ON GENERALIZATIONS OF HADAMARD PRODUCTS OF FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract: In the present paper, we define two classes of analytic functions with negative coefficients. Some interesting properties of generalizations of the Hadamard product in these classes are given.

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Introduction

Let T(n) denote the class of functions f(z) of the form

\[ f(z) = z - \sum_{k=n}^{\infty} a_k z^k \quad (a_k \geq 0, n \in \mathbb{N} \setminus \{1\} = \{2,3,4,\ldots\}) \tag{1.1} \]

which are analytic in the unit disc \( U = \{ z : |z| < 1 \} \).

A function \( f(z) \) in \( T(n) \) is said to be in the class \( T_n(\lambda, \alpha) \) if it satisfies the condition

\[ \Re \left\{ \frac{zf'(z)}{\lambda zf''(z) + (1-\lambda)f(z)} \right\} > \alpha \tag{1.2} \]

for some \( \alpha(0 \leq \alpha < 1) \), \( \lambda(0 \leq \lambda < 1) \) and for all \( z \in U \).

Also, let \( C_n(\lambda, \alpha) \) denote the subclass of \( T(n) \) of all functions \( f(z) \) satisfying the following condition

\[ \Re \left\{ \frac{f'(z) + zf''(z)}{f'(z) + \lambda zf(z)} \right\} > \alpha \tag{1.3} \]

for some \( \alpha(0 \leq \alpha < 1) \), \( \lambda(0 \leq \lambda < 1) \), and for all \( z \in U \).

In particular, the classes \( C_2(\lambda, \alpha) \) and \( T_2(\lambda, \alpha) \) were studied by Altinates and Owa [1]. Putting \( \lambda = 0 \), we obtain the classes \( T_n(0, \alpha) =: T_n(\alpha) \), and \( C_n(0, \alpha) =: C_n(\alpha) \), which were investigated by Choi and Kim [2]. They are subclasses of the classes of functions starlike of order \( \alpha \) and convex of order \( \alpha \), respectively, (see, for details, Duren [3] and also Srivastava and Owa [4]).

Note that

\[ f(z) \in C_n(\lambda, \alpha) \quad \text{if and only if} \quad zf'(z) \in T_n(\lambda, \alpha). \]

Let \( f_j(z)(j = 1,2) \) in \( T(n) \) be given by

\[ f_j(z) = z - \sum_{k=n}^{\infty} a_{k,j} z^k (n \geq 2, j = 1,2). \tag{1.4} \]

Then the Hadamard product (or convolution) \( f_1 \ast f_2 \) is defined by

\[ (f_1 \ast f_2)(z) = z - \sum_{k=n}^{\infty} a_{k,1} a_{k,2} z^k. \tag{1.5} \]

For any real numbers \( p \) and \( q \), we define the generalized Hadamard product \( f_1 \Delta f_2 \) by

\[ (f_1 \Delta f_2)(p,q; z) = z - \sum_{k=n}^{\infty} (a_{k,1})^p (a_{k,2})^q z^k. \tag{1.6} \]
In the special case, if we take \( p=q=1 \), then
\[
(f_1 \Delta f_2)(1,1;z) = (f_1 \ast f_2)(z)(z \in U). \tag{1.7}
\]

In the present paper, we make use of the generalized Hadamard product with a view to proving interesting characterization theorems involving the classes \( T_n(\lambda, \alpha) \) and \( C_n(\lambda, \alpha) \).

**2 Main Results**

In order to prove our results for functions to the general classes \( T_n(\lambda, \alpha) \) and \( C_n(\lambda, \alpha) \), we shall need the following Lemmas given by Altintas and Owa [1].

**Lemma 1**

A function \( f(z) \) defined by (1.1) is in the class if \( T_n(\lambda, \alpha) \) and only if
\[
\sum_{k=n}^{\infty} (k - \alpha(k+1-\lambda))a_k \leq 1 - \alpha. \tag{2.1}
\]

**Lemma 2**

A function \( f(z) \) defined by (1.1) is in the class \( C_n(\lambda, \alpha) \) if and only if
\[
\sum_{k=n}^{\infty} k(k+1-\lambda)a_k \leq 1 - \alpha. \tag{2.2}
\]

**Remark 1**

As pointed out earlier by Altintas and Owa [1], Lemma 1 and Lemma 2 follow immediately from a result due to Altintas and Owa [1] upon setting \( a_i=0 \) (in case \( n=2 \) set \( a_i \neq 0 \) because \( f(0)=0, f'(0)=1 \) which satisfy normalization).

