Uniqueness of the Rayleigh Wave Speed

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Abstract: A simple proof is presented to show that the Rayleigh equation has a unique root in the interval (0,1).

Keywords: Rayleigh wave, Rayleigh equation, uniqueness of solution

1. INTRODUCTION

Rayleigh wave plays an important role in seismic phenomena. A Rayleigh wave is a surface wave in the sense that the amplitude is significant near the surface and decays exponentially as we go down the earth. Rayleigh discussed the theory of this wave in [1] and derived the following "Rayleigh Equation"

\[
(2 - \frac{c_L^2}{c_r^2})^2 = 4 \sqrt{\frac{1 - c_L^2}{c_r^2}} \sqrt{1 - \frac{c_L^2}{c_r^2}},
\]

where \(c_L\), \(c_r\) respectively denote phase speeds of the longitudinal wave (or P wave) and the transverse wave (or S wave) in the medium. The phase speed \(c\) of the Rayleigh wave is to be determined from Eq. (1). The equation possesses an unphysical root \(c = 0\). Since \(c_r < c_L\), it is clear that, for a meaningful theory, a root of Eq. (1) must exist in the interval \((0, c_r)\). Let \(x = \frac{c^2}{c_r^2}\), \(b = (c_r/c_L)^2\) and define

\[
f(x) := (2 - x)^2 - 4 \sqrt{1-x} \sqrt{1-bx}.
\]

By squaring both sides of (1), rearranging terms and canceling a factor \(c_L^2/c_r^2\) we get a cubic in \(x = c^2/c_r^2\),

\[
x^3 - 8x^2 + (24 - 16b)x - 16(1-b) = 0.
\]

Let us define the left side of (3) as \(g(x)\). It is clear that any zero of \(f\) other than \(x = 0\), will be a zero of \(g\) but the converse may not be true. The discriminant of (3) is

\[
256(64b^3 - 107b^2 + 62b - 11).
\]

Eq. (3) will have all three roots real if the discriminant (4) is non-negative. This happens if \(b \geq 0.3215\). Let \(x_2, x_3\) be the roots of (3) other than \(x_1\). Since \(0 < x_1 < 1\), it is clear that

\[
7 < x_2 + x_3 < 8,
\]

also \(x_2x_3 > 16(1-b)\). Theoretical bounds for \(b\) are \(0 < b < 1\), however for all known materials \(b < 1/2\). In this case we have

\[
x_2x_3 > 8.
\]

From (5) and (6) it follows that each of \(x_2\) and \(x_3\) is greater than 1, hence \(x_1\) is the only real root in \((0,1)\). On the other hand if \(0 < b < 0.3215\), \(x_2, x_3\) will be a pair of complex conjugate roots, leaving \(x_1\) as the only real root of the equation.

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The above proof has the drawback of not being valid for \(1/2 \leq b < 1\). Achenbach [7] dealt with this problem by defining a function \(R(s)\) of a complex variable \(s\),

\[
R(s) = (2s^2 - s_T^2)^2 + 4s^2(s_s^2 - s^2)^2(s_T^2 - s^2)^2
\]

where

\[
s = \frac{1}{c}, s_L = \frac{1}{c_L}, s_T = \frac{1}{c_T},
\]

and considering zeros of the function by applying the argument principle. However Achenbach's proof is beyond comprehension of most undergraduate students.

The quest for a formula for the Rayleigh wave speed continues in the modern times [8-11]. For example, Vinh and Ogden [10] have the question of uniqueness of the root of (3) in \((0,1)\) treated by considering zeros of

\[
g'(x) = 3x^2 - 16x + 8(3 - 2b).
\]

If \(b > 1/6\), \(g'(x)\) has two distinct zeros denoted by \(l\) and \(m\) such that

\[
lm = 8(3 - 2b)/3 > 8/3,
\]

since \(0 < b < 1\). Hence

\[
0 < l < 1 < m \text{ or } 1 < l < m.
\]

Vinh and Ogden [10] concluded from (9) that uniqueness of solution of Eq. (3) in the interval \((0,1)\) is ensured. However it appears that the option \(0 < l < 1 < m\) does not justify this conclusion, because the curve \(y = g(x)\) may attain a local maximum at \(l\) and still cross the \(x - \) axis at a point before \(x = 1\).

In this article, we shall present a short and simple proof of the uniqueness of the real root of the Rayleigh equation which is valid for \(0 < b < 1\). Basic idea of this proof may be stated in just one sentence, i.e., “Two real roots in \((0,1)\) imply all three roots in this interval, which is impossible.”

2. UNIQUENESS OF REAL ROOT

Since \(g(0) = -16(1-b) < 0\) and \(g(1) = 1 > 0\), it follows that \(g\) has a real zero, \(x_1\), in \((0,1)\). Denote zeros of \(g\) by \(x_1, x_2\) and \(x_3\). Assume that the first two zeros are real and both lie in \((0,1)\). Then the third zero will also be real. We will show that the assumption of two zeros being in the interval \((0,1)\) implies that the third zero will also be in the same interval. There are three possibilities.

1. All zeros are distinct. Then \(g(x_1) > 0, g(x_2) < 0\). Since \(g(1) > 0\), it follows there must be a zero of \(g\) in \((x_2,1)\). This must be \(x_3\).

2. One of \(x_1, x_2\) is a simple zero while the other has multiplicity two.

3. \(x_1\) has multiplicity three.

In each case, all three zeros lie in \((0,1)\), consequently \(x_1 + x_2 + x_3 < 3\) which is false because from (3) this sum must be \(8\). This contradiction proves that \(x_1\) is the unique zero of \(g\) in \((0,1)\). Since \(f\) has a zero in \((0,1)\) which must be a zero of \(g\), because of the above uniqueness, \(x_1\) must be the only zero of \(f\) in \((0,1)\). Hence Rayleigh equation has a unique root such that \(0 < c < c_T\).

Applying the above argument to a polynomial equation of degree 3, we have the following

**Theorem.** Let the equation

\[
a_0z^3 + a_1z^2 + a_2z + a_3 = 0, \quad a_0 > 0, a_1 \neq 0,
\]

be such that \(f(0)f(-a_1/(3a_0)) < 0\), then two roots of the equation, real or complex, lie in the half plane \(\text{Re} z \geq -a_1/(3a_0)\), if \(a_1 < 0\) or \(\text{Re} z \leq -a_1/(3a_0)\), if \(a_1 > 0\).

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REFERENCES
