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ON THE FEKETE-SZEGO PROBLEM FOR SOME SUBCLASSES OF ANALYTIC FUNCTIONS DEFINED BY CONVOLUTION

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Abstract: In this present investigation, the authors obtain Fekete-Szego inequality for certain normalized analytic functions *f(z)* defined on the open unit disk. As a special case of this result, Fekete-Szego inequality for a class of functions defined through fractional derivatives is obtained. The motivation of this paper is to give a generalization of the Fekete-Szego inequalities obtained by Srivastava and Mishra and Ma and Minda.

Keywords: Analytic functions, Starlike functions, subordination, coefficient problem.

Introduction

Let A denote the class of function $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad \ldots (1.1)$$

which are analytic in the open unit disc $\Delta = \{z : |z| < 1\}$.

Let S be the subclass of A consisting of univalent functions. Let $\phi(z)$ be an analytic function with positive real part on Δ with $\phi(0)=1$, $\phi'(0)>0$ which maps the open unit disc onto a region starlike with respect to 1 and is symmetric with respect to the real axis. Let $S^*(\phi)$ be the class of functions in $f \in A$ for which $\frac{zf''(z)}{f'(z)} \prec \phi(z), \ (z \in \Delta)$ and $C(\phi)$ be the class of functions $f \in A$ for which $1+\frac{zf''(z)}{f'(z)} \prec \phi(z), \ (z \in \Delta)$. These classes were introduced and studied by Ma and Minda [1]. They have obtained the Fekete-Szego inequality for the functions in the class $C(\phi)$. Since $f \in C(\phi)$ if and only if $zf' \in S^*(\phi)$, we get the Fekete-Szego inequality for functions in the class $S^*(\phi)$. Fekete-Szego problem for different subclasses is studied by Ravichandran *et al.* [2, 3, 4] Shanmugam and Sivasubramanian [5], and also by Shanmugam *et al.* [6, 7]. For a brief history of Fekete-Szego problem for the class of starlike, convex and close-to-convex functions, see the recent paper by Srivastava and Owa [8] (see also the references cited by them).

For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} g_n z^n \in A$, the Hadamard. Product (or convolution product) is given by $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n g_n z^n$. For various choices of g(z) we get different operators considered earlier by Dziok and Srivastava [9], Carlson and Shaffer [10], Ruscheweyh [11], Salagean [12], Cho and Kim [13] and Cho and Srivastava [14]. Motivated essentially by the above works, we obtain the Fekete-Szego inequalityfor the estimate $\frac{(f*g)(z)}{(f*h)(z)}$ where g and h are fixed functions such that $g_n > 0$, $h_n > 0$ with $g_n - h_n > 0$ where $h(z) = z + \sum_{n=2}^{\infty} h_n z^n \in A$. For special

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choices of g and h we get all the operators which we have mentioned earlier. Also, for various choices of $g(z)$ and $h(z)$ we get various subclasses of A.

1. For $g(z)=\frac{z}{(1-z)^2}$ and $h(z)=\frac{z}{(1-z)}$,

   we get $\frac{(f*g)(z)}{(f*h)(z)}\equiv\frac{zf'(z)}{f(z)}$

2. For $g(z)=\frac{z+z^2}{(1-z)^3}$ and $h(z)=\frac{z}{(1-z)^2}$,

   we get $\frac{(f*g)(z)}{(f*h)(z)}\equiv 1+\frac{zf''(z)}{f'(z)}$.

In the present paper, we obtain the Fekete-Szego inequality for functions in a more general class $M_{g,h}(\phi)$ of functions, which we define below. The motivation of this paper is to give a generalization of the Fekete-Szego inequalities of Srivastava and Mishra [15].

Definition 1.1.