Applying Lemma 1, we shall prove.

**Theorem 1**

Let functions \( f_j(z)(j=1,2) \) defined by (1.4) be in the classes \( T_n(\lambda, \alpha_j) \) (respectively, then
\[
(f_1 \Delta f_2)(\frac{1}{p}, \frac{p-1}{p};z) \in T_n(\lambda, \beta_1), \tag{2.3}
\]

where \( p > 1 \) and
\[
\beta = \min \left\{ \frac{(k+1-\lambda) - \lambda}{k^2}, \frac{k+1-\lambda}{k} \right\}. \tag{2.4}
\]

**Proof**

Since \( f_j(z) \in T_n(\lambda, \alpha_j), f_j(z) \) by using Lemma 1 we have
\[
\sum_{k=n}^{\infty} \frac{k - \alpha_j(k+1-\lambda)}{1 - \alpha_j}a_{k,j} \leq 1(j = 1, 2) \]

Moreover,
\[
\sum_{k=n}^{\infty} \frac{k - \alpha_1(k+1-\lambda)}{1 - \alpha_1}a_{k,1} \leq 1 \tag{2.5}
\]

and
\[
\sum_{k=n}^{\infty} \frac{k - \alpha_2(k+1-\lambda)}{1 - \alpha_2}a_{k,2} \leq 1 \tag{2.6}
\]

By the Holder inequality, we get
\[
\sum_{k=n}^{\infty} \left( \frac{\alpha_j(k+1-\lambda)}{1 - \alpha_j} \right)^{1/p} \left( \frac{k - \alpha_2(k+1-\lambda)}{1 - \alpha_2} \right)^{p-1} \leq 1. \tag{2.7}
\]
Since

\[(f_1^\Delta f_2)^{1/p}(z) = z - \sum_{k=n}^{\infty}(a_{k,1}^{1/p}(a_{k,2}^{1/p} z^k) \leq \{k \geq 2\} \quad (2.8)}\]

we see that

\[
\sum_{k=n}^{\infty} \left(\frac{k - \beta_1 (k\lambda + 1 - \lambda)}{1 - \beta_1} \right)^{1/p}(a_{k,2}^{1/p} z^k) \leq 1 (k \geq 2) \quad (2.9)
\]

with

\[
\beta_1 = \min_{k \geq 2} \left\{ \frac{(k)!(1 - \lambda)}{(k - \alpha_x (k\lambda + 1 - \lambda))^{1/p}(1 - \alpha_1)} \right\}.
\]

Thus, by Lemma 1, the proof of Theorem 1 is complete.

**Corollary 1**

If the functions \(f_1(z) (j = 1, 2)\) defined by (1.4) are in the class \(T_n(\lambda, \alpha)\), respectively, then

\[(f_1^\Delta f_2)^{1/p}(z) \in T_n(\lambda, \alpha) \quad (p > 1). \quad (2.10)\]

**Proof**

In view of Lemma 1, Corollary 1 follows readily from Theorem 1 in the special case \(\alpha_j = \alpha\).

**Theorem 2**

If the functions \(f_1(z) (j = 1, 2)\) defined by (1.4) are in the class \(C_n(\lambda, \alpha_j)\), respectively, then

\[(f_1^\Delta f_2)^{1/p}(z) \in C_n(\lambda, \alpha) \quad (p > 1). \quad (2.11)\]

where \(p > 1\) and \(\beta_1\) is defined by (2.4).

**Proof**

Since \(f_1(z) \in C_n(\lambda, \alpha_j)\), by using Lemma 2, we get

\[
\sum_{k=n}^{\infty} k - \alpha_j (k\lambda + 1 - \lambda) \leq 1 (j = 1, 2, n \geq 2) \quad (2.12)
\]

Thus the proof of Theorem 2 is similar to that of Theorem 1, where Lemma 2 is used instead of Lemma 1.

