}$, i.e. $f\in\mathcal{S}^*(1/2)$.

Remark 3.6. When we replace $\alpha=1$ and $\beta=\frac{1}{2}$, Theorem 1.7 of Singh et al [11], we obtain the following result.

If $f\in\mathcal{A}$, satisfies the condition

$$\Re\left[\left(1+\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}\right)\right]>0, z\in\mathrm{E},$$

then $f\in\mathcal{S}^*(1/2)$.

To compare this result with Corollary 3.3, we plot $h_3(\mathrm{E})$ in Fig. 3 and we see that according to the result of Singh et al [11], for the starlikeness of order $1/2$ of $f(z)$, the differential operator $\left(1+\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}\right)$ can vary in the right half complex plane whereas according to the result in Corollary 3.3, the same operator can vary over the portion of the plane bounded by the curve $h_3(z)$ (whole shaded region) for the same conclusion. Thus shaded portion in the left half plane as shown in Fig. 3, is the extension of the region of variability of this operator for the same implication.

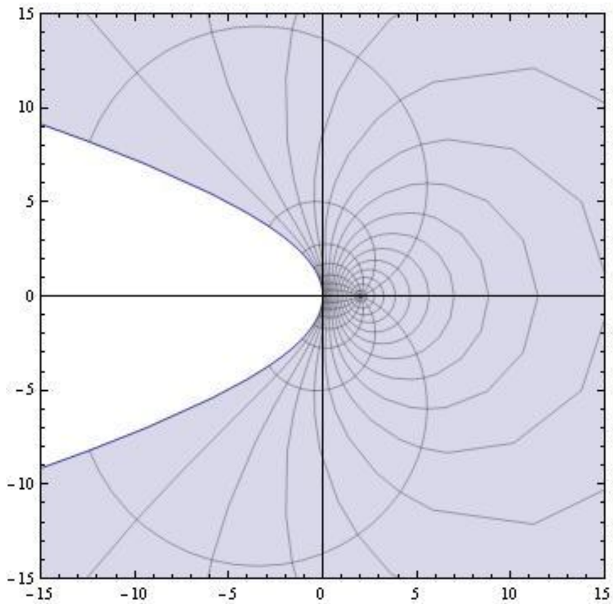

Fig. 3.

4. REFERENCES

1. Fukui, S. On α-convex functions of order *β*. *Internat. J. Math. & Math. Sci.* 20 (4): 769–772 (1997).
2. Lewandowski, Z., S.S. Miller & E. Zlotkiewicz. Generating functions for some classes of univalent functions. *Proc. Amer. Math. Soc.* 56: 111–117 (1976).
3. Li, J.-L. & S. Owa. Sufficient conditions for starlikeness, *Indian J. Pure Appl. Math.* 33: 313–318 (2002).
4. Miller, S.S., P.T. Mocanu & M.O. Reade. All α-convex functions are univalent and starlike. *Proc. Amer. Math. Soc*. 37: 553–554 (1973).
5. Miller, S.S., P.T. Mocanu, & M.O. Reade. Bazilevic functions and generalized convexity. *Rev. Roumaine Math. Pures Appl.* 19: 213–224 (1974).
6. Miller, S.S. & P.T. Mocanu. *Differential Suordinations: Theory and Applications. Series on Monographs and Textbooks in Pure and Applied Mathematics* (No. 225). Marcel Dekker, New York (2000).
7. Mocanu, P.T. Alpha-convex integral operators and strongly starlike functions. *Studia Univ. Babes-Bolyai Math.* 34 (2): 18–24 (1989).
8. Nunokawa, M., N.E. Cho, O.S. Kwon, S. Owa, & S. Saitoh. Differential inequalities for certain analytic functions. *Compt. Math. Appl.* 56: 2908–2914 (2008).

9. Padmanabhan, K.S. On sufficient conditions for starlikeness. Indian *J. Pure Appl. Math.* 32 (4): 543–550 (2001).
10. Ravichandran, V., C. Selvaraj & R. Rajalakshmi. Sufficient conditions for starlike functions of order α. *J. Inequal. Pure and Appl. Math.* 3 (5): Art. 81: 1–6 (2002).
11. Singh, S., S. Gupta, & S. Singh. Starlikeness of analytic maps satisfying a differential inequality. *General Mathematics* 18 (3): 51–58 (2010).