Let $\phi(z)$ be a univalent starlike function with respect to 1 which maps the open unit disk Δ onto a region in the right half plane and is symmetric with respect to the real axis, $\phi(0)=1$, $\phi'(0)>0$. A function $f\in A$ is in the class $M_{g,h}(\phi)$ if

$$\frac{(f*g)(z)}{(f*h)(z)}\prec\phi(z),\ g_n>0,\ h_n>0,\ g_n-h_n>0$$

To prove our main result, we need the following:

Lemma 1.2

If $p_1(z)=1+c_1z+c_2z^2+\dots$ is an analytic function with positive real part, then

$$\left|c_2-vc_1^2\right|\le\begin{cases}-4v+2 & \text{if } v\le 0\\ 2 & \text{if } 0\le v\le 1\\ 4v-2 & \text{if } v\ge 1\end{cases}\cdot$$

When $v<0$ or $v>1$, equality holds if and only if $p_1(z)$ *is* $\frac{(1+z)}{(1-z)}$ *or one of its rotations. If $0<v<1$, then* equality holds if and only if

$$p_1(z)=\left(\frac{1}{2}+\frac{\lambda}{2}\right)\frac{(1+z)}{(1-z)}+\left(\frac{1}{2}-\frac{\lambda}{2}\right)\frac{(1-z)}{(1+z)}\quad(0\le\lambda\le 1)$$

or one of its rotations.

If $v=1$, then equality holds if $p_1(z)$ *is a reciprocal of one of the functions such that the equality holds in the case of $v=0$.*

Although the upper bounds are sharp, it can be improved as follows.

$$|c_2-v\,c_1^{\,2}|+v|c_1|^2\le 2\quad(0<v\le 1/2)$$

and

$$|c_2-v\,c_1^{\,2}|+(1-v)|c_1|^2\le 2\quad(1/2<v\le 1).$$

Fekete-Szego Problem

Theorem 2.1

Let $\phi(z)=1+B_1z+B_2z^2+\dots$. If f(z) is given by (1.1) belongs to the class $M_{g,h}(\phi)$, then

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\frac{B_2}{g_3-h_3}-\frac{\mu B_1 2}{(g_2-h_2)^2}+\frac{(g_2h_2-h_2^2)B_1^{\,2}}{(g_3-h_3)(g_2-h_2)^2} & \text{if } \mu\le\sigma_1\\ \frac{B_1}{2(g_3-h_3)} & \text{if } \sigma_1\le\mu\le\sigma_2\\ \frac{-B_2}{g_3-h_3}+\frac{\mu B_1 2}{(g_2-h_2)^2}-\frac{(g_2h_2-h_2^2)B_1^{\,2}}{(g_3-h_3)(g_2-h_2)^2} & \text{if } \mu\ge\sigma_2\end{cases}$$

where

$$\sigma_1=\frac{(g_2-h_2)^2(B_2-B_1)+h_2(g_2-h_2)B_1^{\,2}}{(g_3-h_3)B_1^{\,2}},$$

$$\sigma_2 = \frac{(g_2-h_2)^2(B_2+B_1)+h_2(g_2-h_2)B_1^{\ 2}}{(g_3-h_3)B_1^{\ 2}}$$

and $0 \le \mu \le 1$.

Proof:

For $f \in M_{g,h}(\phi)$, let

$$p(z) = \frac{(f*g)(z)}{(f*h)(z)} = 1 + b_1 z + b_2 z^2 + \ldots \tag{2.1}$$

From (2.1), we obtain

$$a_2 g_2 - a_2 h_2 = b_1 \text{ and } (g_3-h_3)a_3 = b_2 + a_2^{\ 2}(g_2h_2 - h_2^{\ 2}).$$

Since $\phi(z)$ is univalent and $p \prec \phi$, the function

$$p_1(z) = \frac{1+\phi^{-1}(p(z))}{1-\phi^{-1}(p(z))} = 1 + c_1 z + c_2 z^2 + \ldots$$

is analytic and has positive real part in the open unit disk. Also, we have

$$p(z) = \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) \tag{2.2}$$

and from the equation (2.2), we obtain

$$b_1 = \frac{1}{2}B_1 c_1 \text{ and } b_2 = \frac{1}{2}B_1\left[c_2 - \frac{1}{2}c_1^{\ 2}\right] + \frac{1}{4}B_2 c_1^2.$$

