Proceedings of the Pakistan Academy of Sciences 48 (2): 107–109, 2011

ISSN: 0377 - 2969

Pakistan Academy of Sciences

Original Article

Uniqueness of the Rayleigh Wave Speed

Faiz Ahmad*

Centre for Advanced Mathematics and Physics,
National University of Sciences and Technology,
Sector H-12, Islamabad

Abstract: A simple proof is presented to show that the Rayleigh equation has a unique root in the interval (0,1).

Keywords. Rayleigh wave, Rayleigh equation, uniqueness of solution

1. INTRODUCTION

Rayleigh wave plays an important role in seismic phenomena. A Rayleigh wave is a surface wave in the sense that the amplitude is significant near the surface and decays exponentially as we go down the earth. Rayleigh discussed the theory of this wave in [1] and derived the following " Rayleigh Equation"

$$(2-\frac{c^2}{c_T^2})^2 = 4\sqrt{1-\frac{c^2}{c_T^2}}\sqrt{1-\frac{c^2}{c_L^2}}, \qquad (1)$$

where c_L, c_T respectively denote phase speeds of the longitudinal wave (or P wave) and the transverse wave (or S wave) in the medium. The phase speed c of the Rayleigh wave is to be determined from Eq. (1). The equation possesses an unphysical root $c=0$. Since $c_T < c_L$, it is clear that, for a meaningful theory, a root of Eq. (1) must exist in the interval $(0, c_T)$. Let $x = c^2/c_T^2$, $b=(c_T/c_L)^2$ and define

$$f(x) := (2-x)^2 - 4\sqrt{1-x}\sqrt{1-bx}. \qquad (2)$$

To first order in $\varepsilon > 0$, , $f(\varepsilon) = -2\varepsilon(1-b)$, which is negative since $0<b<1$. Also $f(1)=1$. Hence $f(x)$ possesses a zero, say x_1, in the interval $(0,1)$. A physically important question arises whether this is the only real zero in this interval. In the classical text-books on the subject [2-6], this problem is treated in the following manner.

By squaring both sides of (1), rearranging terms and canceling a factor c^2/c_T^2 we get a cubic in $x = c^2/c_T^2$,

$$x^3 - 8x^2 + (24-16b)x - 16(1-b) = 0. \qquad (3)$$

Let us define the left side of (3) as $g(x)$. It is clear that any zero of f other than $x=0$, will be a zero of g but the converse may not be true. The *discriminant* of (3) is

$$256(64b^3 - 107b^2 + 62b - 11). \qquad (4)$$

Eq. (3) will have all three roots real if the discriminant (4) is non-negative. This happens if $b \geq 0.3215$. Let x_2, x_3 be the roots of (3) other than x_1. Since $0 < x_1 < 1$, it is clear that

$$7 < x_2 + x_3 < 8, \qquad (5)$$

also $x_2 x_3 > 16(1-b)$. Theoretical bounds for b are $0<b<1$, however for all known materials $b<1/2$. In this case we have

$$x_2 x_3 > 8. \qquad (6)$$

From (5) and (6) it follows that each of x_2 and x_3 is greater than 1 , hence x_1 is the only real root in $(0,1)$. On the other hand if $0 < b < 0.3215$, x_2, x_3 will be a pair of complex conjugate roots, leaving x_1 as the only real root of the equation.

Received June 2010, Accepted June 2011
*Email: faizmath@hotmail.com

The above proof has the drawback of not being valid for $1/2 \le b < 1$. Achenbach [7] dealt with this problem by defining a function $R(s)$ of a *complex* variable s,

$$R(s) = (2s^2 - s_T^2)^2 + 4s^2(s_L^2 - s^2)^{\frac{1}{2}}(s_T^2 - s^2)^2 \qquad (7)$$

where

$$s = \frac{1}{c}, s_L = \frac{1}{c_L}, s_T = \frac{1}{c_T},$$

and considering zeros of the function by applying the argument principle. However Achenbach's proof is beyond comprehension of most undergraduate students.