**Corollary 2**

If the functions \(f_1(z) (j = 1, 2)\) defined by (1.4) are in the class \(C_n(\lambda, \alpha)\), respectively, then

\[(f_1^\Delta f_2)^{1/p}(z) \in C_n(\lambda, \alpha) \quad (p > 1). \quad (2.13)\]

**Theorem 3**

Let functions \(f_1(z) (j = 1, 2, \ldots, m)\) defined by (1.4) be in the classes \(T_n(\lambda, \alpha_j)\), respectively, and let \(F_m(z)\) be defined by

\[
F_m(z) = z - \sum_{k=0}^{\infty} \left(\sum_{j=1}^{m} (a_{k,j})^p z^k \right) \quad (p \geq 2, z \in U). \quad (2.14)
\]

then

\[F_m(z) \in T_n(\lambda, \beta_m), \quad (2.15)\]

where

\[
\beta_m = \left\{ \frac{\left[1 - \alpha \right]}{\left[1 - \alpha \right]} \right\} \quad (2.16)
\]

\[\alpha = \min_{1 \leq j \leq m} \alpha_j, \quad (2.17)\]
and
\[
\left( \frac{n - \alpha(\lambda, n + 1 - \lambda)}{1 - \alpha} \right)^p \geq n^m. \tag{2.18}
\]

**Proof**

Since \( f_j(z) \in C_n(\lambda, \alpha_j) \), using Lemma 1, we observe that
\[
\sum_{k=n}^{\infty} k - \alpha_j(\lambda, k + 1 - \lambda) (1 - \alpha_j) a_{k,j} \leq 1 \quad (j = 1, 2, \ldots, m)
\]
and
\[
\sum_{k=n}^{\infty} k - \alpha_j(\lambda, k + 1 - \lambda) (1 - \alpha_j) a_{k,j} \leq 1 \quad (p > 1).
\]

Thus we have
\[
\sum_{k=n}^{\infty} \left\{ \sum_{j=1}^{m} \frac{k - \alpha_j(\lambda, k + 1 - \lambda)}{(1 - \alpha_j)} \right\}^p (\alpha, \beta_{k,j}) \leq 1.
\]
Let \( \beta_n \) be defined by (2.16). Since
\[
\frac{\partial \beta_n}{\partial k} \geq 0 \quad (p \geq 2, k \geq n),
\]
we have
\[
\beta_n \leq \beta_k \quad (k \geq n). \tag{2.19}
\]
Thus by Lemma 1 and (2.17), we find that
\[
\sum_{k=n}^{\infty} \left( \frac{k - \beta_n(\lambda, k + 1 - \lambda)}{1 - \beta_n} \right) (\sum_{j=1}^{m} (a_{k,j})^p) \leq \sum_{j=1}^{m} \left( \frac{k - \beta_k(\lambda, k + 1 - \lambda)}{1 - \beta_k} \right) (\sum_{j=1}^{m} (a_{k,j})^p)
\]
\[
\leq \sum_{k=n}^{\infty} \frac{1}{k - \alpha(\lambda, k + 1 - \lambda)} (\sum_{j=1}^{m} (a_{k,j})^p)
\]
\[
\leq \sum_{k=n}^{\infty} \frac{1}{k - \alpha(\lambda, k + 1 - \lambda)} (\sum_{j=1}^{m} (a_{k,j})^p) \leq 1. \tag{2.20}
\]

By (2.18), we see that \( 0 \leq \beta_n \leq 1 \). Thus the proof of Theorem 3 is complete.

**Theorem 4**

Let functions \( f_j(z)(j=1,2,\ldots,m) \) defined by (1.4) be in the class \( C_n(\lambda, \alpha_j) \), respectively, and let \( F_m(z) \) be defined by (2.14). Then
\[
F_m(z) \in C_n(\lambda, \alpha_j) \quad (z \in U). \tag{2.21}
\]
where \( \beta_n \) is defined by (2.16).

**Proof**

Since \( f_j(z) \in C_n(\lambda, \alpha_j) \), by using Lemma 2, we obtain
\[
\sum_{k=n}^{\infty} \frac{k - \alpha_j(\lambda, k + 1 - \lambda)}{1 - \alpha_j} a_{k,j} \leq 1. \tag{2.22}
\]

Thus the proof of Theorem 4 is analogous to Theorem 3. The details may be omitted.

**Remark 2**

Putting \( \lambda = 0 \) in all results, we get the results obtained by Choi and Kim [2].


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**References**