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A Note on Laskerian Rings

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Abstract: Let D be an integral domain with quotient field K and $\overline{D}$ is its integral closure. (1) If $\overline{D}$ is a one dimensional Laskerian ring such that each primary ideal of $\overline{D}$ is a valuation ideal, then each overring of D is Archimedean. (2) If D is not a field, then D is a Dedekind domain if and only if D is a Laskerian almost Dedekind domain. (3) $\overline{D}$ is one dimensional Laskerian and each primary ideal of $\overline{D}$ is a valuation ideal if and only if $\overline{D}$ is one dimensional Prufer and $\overline{D}$ has finite character. In this case D is Laskerian. (4) $\overline{D}$ is one dimensional Prufer (respectively almost Dedekind) if and only if every valuation ring of K lying over D is Laskerian (respectively strongly Laskerian). (5) The complete integral closure of a pseudo-valuation domain (D, M) is Laskerian of dimension at most one.

Keywords: Laskerian ring, overrings, complete integral closure, pseudo-valuation domain

1. INTRODUCTION

In 1905, Emanuel Lasker introduced the notion of primary ideal, which corresponds to an irreducible variety and plays a role similar to prime powers in the prime decomposition of an integer. He proved the primary decomposition theorem for an ideal of a polynomial ring in terms of primary ideals [1]. Emmy Noether, in her seminal paper [2], proved that in a commutative ring satisfying the ascending chain conditions on ideals, every ideal is the intersection of finite number of irreducible ideals (an irreducible ideal of a Noetherian ring is a primary ideal). She established several intersection decompositions. Among rings without the necessary ascending chain conditions, the rings in which such decomposition holds for all ideals are called Laskerian rings.

Throughout this note all rings are commutative with identity. The letter D denotes an integral domain with quotient field K. By an overring of D we mean a ring between D and K. We use $\overline{D}$ to denote integral closure of D in K, $\dim D$ to represent Krull dimension of D. By [3, Page 360], D is said to have valuative dimension n, represented as $\dim_v D = n$, if each valuation overring of D has dimension at most n and if there exists a valuation overring of D of dimension n. A commutative ring R with identity is Laskerian if each ideal of R admits a shortest primary representation; R is strongly Laskerian, if R is Laskerian and each primary ideal of R contains a power of its radical [3, Page 455]. It is equivalent to say that a commutative ring R with identity is Laskerian (respectively strongly Laskerian) if every ideal of R can be represented as a finite intersection of primary ideals (respectively strongly primary ideals), whereas an ideal Q of R is primary if each zero divisor of the ring R/Q is nilpotent and Q is strongly primary if Q is primary and satisfies $\left(\sqrt{Q}\right)^n \subset Q$ for some n. Following [4, Page 505], R is a zero divisor ring ($Z.D$ ring), if $Z_R(R/I)$, the set of zero divisors of R/I, is a finite union of prime ideals for all ideals I of R.

In general, *Artinian* $\Rightarrow$ *Noetherian* $\Rightarrow$ *Strongly Laskerian* $\Rightarrow$ *Laskerian* $\Rightarrow$ *Z.D ring,* But none of the above implication is reversible.

An integral domain D is said to be a Prufer domain if D_P is a valuation ring for every prime

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ideal P of D. By [3, Page 434], a domain D is an almost Dedekind domain if D_M is a Noetherian valuation ring for each maximal ideal M of D and its dimension is at most one. If D is a Prufer domain, then D is Laskerian if and only if dim $D \leq 1$ and each nonzero element of D belongs to only a finitely many maximal ideals of D [3, Page 456].

In this note we discuss the overrings, integral closure and complete integral closure of an integral domain in Laskerian perspective. First we transformed $(\Rightarrow)$ of : Integral closure $\overline{D}$ of an integral domain D is a Dedekind domain $\Leftrightarrow$ every overring of D satisfies $ACCP$ (or atomic) [5, Theorem 1], as: If integral closure $\overline{D}$ of an integral domain D is a one dimensional Laskerian ring such that each primary ideal of $\overline{D}$ is a valuation ideal, then each overring of D is Archimedean. In the second proposition we prove the necessary and sufficient condition for an almost Dedekind domain to be a Dedekind domain through the Laskerian property. We also show that $\overline{D}$ is a one dimensional Laskerian and each primary ideal of $\overline{D}$ is a valuation ideal if and only if $\overline{D}$ is a one dimensional Prufer and $\overline{D}$ has finite character. In this case D is Laskerian. With the inspiration; a valuation ring V is a Laskerian ring (respectively a strongly Laskerian ring) if and only if V has rank at most one (respectively V is discrete of rank at most one) (see [3,Page 456]), we establish that every valuation ring of K lying over D is Laskerian (respectively strongly Laskerian) if and only if $\overline{D}$ is a one dimensional Prufer domain (respectively an almost Dedekind domain). Further we observed that the complete integral closure D_0 of a pseudo-valuation domain (PVD) (D,M) (i.e. in D every prime ideal is strongly prime) is Laskerian of dimension ≤ 1 and its maximal ideal is contained in M.