Hence,

$$a_3 - \mu a_2^{\ 2} = \frac{B_1}{2(g_3-h_3)}(c_2 - v c_1^{\ 2})$$

where

$$v = \frac{1}{2}\left[1 - \frac{B_2}{B_1} + \frac{h_2^{\ 2} - g_2h_2 + \mu(g_3-h_3)}{(g_2-h_2)^2}B_1\right].$$

Our result now follows as an application of Lemma 1.2. We define the functions $K^{\phi n}$ $(n=2,3,4...)$ by

$$\frac{(K^{\phi n}*g)(z)}{(K^{\phi n}*h)(z)} = \phi(z^{n-1}),\ K^{\phi n}(0) = 0 = [K^{\phi n}]'(0) - 1$$

and the function F^{λ} and G^{λ} $(0 \le \lambda \le 1)$ by

$$\frac{(F^{\lambda}*g)(z)}{(F^{\lambda}*h)(z)} = \phi\left(\frac{z(z+\lambda)}{1+\lambda z}\right),\ F^{\lambda}(0) = 0 = [F^{\lambda}]'(0) - 1$$

and

$$\frac{(G^{\lambda}*g)(z)}{(G^{\lambda}*h)(z)} = \phi\left(-\frac{z(z+\lambda)}{1+\lambda z}\right),\ G^{\lambda}(0) = 0 = [G^{\lambda}]'(0) - 1$$

provided they exist

Clearly the functions $K^{\phi n}(n=2,3,4...)$, F^{λ} and G^{λ} $(0 \le \lambda \le 1)$, $\in M_{g,h}(\phi)$. Also, we write $K^{\phi} := K^{\phi 2}$.

If $\mu < \sigma_1$ or, $\mu < \sigma_2$ then equality holds if and only if f is K^{ϕ} or one of its rotations. When $\sigma < \mu < \sigma_2$, equality holds if and only if f is K^{ϕ}_3 or one of its rotations. If $\mu < \sigma_1$, then equality holds if and only if f is F^{λ} or one of its rotations. If $\mu < \sigma_2$, then then equality holds if and only if f is G^{λ} or one of its rotations.

If $\sigma_1 \le \mu \le \sigma_2$, Theorem 2.1 can be improved. Let σ_3 be given by

$$\sigma_3 := \frac{(g_2-h_2)B_2 - h_2(g_2-h_2)B_1^{\ 2}}{(g_3-h_3)B_1^{\ 2}}$$

If $\sigma_1 \le \mu \le \sigma_3$, then

$$\left|a_3 - \mu a_2^{\ 2}\right| + \frac{(g_2-h_2)^2(B_1-B_2) + \mu(g_3-h_3) - h_2(g_2-h_2)B_1^{\ 2}}{(g_2-h_2)(g_3-h_3)B_1^{\ 2}}\left|a_2^{\ 2}\right| \le \frac{B_1}{2(g_3-h_3)}$$

If $\sigma_3 \le \mu \le \sigma_2$, then

$$\left|a_3 - \mu a_2^2\right| + \frac{(g_2-h_2)^2(B_1+B_2)+\mu(g_3-h_3)-h_2(g_2-h_2)B_1^{\,2}}{(g_2-h_2)(g_3-h_3)B_1^{2}}$$

$$|a_2^2| \le \frac{B_1}{2(g_3-h_3)}$$

For $g(z) = \frac{z}{(1-z)^2}$ and $h(z) = \frac{z}{(1-z)}$ in Theorem 2.1, we get the following result for the class $S^*(\phi)$.

Corollary 2.2

Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$. If f(z) is given by (1.1) belongs to the class $S^*(\phi)$, then

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{B_2}{2} - \mu\ B_1 2 + \frac{B_1^{\,2}}{2} & \text{if } \mu \le \sigma_1 \\ \frac{B_1}{2} & \text{if } \sigma_1 \le \mu \le \sigma_2 \\ \frac{-B_2}{2} + \mu\ B_1 2 - \frac{B_1^{\,2}}{2} & \text{if } \mu \ge \sigma_2 \end{cases}$$

where

$$\sigma_1 = \frac{B_2 - B_1 + B_1^{\,2}}{2B_1^{2}},$$

$$\sigma_2 = \frac{B_2 + B_1 + B_1^{\,2}}{2B_1^{2}}$$

and $0 \le \mu \le 1$.

