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THERMAL EFFECTS ON RAYLEIGH WAVES IN A ROTATING TRANSVERSELY ISOTROPIC MATERIAL

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Abstract: Thermal effects on the speed of Rayleigh waves in a rotating transversely isotropic material were studied. A very simple technique was adopted to solve the secular equation.

Keywords: Rayleigh waves, transversely isotropic material, orthotropic elastic solids

Introduction

In 1885, for the first time, Rayleigh [1] studied the waves propagated along the plane surface of elastic solid Therefore, surface waves are known by his name. Subsequently, a number of researchers [2-9] studied the speed of Rayleigh waves by applying different techniques in different kinds of materials. Recently, Pham and Ogden [10] discussed the Rayleigh waves' speed in orthotropic elastic solids.

In this article, rotational and thermal effects on the speed of Rayleigh waves in a transversely isotropic medium were studied.

Boundary Value Problem and Secular Equation

Consider the semi-infinite stress-free surface of transversely isotropic material. We choose the rectangular co-ordinate system in such a way that x_3-axis is normal to the boundary and the body occupies region x_3 d" 0.

By following Pham and Ogden [10] we consider the plane harmonic waves in x_1-direction in x_1 x_3-plane with displacement components (u_1, u_2, u_3) such that

$$u_i = u_i (x_1, x_3, t)\ ,\ i = 1,3,\ \ u_2 = 0 \tag{1}$$

The generalized Hook's law for transversely isotropic body may be written as

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0\\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0\\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & c_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & c_{44} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{c_{11}-c_{12}}{2}\end{bmatrix} \begin{bmatrix}\in_{11}\\ \in_{22}\\ \in_{33}\\ 2\in_{23}\\ 2\in_{13}\\ 2\in_{12}\end{bmatrix} \tag{2}$$

where $\in_{ij}$ is the strain tensor such that

σ_{ij} is the strees tensor and $c_{ii} > 0$, $i = 1,3,4$; $c_{11}c_{33}$-$cc^2_{13} > 0$, which are the necessary and sufficient conditions for the strain energy of the material to be positive definite.

Let the material be subjected to a temperature change T-T_0, where T_0 is the reference temperature and $|T-T_0| << T_0$, then by using Khan and Ahmed [12] we have

$$\sigma_{ij} = c_{ijkl}\in_{kl} - \beta_{ij}(T - T_0)$$

$$\frac{\partial}{\partial x_i}(K_{ij}\frac{\partial}{\partial x_j}) = \rho C_v \frac{\partial T}{\partial t} + T_0 \ \beta_{ij}\frac{\partial \in_{ij}}{\partial t} \qquad (4)$$

where K_{ij}, β_{ij}, C_v are the conductivity tensor, thermal moduli, and specific heat at constant deformation, respectively.

For transversely isotropic material [12]

$$K_{ij} = K_1\,\delta_{ij} + (K_2 - K_1)\,\delta_{i3}\,\delta_{j3}$$

$$\beta_{ij} = \beta_1\,\delta_{ij} + (\beta_2 - \beta_1)\,\delta_{i3}\,\delta_{j3} \qquad (5)$$

where β_1 and β_2 are the thermal moduli w.r.t the plane and the axis of symmetry, respectively, and are given by

$$\beta_1 = \alpha_1(c_{11} + c_{12}) + \alpha_2 c_{13}$$

$$\beta_2 = 2\alpha_1 c_{13} + \alpha_2 c_{13} \qquad (6)$$

where α_1, α_2 are the coefficients of the thermal expansions w.r.t the plane and axis of symmetry, respectively.

We assume using Khan and Ahmed [12]

$$T - T_0 = A_1 U_{1.1} + B_1 u_{3.3} \qquad (7)$$

For our given problem, using (1), (2), (3), (4), (5) and (7)

$$\sigma_{11} = (c_{11} - \beta_{11}A_1)u_{1.1} + (c_{13} - \beta_{11}B_1)u_{3.3}$$

$$\sigma_{33} = (c_{13} - \beta_{33}A_1)u_{1.1} + (c_{33} - \beta_{33}B_1)u_{3.3} \qquad (8)$$

$$\sigma_{13} = c_{44}(u_{1.3} + u_{3.1})$$

If a homogeneous elastic body is rotating about an axis, we may choose x_3-axis, with a constant angular velocity Ω. Then equations of motion for infinitesimal deformation may be written as follows [11]:

$$\sigma_{ij,j} = \rho\left\{\ddot{u}_i + \Omega_j u_j \Omega_i - \Omega^2 u_i + 2\varepsilon_{ijk}\Omega_j \dot{u}_k\right\}. \qquad (9)$$

where $\Omega = \Omega\,(0,0,1)$.

