Proc. Pakistan Acad. Sci. 43(4):269-273.2006

ON GENERALIZATIONS OF HADAMARD PRODUCTS OF FUNCTIONS WITH NEGATIVE COEFFICIENTS

H. E. Darwish

Department of Mathematics, Faculty of Science, University of Mansoura, Mansoura, Egypt

Received April 2006, accepted June 2006

Communicated by Prof. Dr. M. Iqbal Choudhary

Abstract: In the present paper, we define two classes of analytic functions with negative coefficients. Some interesting properties of generalizations of the Hadamard product in these classes are given.

Keywords: Hadamard product, analytic, AMS (2000) Subject Classification 30C 45

Introduction

Let T(n) denote the class of functions f(z) of the form

$$f(z) = z - \sum_{k=n}^{\infty} a_k z^k \, (a_k \geq 0, n \in N \setminus \{1\} = \{2,3,K\}) \quad (1.1)$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$.

A function f(z) in T(n) is said to be in the class $T_n(\lambda, \alpha)$ if it satisfies the condition

$$\mathrm{Re}\left\{\frac{zf'(z)}{\lambda zf'(z) + (1-\lambda)f(z)}\right\} > \alpha \quad (1.2)$$

for some $\alpha(0 \leq \alpha < 1)$, $\lambda(0 \leq \lambda < 1)$ and for all $z \in U$.

Also, let Cn (λ, α) denote the subclass of T(n) of all functions f(z) satisfying the following condition

$$\mathrm{Re}\left\{\frac{f'(z) + zf''(z)}{f'(z) + \lambda zf''(z)}\right\} > \alpha \quad (1.3)$$

for some $\alpha(0 \leq \alpha < 1)$, $\lambda(0 \leq \lambda < 1)$, and for all $z \in U$.

In particular, the classes $C_2(\lambda,\alpha)$ and $T_2(\lambda,\alpha)$ were studied by Altinates and Owa [1]. Putting λ=0, we obtain the classes $T_n(0,\alpha) =: T_n(\alpha)$, and $C_n(0,\alpha) =: C_n(\alpha)$, which were investigated by Choi and Kim [2]. They are subclasses of the classes of functions starlike of order α and convex of order α, respectively, (see, for details, Duren [3] and also Srivastava and Owa [4]).

Note that

$f(z) \in C_n(\lambda, \alpha)$ if and only if $zf'(z) \in T_n(\lambda, \alpha)$.

Let $f_j(z)(j = 1,2)$ in T(n) be given by

$$f_j(z) = z - \sum_{k=n}^{\infty} a_{k,j} z^k \, (n \geq 2, j = 1,2). \quad (1.4)$$

Then the Hadamard product (or convo-lution) $f_1 * f_2$ is defined by

$$(f_1 * f_2)(z) = z - \sum_{k=n}^{\infty} a_{k,1} a_{k,2} z^k. \quad (1.5)$$

For any real numbers *p* and *q*, we define the generalized Hadamard product $(f_1 \Delta f_2)$ by

$$(f_1 \Delta f_2)(p, q; z) = z - \sum_{k=n}^{\infty} (a_{k,1})^p (a_{k,2})^q z^k. \quad (1.6)$$

E-mail: Darwish333@yahoo.com

In the special case, if we take p=q=1, then

$$(f_1 \Delta f_2)(1,1;z) = (f_1 * f_2)(z)(z \in U). \quad (1.7)$$

In the present paper, we make use of the generalized Hadamard product with a view to proving interesting characterization theorems involving the classes $T_n(\lambda,\alpha)$ and $C_n(\lambda,\alpha)$.

2 Main Results

In order to prove our results for functions to the general classes $T_n(\lambda,\alpha)$ and $C_n(\lambda,\alpha)$, we shall need the following Lemmas given by Altintas and Owa [1].

Lemma 1

A function f(z) defined by (1.1) is in the class if $T_n(\lambda,\alpha)$ and only if

$$\sum_{k=n}^{\infty}[k-\alpha(\lambda k+1-\lambda)]a_k \le 1-\alpha. \quad (2.1)$$

Lemma 2

A function f(z) defined by (1.1) is in the class $C_n(\lambda,\alpha)$ if and only if

$$\sum_{k=n}^{\infty}k[k-\alpha(\lambda k+1-\lambda)]a_k \le 1-\alpha. \quad (2.2)$$

Remark 1

As pointed out earlier by Altintas and Owa [1], Lemma 1 and Lemma 2 follow immediately from a result due to Altintas and Owa [1] upon setting a_k=0(k=2,3,...,n-1 (in case n=2 set $a_1 \neq 0$ because f(0)=0, f′(0)=1 which satisfy normalization).

Applying Lemma 1, we shall prove.