2. MAIN RESULTS

Following [6], an integral domain D is Archimedean in case $\cap_{n\geq 1} Dr^n = 0$ for each nonunit $r \in D$. The most natural examples of Archimedean domains are completely integrally closed domains, one dimensional integral domains and Noetherian integral domains. An ideal A of a domain D is a valuation ideal if there exist a valuation ring $D_v \supset D$ and an ideal A_v of D_v such that $A_v \cap D = A$.

Lemma 2.1

Let D be an integral domain. If D is Laskerian, such that every primary ideal is a valuation ideal, then D is Prufer.

Proof: Since D is Laskerian, it is clear that each ideal of D has finitely many minimal prime divisors. Since a ring with later property has Noetherian spectrum if and only if ascending chain condition for prime ideals is satisfied in D, therefore by [7, Theorem 3.8] D is a Prufer domain.

Proposition 2.2

Let D be an integral domain such that its integral closure $\overline{D}$ is a one dimensional Laskerian ring and each primary ideal of $\overline{D}$ is a valuation ideal, then each overring of D is Archimedean.

Proof: If integral closure $\overline{D}$ of an integral domain D is a one dimensional Laskerian ring, such that each primary ideal of $\overline{D}$ is a valuation ideal, then by [3, Theorems 36.2 and 30.8] , $\dim_v D = \dim_v \overline{D} = \dim \overline{D} = \dim D \leq 1$. So by [8, Corollary 1.4], each overring of D is Archimedean.

Remark 2.3

Each overring of an integral domain D is Noetherian if and only if D is Noetherian and $\dim D \leq 1$ [3, Page 493, Exercise 16]. This cannot be generalized in Laskerian domains. Because, if D is a non Noetherian Laskerian integral domain such as a one dimensional valuation ring (the case when Archimedean and Laskerian behave alike), then overrings of D do not satisfy ACCP [9, Theorem 2.1].

It is easy to demonstrate that a Dedekind domain is a Laskerian domain. We prove in the next proposition that an almost Dedekind domain which is not Dedekind is not a Laskerian domain. This then demonstrates a clear difference between Dedekind domains and non Noetherian almost Dedekind domains. Here the concept of Laskerian domain is expanded to provide a way of measuring how close an almost

Dedekind domain which is not Dedekind is to being Dedekind.

Proposition 2.4

In an integral domain D with identity which is not a field, the following conditions are equivalent:

(1) D is a Dedekind domain.

(2) D is a Laskerian almost Dedekind domain.

Proof: (1) $\Rightarrow$ (2): D is Dedekind $\Rightarrow D$ is Noetherian. If M is maximal ideal of D, then D_M is a nontrivial Noetherian valuation ring. Therefore by [3, Theorem 17.5], D_M is rank one discrete and D is almost Dedekind.

(2) $\Rightarrow$ (1): If D is an almost Dedekind then dim $(D)=1$. In a Prufer domain, the Laskerian property in D is equivalent to the condition that each nonzero element of D belongs to only finitely many maximal ideals of D [3, Page 456, Exercise 9]. Hence by [3, Theorem 37.2] D is a Dedekind domain.

Mori and Nagata have proved that if D is Noetherian and one or two dimensional, then $\overline{D}$ is Noetherian [10]. Hence if D is Noetherian and one dimensional, then $\overline{D}$ is a Dedekind domain. On the other hand if the integral closure $\overline{D}$ of D is Dedekind, it is not necessary that D is Noetherian. For example if $R = \mathrm{Q} + X\overline{\mathrm{Q}}[X] = \{a_0 + \sum a_i X^i \mid a_0 \in \mathrm{Q},\ a_i \in \overline{\mathrm{Q}}\}$, where Q is the field of rational and $\overline{\mathrm{Q}}$ is algebraic closure of Q, then $\overline{R} = \overline{\mathrm{Q}}[X]$ is a PID but R is not Noetherian. In [11, Lemma 2] it is proved that if integral closure $\overline{D}$ of an integral domain D is a Dedekind domain then D is Laskerian. In the next proposition we observe that D remains Laskerian if integral closure of D is one dimensional Prufer with finite character.

Proposition 2.5

Let $\overline{D}$ be integral closure of an integral domain D. $\overline{D}$ is one dimensional Laskerian and each primary ideal of $\overline{D}$ is a valuation ideal $\Leftrightarrow$ $\overline{D}$ is one dimensional Prufer and $\overline{D}$ has finite character. When this is the case, D is Laskerian.

Proof: By Lemma 2.1 $\overline{D}$ is Prufer. Since $\overline{D}$ is Laskerian (by [12, Theorem 4]) it has Noetherian spectrum which means that every non zero element of $\overline{D}$ belongs to finite number of maximal ideals of $\overline{D}$, that is, $\overline{D}$ has finite character.