In view of (9), the equations of motion for our problem are:

$$\sigma_{11,1} + \sigma_{13,3} = \rho\,\{\ddot{u}_1 - \Omega^2 u_1\}$$

$$\sigma_{31,1} + \sigma_{33,3} = \rho\,\ddot{u}_3 \qquad (10)$$

In view of (8), (10) can be written as

$$(c_{11} - \beta_{11}A_1)u_{1,11} + c_{44}u_{1,33}$$

$$+ (c_{13} + c_{44} - \beta_{11}B_1)u_{3,31} = \rho(\ddot{u}_1 - \Omega^2 u_1)$$

$$c_{44}u_{3,11} + (c_{33} - \beta_{33}B_1)u_{3,33}$$

$$+ (c_{13} + c_{44} - \beta_{33}A_1)u_{1,13} = \rho\,\ddot{u}_3 \qquad (11)$$

The boundary conditions of zero traction are

$\sigma_{3i} = 0$, $i = 1,3$ *on the plane* $x_3 = 0$ (12)

The usual requirements that the displacements and the stress components decay away from the boundary implies

$$u_i \to 0,\ \sigma_{ij} \to 0,\ (i,j = 1,3) \text{ as } x_3 \to -\infty \qquad (13)$$

Considering the harmonic waves propagating in x_1-direction, by following Pham and Ogden [10], we write:

$$u_j = \varphi_j(y)\exp[ik(x_1 - ct)]\ ;\ j = 1,3\ ,\ y = kx_3 \qquad (14)$$

where k is the wave number and c is the wave speed and φ_j, j=1,3 are the functions to be determined.

Substituting (14) into (11), we get

$$\left[k^2(c_{11}-\beta_{11}A_1-\rho c^2)-\rho\Omega^2\right]\varphi_1-c_{44}k^2\varphi_1''$$
$$-ik^2(c_{44}+c_{13}-\beta_{11}B_1)\phi_3'=0$$

$$(c_{44}-\rho c^2)\varphi_3-(c_{33}-\beta_{33}B_1)\varphi_3''$$
$$-i(c_{44}+c_{13}-\beta_{33}A_1)\phi_1'=0 \quad (15)$$

and the boundary condition (12) in terms of ϕ_j becomes

$$ic_{13}\phi_1+c_{33}\phi_3'=0$$
$$\phi_1'+i\phi_3=0\text{, on the plane } x_3=0 \quad (16)$$

while from (13)

$$\phi_j,\phi_j'\to 0 \text{ as } x_3\to-\infty \quad (17)$$

Taking Laplace Transfrom of (15) and using (16)

$$\left[k^2\left(-c_{44}s^2-\rho c^2+c_{11}-\beta_{11}A\right)-\rho\Omega^2\right]\bar\phi_1(s)-i\,k^2(c_{44}+c_{13}$$
$$-\beta_{11}B_1)s\bar\phi_3(s)=k^2\left[-c_{44}s\phi_1(0)-i\,(c_{13}-\beta_{11}B_1)\phi_3(0)\right.$$
$$i(c_{13}+c_{44}-\beta_{33}A_1)s\bar\phi_1(s)+\left[(c_{33}-\beta_{33}B_1)s^2\right.$$
$$\left.-c_{44}+\rho c^2)\right]\bar\phi_3(s)=ic_{44}\phi_1(0)+(c_{33}-\beta_{33}B_1)\phi_3(0)s \quad (18)$$

$$\Rightarrow\bar\phi_1(s)=\frac{P}{Q} \quad (19)$$

where

$$P=k^2\left[c_{44}(c_{33}-\beta_{33}B_1)\varphi_1(0)s^3-ic_{44}(c_{33}-\beta_{33}B_1)\right.$$
$$\varphi_3(0)s^2+c_{44}(\rho c^2+c_{13}-\beta_{11}B_1)\varphi_1(0)s$$
$$\left.+i(c_{13}-\beta_{11}B_1)(\rho c^2-c_{44})\varphi_3(0)\right] \text{ and}$$

$$Q=k^2c_{44}(c_{33}-\beta_{33}B_1)s^4+\left[k^2\{(c_{13}+c_{44}-\beta_{11}B_1)\right.$$
$$(c_{13}+c_{44}-\beta_{33}A_1)(c_{33}-\beta_{33}B_1)(c_{11}-\rho c^2-\beta_{11}A_1)+$$
$$\left.c_{44}(\rho c^2-c_{44})\}+(c_{33}-\beta_{33}B_1)\rho\Omega^2\right]s^2$$
$$+\left[k^2(c_{11}-\rho c^2-\beta_{11}A_1)-\rho\Omega^2\right](c_{44}-\rho c^2)$$

Let s_1^2, s_2^2 be the roots for s^2 (where s_1, s_2 have positive real parts)

$$Q=0, \quad (20)$$

This implies

$$\bar\phi_1(s)=\frac{A}{s-s_1}+\frac{B}{s-s_2}+\frac{C}{s+s_1}+\frac{D}{s+s_2} \quad (21)$$

where A, B, C, D are the constants to be determined.