Theorem 1

Let functions $f_j(z)$(j=1,2) defined by (1.4) be in the classes $T_n(\lambda,\alpha_j)$ (respectively, then

$$(f_1 \Delta f_2)(\frac{1}{p},\frac{p-1}{p};z) \in T_n(\lambda,\beta_1), \quad (2.3)$$

where p > 1 and

$$\beta_1 = \min_{k\ge n}\left\{1-\frac{(k-1)(1-\lambda)}{(\frac{k-\alpha_1(\lambda k+1-\lambda)}{1-\alpha_1})^{\frac{1}{p}}(\frac{k-\alpha_2(\lambda k+1-\lambda)}{1-\alpha_2})^{1-\frac{1}{p}}-1-\lambda(k-1)}\right\}. \quad (2.4)$$

Proof

Since $f_j(z) \in T_n(\lambda,\alpha_j)$, fj(z) by using Lemma 1 we have

$$\sum_{k=n}^{\infty}(\frac{k-\alpha_j(k\lambda+1-\lambda)}{1-\alpha_j})a_{k,j} \le 1(j=1,2)$$

Moreover,

$$\sum_{k=n}^{\infty}(\frac{k-\alpha_1(k\lambda+1-\lambda)}{1-\alpha_1}a_{k,1})^{1/p} \le 1 \quad (2.5)$$

and

$$\sum_{k=n}^{\infty}(\frac{k-\alpha_2(k\lambda+1-\lambda)}{1-\alpha_2}a_{k,2})^{1/p} \le 1 \quad (2.6)$$

By the Holder inequality, we get

$$\sum_{k=n}^{\infty}(\frac{k-\alpha_1(k\lambda+1-\lambda)}{1-\alpha_1})^{1/p}(\frac{k-\alpha_2(k\lambda+1-\lambda)}{1-\alpha_2})^{\frac{P-1}{P}}(a_{k,1})^{1/p}(a_{k,2})^{\frac{P-1}{P}} \le 1. \quad (2.7)$$

Since

$$(f_1 \Delta f_2)(\frac{1}{p},\frac{p-1}{p};z) = z - \sum_{k=n}^{\infty} (a_{k,1})^{1/p} (a_{k,2})^{p-1/p} z^k \leq (k \geq 2) \quad (2.8)$$

we see that

$$\sum_{k=n}^{\infty} (\frac{k-\beta_1(k\lambda+1-\lambda)}{1-\beta_1} (a_{k,1})^{1/p} (a_{k,2})^{\frac{P-1}{P}} \leq 1 (k \geq 2) \quad (2.9)$$

with

$$\beta_1 = \min_{k \geq n} \left\{ 1 - \frac{(k-1)(1-\lambda)}{(\frac{k-\alpha_1(\lambda k+1-\lambda)}{1-\alpha_1})^{\frac{1}{p}} (\frac{k-\alpha_2(\lambda k+1-\lambda)}{1-\alpha_2})^{1-\frac{1}{p}} - 1 - \lambda(k-1)} \right\}.$$

Thus, by Lemma 1, the proof of Theorem 1 is complete.

Corollary 1

If the functions $f_j(z)(j=1,2)$ defined by (1.4) are in the class $T_n(\lambda,\alpha)$ T, respectively, then

$$(f_1 \Delta f_2)(\frac{1}{p},\frac{p-1}{p};z) \in T_n(\lambda,\alpha) \quad (p > 1). \quad (2.10)$$

Proof

In view of Lemma 1, Corollary 1 follows readily from Theorem 1 in the special case $\alpha_j=\alpha$.

Theorem 2

If the functions $f_j(z)(j = 1, 2$ defined by (1.4) are in the class $C_n(\lambda,\alpha j)$, respectively, then

$$(f_1 \Delta f_2)(\frac{1}{p},\frac{p-1}{p};z) \in C_n(\lambda,\alpha) \quad (p > 1). \quad (2.11)$$

where $p > 1$ and $\beta 1$ is defined by (2.4).

Proof

Since $f_j(z) \in C_n(\lambda,\alpha_j)$, by using Lemma 2, we get

$$\sum_{k=n}^{\infty} k(\frac{k-\alpha_j(\lambda k+1-\lambda)}{1-\alpha_j}) a_{k,j} \leq 1 (j=1,2, n \geq 2). \quad (2.12)$$

Thus the proof of Theorem 2 is similar to that of Theorem 1, where Lemma 2 is used instead of Lemma 1.