The result is sharp.

For $g(z) = \frac{z+z^2}{(1-z)^3}$ and $h(z) = \frac{z}{(1-z)^2}$ in Theorem 2.1, we get the following result for the class $C(\phi)$ obtained by Ma and Minda [1].

Corollary 2.3

Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$. If f (z) is given by (1.1) belongs to the class $C(\phi)$, then

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{1}{6}\left(\frac{B_2}{2} - \mu\ B_1 2 + \frac{B_1^{\,2}}{2}\right) & \text{if } \mu \le \sigma_1 \\ \frac{B_1}{6} & \text{if } \sigma_1 \le \mu \le \sigma_2 \\ \frac{1}{6}\left(\frac{-B_2}{2} + \mu\ B_1 2 - \frac{B_1^{\,2}}{2}\right) & \text{if } \mu \ge \sigma_2 \end{cases}$$

where

$$\sigma_1 = \frac{2(B_2 - B_1 + B_1^{\,2})}{3B_1^{2}},$$

$$\sigma_2 = \frac{2(B_2 + B_1 + B_1^{\,2})}{3B_1^{2}}$$

and $0 \le \mu \le 1$. The result is sharp.

Applications to functions defined by fractional Derivative

We start with the following:

Definition 3.1

([see [16] and [17]; also see [8,18])

Let *f(z)* be analytic in a simply connected region of the z-plane containing the origin. The fractional derivative of order δ is defined, for a function $f(z)$, by

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\delta}} dt \ (0 \le \delta < 1),$$

where $f(z)$ is constrained, and the multiplicity of $(z-t)^{-\delta}$ is removed by requiring that log (z-ζ) is real for z-ζ >0.

Using the above Definition 3.1 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [16] the operator $\Omega^{\delta}: A \to A$ defined by

$$(\Omega^{\delta} f)(z) : \Gamma(2-\delta) z^{\delta} D_z^{\delta} f(z) \ (\delta \neq 2,3,4\ldots).$$

If

$$g(z) = z + \sum_{n=2}^{\infty} n \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)} z^n \text{ and}$$

$$h(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)} z^n,$$ Theorem 2.1 reduces to the following result in terms of the fractional derivative.

Corollary 3.2

Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$. If f (z) is given by (1.1) belongs to the class $M_{g,h}(\phi)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{(2-\delta)(3-\delta)}{6}\left(\frac{B_2}{2} - \mu \frac{3(2-\delta)}{2(3-\delta)} B_1 2 + \frac{B_1^{\,2}}{2}\right) & \text{if } \mu \leq \sigma_1 \\ \frac{(2-\delta)(3-\delta)}{6} \frac{B_1}{6} & \text{if } \sigma_1 \leq \mu \leq \sigma_2 \\ \frac{(2-\delta)(3-\delta)}{6}\left(\frac{-B_2}{2} + \mu \frac{3(2-\delta)}{2(3-\delta)} B_1 2 - \frac{B_1^{\,2}}{2}\right) & \text{if } \mu \geq \sigma_2 \end{cases}$$

where

$$\sigma_1 = \frac{2(3-\delta)}{3(2-\delta)} \frac{2(B_2 - B_1 + B_1^{\,2})}{3B_1^2},$$

$$\sigma_2 = \frac{2(3-\delta)}{3(2-\delta)} \frac{2(B_2 + B_1 + B_1^{\,2})}{3B_1^2}$$

and $0 \leq \mu \leq 1$. The result is sharp.

Remark 3.3

For the choices $B_1 = \frac{8}{\pi^2}$ and $B_2 = \frac{16}{3\pi^2}$, Corollary 3.2 reduces to a result obtained by Srivastava and Mishra [15, Theorem 8, p.64] for which $(\Omega^{\delta} f)(z)$ is a parabolic starlike function (see [19] and [20]).

Remark 3.4

For the choices $B_1 = \frac{8}{\pi^2}$, $B_2 = \frac{16}{3\pi^2}$ and $\delta = 1$, Corollary 3.2 reduces to a result obtained by Ma and Minda [21].

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