Taking inverse Laplace transform and applying (17), we have

$\phi_1(y)=A_1\exp(s_1y)+A_2\exp(s_2y)$ where $y=kx_3$ (22)

In view of (22), (15) and (17) can be written as

$$\phi_3(y)=\alpha_1A_1\exp(s_1y)+\alpha_2A_2\exp(s_2y) \quad (23)$$

where

$$\alpha j=\frac{\left[k^2(c_{11}-\rho c^2-\beta_{11}A_1)-\rho\Omega^2\right]-c_{44}s_j^2)}{ik^2(c_{13}+c_{44}-\beta_{11}B_1)s_j},\ j=1,2 \quad (24)$$

From (20), we get

$$s_1^2+s_2^2=-\frac{\left[k^2\left\{(c_{13}+c_{44}-\beta_{11}B_1)(c_{13}+c_{44}-\beta_{33}A_1)-(c_{33}-\beta_{33}B_1)(c_{11}-\rho c^2-\beta_{11}A_1)+c_{44}(\rho c^2-c_{44})\right\}+(c_{33}-\beta_{33}B_1)\rho\Omega^2\right]}{k^2c_{44}(c_{33}-\beta_{33}B_1)}$$

$$s_1^2s_2^2=\frac{\left[k^2(c_{11}-\rho c^2-\beta_{11}A_1)-\rho\Omega^2\right](c_{44}-\rho c^2)}{k^2c_{44}(c_{33}-\beta_{33}B_1)} \tag{25}$$

In view of (22) and (23), eq. (16) gives

$$\begin{aligned}&\left[i(c_{13}-\beta_{33}A_1)+(c_{33}-\beta_{33}B_1)\alpha_1s_1\right]A+\\&\left[i(c_{13}-\beta_{33}A_1)+(c_{33}-\beta_{33}B_1)\alpha_2s_2\right]\mathrm{B}=0\\&(s_1+i\alpha_1)A+(s_2+i\alpha_2)B=0\end{aligned} \tag{26}$$

For non-trivial solution of (26), the determinant of co-efficient must vanish i.e.

$$\begin{aligned}&\left[i(c_{13}-\beta_{33}A_1)+(c_{33}-\beta_{33}B_1)\alpha_1s_1\right](s\ +i\alpha_2)\\&-\left[i(c_{13}-\beta_{33}A_1)+(c_{33}-\beta_{33}B_1)\alpha_2s_2\right](s_1+i\alpha_1)=0\end{aligned} \tag{27}$$

Simplifying (27) and making use of (24) and (25), we get

$$\begin{aligned}&(c_{44}-\rho c^2)\Big[k^2(c_{13}-\beta_{33}A_1)(c_{13}-\beta_{11}B_1)\\&-\left\{k^2(c_{11}-\beta_{11}A_1-\rho c^2)-\rho\Omega^2\right\}(c_{33}-\beta_{33}B_1)\Big]\\&+\rho c^2k\sqrt{c_{44}(c_{33}-\beta_{33}B_1)}\\&\sqrt{k^2(c_{11}-\rho c^2-\beta_{11}A_1)-\rho\Omega^2}\ \sqrt{c_{44}-\rho c^2}=0\end{aligned} \tag{28}$$

which may be written as:

$$\frac{\rho c^2}{c_{11}-\beta_{11}A_1-\frac{\rho\Omega^2}{k^2}}\sqrt{\frac{c_{33}-\beta_{33}B_1}{c_{44}}}\sqrt{\frac{\dfrac{c_{44}}{c_{11}-\beta_{11}A_1-\frac{\rho\Omega^2}{k^2}}-\dfrac{\rho c^2}{c_{11}-\beta_{11}A-\frac{\rho\Omega^2}{k^2}}_1}{1-\dfrac{\rho c^2}{c_{11}-\beta_{11}A_1-\frac{\rho\Omega^2}{k^2}}}}$$

$$\left[1-\frac{(c_{13}-\beta_{33}A_1)(c_{13}-\beta_{11}B_1)}{(c_{33}-\beta_{33}B_1)(c_{11}-\beta_{11}A_1)-\frac{\rho\Omega^2}{k^2}}-\frac{\rho c^2}{c_{11}-\beta_{11}A_1-\frac{\rho\Omega^2}{k^2}}\right]=0$$

Let u = $\frac{\rho c^2}{c_{11}-\beta_{11}A_1-\frac{\rho\Omega^2}{k^2}}$, a = $\frac{c_{44}}{c_{33}-\beta_{33}B_1}$, b = $\frac{c_{44}}{c_{11}-\beta_{11}A_1-\frac{\rho\Omega^2}{k^2}}$, p = $\frac{(c_{13}-\beta_{33}A_1)(c_{13}-\beta_{11}B_1)}{(c_{33}-\beta_{33}B_1)(c_{11}-\beta_{11}A_1-\frac{\rho\Omega^2}{k^2})}$

Therefore, the above equation becomes

$$\begin{aligned}&\text{u-}\sqrt{\frac{1}{a}}\sqrt{\frac{b-u}{1-u}}\,(1-p-u)=o\\&\Rightarrow(1-a)u^3+\left[a-2(1-p)-b\right]u^2\\&+\left[(1-p)^2+2b(1-p)\right]u-b(1-p)^2=0\end{aligned} \tag{29}$$

The last equation can be solved by Cardan's rule for u and hence the velocity c can be determined.

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