Corollary 2

If the functions $f_j(z)$ $(j = 1,2)$ defined by (1.4) are in the class $C_n(\lambda,\alpha)$, respectively, then

$$(f_1 \Delta f_2)(\frac{1}{p},\frac{p-1}{p};z) \in C_n(\lambda,\alpha) \quad (p > 1). \quad (2.13)$$

Theorem 3

Let functions $f_j(z)(j=1,2,...m)$ defined by (1.4) be in the classes $T_n(\lambda,\alpha_j)$, respectively, and let $Fm(z)$ be defined by

$$F_m(z) = z - \sum_{k=n}^{\infty} (\sum_{j=1}^{\infty} (a_{k,j})^p) z^k \, (p \geq 2, z \in U). \quad (2.14)$$

then

$$Fm(z) \in T_n(\lambda,\beta_n), \quad (2.15)$$

where

$$\beta_n = 1 - (n-1)(1-\lambda) / (\frac{1}{m} (\frac{[n-\alpha(\lambda n+1-\lambda)]}{1-\alpha})^p - (\lambda n+1-\lambda)), \quad (2.16)$$

$$\alpha = \min_{1 \leq j \leq m} \alpha_j, \quad (2.17)$$

and

$$\left(\frac{n-\alpha(\lambda n+1-\lambda)}{(1-\alpha)}\right)^{p} \geq nm. \tag{2.18}$$

Proof

Since $f_j(z)\in C_n(\lambda,\alpha_j)$, using Lemma 1, we observe that

$$\sum_{k=n}^{\infty}\frac{k-\alpha_j(\lambda k+1-\lambda)}{1-\alpha_j})a_{k,j} \leq 1(j=1,2,..,m)$$

and

$$\sum_{k=n}^{\infty}\frac{k-\alpha_j(\lambda k+1-\lambda)}{1-\alpha_j})^{p}(a_{k,j})^{p}$$

$$\leq (\sum_{k=n}^{\infty}\frac{k-\alpha_j(\lambda k+1-\lambda)}{1-\alpha_j}a_{k,j})^{p} \leq 1(p>1).$$

Thus we have

$$\sum_{k=n}^{\infty}\left\{\frac{1}{m}\sum_{j=1}^{m}\left(\frac{k-\alpha_j(\lambda k+1-\lambda)}{(1-\alpha_j)}\right)^{p}(\alpha_{k,j})^{p}\right\} \leq 1.$$

Let β_k be defined by (2.16). Since

$\frac{\partial \beta_k}{\partial k} \geq 0(p\geq 2, k\geq n)$, we have

$$\beta_n \leq \beta_k\ (k \geq n). \tag{2.19}$$

Thus by Lemma 1 and (2.17), we find that

$$\sum_{k=n}^{\infty}\frac{(k-\beta_n(\lambda k+1-\lambda)}{1-\beta_n}(\sum_{j=1}^{m}(a_{k,j})^{p})$$

$$\leq \sum_{j=1}^{\infty}\frac{(k-\beta_k(\lambda k+1-\lambda)}{1-\beta_k}(\sum_{j=1}^{m}(a_{k,j})^{p})$$

$$\leq \sum_{k=n}^{\infty}\frac{1}{m}\frac{(k-\alpha(\lambda k+1-\lambda)}{1-\alpha})^{p}(\sum_{j=1}^{m}(a_{k,j})^{p})$$

$$\leq \sum_{k=n}^{\infty}\frac{1}{m}\sum_{j=1}^{\infty}(\frac{(k-\alpha(\lambda k+1-\lambda)}{1-\alpha})^{p}(a_{k,j})^{p} \leq 1. \tag{2.20}$$

By (2.18), we see that $0\leq\beta_n<1$. Thus the proof of Theorem 3 is complete.

Theorem 4

Let functions $f_j(z)(j=1,2,...m)$ defined by (1.4) be in the class $C_n(\lambda,\alpha_j)$, respectively, and let $F_m(z)$ be defined by (2.14). Then

$$F_m(z) \in C_n(\lambda, \alpha_j) \quad (z\in U). \tag{2.21}$$

where β_n is defined by (2.16).

Proof

Since $f_j(z) \in C_n(\lambda, \alpha_j)$, by using Lemma 2, we obtain

$$\sum_{k=n}^{\infty}\frac{k[k-\alpha_j(\lambda k+1-\lambda)]}{1-\alpha_j})a_{k,j} \leq 1 \tag{2.22}$$

Thus the proof of Theorem 4 is analogous to Theorem 3. The details may be omitted.

Remark 2

Putting λ=0 in all results, we get the results obtained by Choi and Kim [2].

Acknowledgments

The author wishes to thank Prof. M. K. Aouf for his kind encouragement and help in the preparation of this paper. Also, the author is thankful to the referee for his valuable suggestions.

References

1. **Altintas, O. and Owa, S**. 1988. On subclasses of univalent functions with negative coefficients. *Pusan Kyougnam Math. J*. 4:41-56.
2. **Choi, J.H. and Kim, Y.C.** 1996. Generalizations of Hadamard products of functions with negative coefficients. *J. Math. Anal. Appl*. 199:495-501.
3. **Duren, P.L.** 1983. *Univalent Functions*. Grundlehren der Mathematischen Wissenschaften, Vol. 259, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo.
4. **Srivastava, H.M. and. Owa, S**. (Eds.). 1992. *Current Topics in Analytic Function Theory*. World Scientific, Singapore, New Jersey, London, Hong Kong.