The quest for a formula for the Rayleigh wave speed continues in the modern times [8-11]. For example, Vinh and Ogden [10] have the question of uniqueness of the root of (3) in $(0,1)$ treated by considering zeros of

$$g'(x) = 3x^2 - 16x + 8(3-2b). \qquad (8)$$

If $b > 1/6$, $g'(x)$ has two distinct zeros denoted by l and m such that

$lm = 8(3-2b)/3 > 8/3$,

since $0 < b < 1$. Hence

$$0 < l < 1 < m \text{ or } 1 \le l < m. \qquad (9)$$

Vinh and Ogden [10] concluded from (9) that uniqueness of solution of Eq. (3) in the interval $(0,1)$ is ensured. However it appears that the option $0 < l < 1 < m$ does not justify this conclusion, because the curve $y = g(x)$ may attain a local maximum at l and still cross the $x-$ axis at a point before $x = 1$.

In this article, we shall present a short and simple proof of the uniqueness of the real root of the Rayleigh equation which is valid for $0 < b < 1$. Basic idea of this proof may be stated in just one sentence, i.e., "Two real roots in $(0,1)$ imply all three roots in this interval, which is impossible."

2. UNIQUENESS OF REAL ROOT

Since $g(0) = -16(1-b) < 0$ and $g(1) = 1 > 0$, it follows that g has a real zero, x_1, in $(0,1)$. Denote zeros of g by x_1, x_2 and x_3. Assume that the first two zeros are real and both lie in $(0,1)$. Then the third zero will also be real. We will show that the assumption of two zeros being in the interval $(0,1)$ implies that the third zero will also be in the same interval. There are three possibilities.

1. All zeros are distinct. Then $g(x_1+) > 0$, $g(x_2+) < 0$. Since $g(1) > 0$, it follows there must be a zero of g in $(x_2,1)$. This must be x_3.
2. One of x_1 , x_2 is a simple zero while the other has multiplicity two.
3. x_1 has multiplicity three.

In each case, all three zeros lie in $(0,1)$, consequently $x_1 + x_2 + x_3 < 3$ which is false because from (3) this sum must be 8. This contradiction proves that x_1 is the unique zero of g in $(0,1)$. Since f has a zero in $(0,1)$ which must be a zero of g , because of the above uniqueness, x_1 must be the only zero of f in $(0,1)$. Hence Rayleigh equation has a unique root such that $0 < c < c_T$.

Applying the above argument to a polynomial equation of degree 3, we have the following

Theorem. *Let the equation*

$$a_0 z^3 + a_1 z^2 + a_2 z + a_3 = 0,\ a_0 > 0, a_1 \ne 0,$$

be such that $f(0) f(-a_1/(3a_0)) < 0$, *then two roots of the equation, real or complex, lie in the half plane* $\operatorname{Re} z \ge -a_{1/}(3a_0)$, *if* $a_1 < 0$ *or* $\operatorname{Re} z \le -a_1/(3a_0)$, *if* $a_1 > 0$.

ACKNOWLEDGEMENTS

The author is grateful to the Higher Education Commission of Pakistan for financial support.

REFRENCES

1. Rayleigh, Lord. On the free vibrations of an infinite plate of homogeneous elastic matter, Proc. London Math. Soc. 20: 225-235 (1889).
2. Love, A.E.H., *A Treatise on the Mathematical Theory of Elasticity*, Dover, New York, p. 306-309 (1944).

3. Nowacki, W. *Dynamics of Elastic Systems*, Translated from the Polish by H. Zorski. John Wiley and Sons, New York, p. 290-293 (1963).
4. Miklowitz, J. *Elastic Waves and Waveguides*, Amsterdam, North-Holland, p. 146-151 (1978).
5. Ewing, W.M., W.S. Jardetzki, & F. Press. *Elastic Waves in Layered Media*. McGraw-Hill, New York, p. 31-34 (1957).
6. Eringen, A.C., & E.S. Suhubi. *Elastodynamics Volume II, Linear Theory*. Academic Press, New York, p. 518-524 (1975).
7. Achenbach, J.D., *Wave Propagation in Elastic Solids.* Amsterdam, North-Holland, p. 187-194 (1980).
8. Nkemzi, D. A new formula for the velocity of Rayleigh waves, *Wave Motion,* 26: 199-205 (1997).
9. Malischewski, P.G. Comment to a new formula for velocity of Rayleigh waves" by D. Nkemzi [*Wave Motion* 26 (1997) 199- 205], *Wave Motion,* 31: 931-939 (2000).
10. Vinh, P.C. & R.W. Ogden. On formulas for the Rayleigh wave speed. *Wave Motion,* 39: 191-197 (2004).
11. Rehman, A., A. Khan & A. Ali. Rayleigh wave speed in rotating incompressible transversely isotropic medium. *World Applied Sciences Journal*, 6(10): 1-5 (2009).