($\Leftarrow$) Since $\overline{D}$ is one dimensional Prufer and has finite character, therefore, by [3, Page 456, Exercise 9], $\overline{D}$ is Laskerian and every primary ideal of $\overline{D}$ is a valuation ideal [7]. Next we show that D is Laskerian. By Proposition 2.2, dim $(D) \leq 1$. Proper prime ideals are maximal in D, then every ideal of D is equal to its kernel [13, Page 738]. Since every proper ideal in D has a finite number of prime divisors, every ideal in D is an intersection of a finite number of pair wise co-maximal primary ideals in D.

Lemma 2.7

Let V be a non trivial valuation ring on the field K. Then V is completely integrally closed $\Leftrightarrow$ V is one dimensional $\Leftrightarrow$ V is Laskerian $\Leftrightarrow$ V is Archimedean.

Proof: A valuation ring V is completely integrally closed if and only if V is one dimensional (cf. [3, Theorem 17.5]). By [3, Page 456], a valuation ring V is Laskerian if and only if V has rank at most one. Since all valuation rings are GCD domains, by (cf. [14, Theorem 3.1]), V is Archimedean if and only if V is completely integrally closed.

Remark 2.8

A rank 2 valuation ring is not Laskerian (according to [3, Page 456]). However any valuation ring is a $Z.D$ ring [4, Page 507].

Theorem 2.9

Let D be an integral domain with quotient field K. Then the integral closure $\overline{D}$ of D is a one dimensional Prufer domain $\Leftrightarrow$ every valuation ring of K lying over D is Laskerian.

Proof: Let $\overline{D}$ be a one dimensional Prufer domain and let D_v be a valuation overring of D. So $\overline{D} \subset D_v \subset K$. Let P_v be the centre of D_v in $\overline{D}$. Since $\overline{D}$ is a one dimensional Prufer domain, therefore $\overline{D}_{P_v}$ is a rank one valuation ring (and hence a maximal ring in K) and hence $\overline{D}_{P_v} = D_v$. Since every valuation ring lying over D is of rank one, it is Laskerian.

Conversely, suppose that every valuation ring V of K lying over D is Laskerian, therefore by Lemma 2.7, V is one dimensional. Hence by [11, Theorem 1] $\overline{D}$ is a Prufer domain.

Theorem 2.10

Every valuation ring of K lying over an integral domain D is strongly Laskerian $\Leftrightarrow$ $\overline{D}$ is an almost Dedekind domain.

Proof: Suppose every valuation ring V of K lying over D is strongly Laskerian, therefore V is discrete rank one (see [3, Page 456]). By [3, Theorem 36.2], $\overline{D}$ is an almost Dedekind domain.

Conversely, suppose $\overline{D}$ is an almost Dedekind domain (hence one dimensional) and let D_v be a valuation overring of D (ring between D and K). Then $\overline{D} \subset D_v \subset K$. If P is the centre of D_v in $\overline{D}$, then $\overline{D}_P \subset D_v$; since $\overline{D}_P$ is discrete rank one, it follows that $\overline{D}_P = D_v$. Therefore every valuation ring of K lying over D is strongly Laskerian (see [3, Page 456]).

Remark 2.11

Theorems 2.9 and 2.10 respectively generalize [5, Theorem 1] as follows: Let D be an integral domain. Then the integral closure of D is a Dedekind domain (respectively Prufer 1-dimensional domain, respectively, an almost Dedekind domain) if and only if every overring of D satisfies ACCP (respectively every valuation overring of D is Laskerian, respectively every valuation overring of D is strongly Laskerian).

By [15], an integral domain D with quotient field K, is said to be pseudo-valuation domain (PVD), if whenever P is a prime ideal in D and $xy \in P$, where $x, y \in K$, then $x \in P$ or $y \in P$ (i.e. in a PVD every prime ideal is strongly prime). An integral domain D with quotient field K, is said to be PVD if for every nonzero $x \in K$, either $x \in D$ or $ax^{-1} \in D$ for every nonunit $a \in D$. A valuation domain is a PVD but the converse is not true necessarily, for example the non- integrally closed PVD $\mathbb{R} + X\mathbb{C}[[X]]$, is not a valuation domain.

Theorem 2.12

Let (D, M) be a PVD. The complete integral closure D_0 of D is quasilocal Laskerian ring of dim≤1.

Proof: If (D, M) is a PVD with quotient field K, then there is a valuation overring V of D such that $Spec(D) = Spec(V)$ [15, Theorem 2.7 and Theorem 2.10], and hence D_0 is the integral closure of V. Thus (i) $D_0 = V_p$ if V has a height-one prime ideal P or (ii) $D_0 = K$ if V does not have a height one prime ideal. In both cases D_0 is a Laskerian ring.

In the following we restate theorem [16, Theorem 4] by replacing Archimedean valuation domain with Laskerian valuation domain.

Theorem 2.13

Suppose D is an integral domain with quotient field K and let L be an extension field of K. If the complete integral closure of D is an intersection of Laskerian valuation domains on K, then the complete integral closure of D in L is an intersection of Laskerian valuation domains on L.

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