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UNSTEADY FALKNER-SKAN FLOW OF A SECOND GRADE FLUID

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Abstract: The non-similar solutions for the unsteady flow of a second grade fluid wereconstructed. The method of perturbation was used for the solution of the governing non-linear equation. The results of Rajagopal *et al.* (Int. J. Non-Linear Mechanics, 1983) can be obtained as a special case of the presented analysis.

***Keywords**:* Unsteady flow, second grade fluid, perturbation method, numerical solution

Introduction

Boundary Layers are thin regions in a flow which are close to the solid boundary and where viscous forces are important. The concept of the boundary layer was first introduced by Schlichting [1]. He presented the boundary layer concept in a paper, Fluid Motion with Very Small Friction (in German), before the Mathematical Congress in Heidelberg.

Prior to Prandtl's historic breakthrough, the science of fluid mechanics had been developing in two rather different directions. Theoretical hydrodynamics evolved from Euler's equations of motion for a non-viscous fluid [2]. Since the results of hydrodynamics contradicted many experimental observations, the practicing engineers developed their own empirical art of hydraulics. This was based on experimental data and differed significantly from the purely mathematical approach of theoretical hydrodynamics. Although the complete equations describing the motion of a viscous fluid [3,4] were known prior to Prandtl, the mathematical difficulties in solving these equations prohibited a theoretical treatment of viscous flows. Prandtl showed that many viscous flows can be analyzed by dividing the flow into two regions, one close to solid boundary, the other covering the rest of the flow. Only in the thin region adjacent to a solid boundary (the boundary layer) is the effect of viscosity important. In the region outside of the boundary layer, the effect of viscosity is negligible and the fluid may be treated as inviscid. Thus, the boundary-layer concept provided the link that had been missing between theory and practice. Furthermore, the boundary-layer concept permitted the solution of viscous flow problems that would have been impossible through application of the Navier-Stokes equations to the complete flow field. So the introduction of the boundary-layer concept marked the beginning of the modern era of fluid mechanics. The flow of non-Newtonian fluid has recently gained considerable importance because of its applications in various branches of Science, Engineering and Technology. With this in view, the purpose of this paper was to extend the work of Rajagopal *et al*. [5] from steady to unsteady case. For this, the governing equations of second grade fluid in unsteady state were considered. Using similarity transformations, the boundary layer partial differential equation is reduced into ordinary differential equation. For the solution of involved non-linear ordinary differential equation, the perturbation technique was used. The zeroeth and first order problems were solved using Runge-Kutta method.

Formulation of the problem

The equations of motion in second grade fluid for an unsteady flow are given as (6)

$$\rho\left[\frac{\partial u}{\partial t}-vw\right]=-\frac{\partial h}{\partial x}+\mu\nabla^2 u+\alpha_1\frac{\partial}{\partial t}\left(\nabla^2 u\right)-\alpha_1 v\nabla^2 w, \quad (1)$$

$$\rho\left[\frac{\partial v}{\partial t}+uw\right]=-\frac{\partial h}{\partial y}+\mu\nabla^2 v+\alpha_1\frac{\partial}{\partial t}\left(\nabla^2 v\right)+\alpha_1 u\nabla^2 w, \quad (2)$$

where the generalized energy

$$h=\frac{\rho}{2}\left(u^2+v^2\right)+p-\frac{1}{4}\left(3\alpha_1+2\alpha_2\right)tr\mathbf{A}_1^2-\alpha_1\left(u\nabla^2 u+v\nabla^2 v\right). \quad (3)$$

Here ρ is the density, u,v,w are the velocity components in $x-$, $y-$ and $z-$ directions, μ is the dynamic viscosity, ∇^2 is the Laplacian, α_1, α_2 are the material parameters of the fluid, ρ is the pressure and t is the time.

The boundary conditions for the flow past a wedge placed symmetrically with respect to the flow direction are

$$u=v=0 \text{ at } y=0,\; u\to U \text{ as } y\to\infty. \quad (4)$$

Let us now introduce the following non-dimensional parameters

$$\bar{x}=\frac{x}{L},\; \bar{y}=\sqrt{R}\frac{y}{L},\; \bar{t}=\frac{tU_0}{L},\; \bar{u}=\frac{u}{U_0},\; \bar{v}=\sqrt{R}\frac{v}{U_0},\; \bar{p}=\frac{p-p_0}{\rho U_0^2}. \quad (5)$$

The non-dimensional form of the Bernoulli's equation can be written as

$$\frac{1}{\rho}\left[\rho U_0^2\bar{p}+p_0\right]+\frac{\left(U_0\bar{U}\right)^2}{2}=c, \quad \text{where } \bar{U}=\frac{U}{U_0}. \quad (6)$$

Differentiating equation (6) with respect to $\bar{x}$ we get

$$-\frac{\partial\bar{p}}{\partial\bar{x}}=\bar{U}\frac{\partial\bar{U}}{\partial\bar{x}}+\frac{\partial\bar{U}}{\partial\bar{t}}. \quad (7)$$

In view of equations (5) – (7), equations (1) and (2) reduce to the following form

$$\frac{\partial\bar{u}}{\partial\bar{t}}+\bar{u}\frac{\partial\bar{u}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{u}}{\partial\bar{y}}=\left[\begin{array}{c}\bar{U}\frac{\partial\bar{U}}{\partial\bar{x}}+\frac{\partial\bar{U}}{\partial\bar{t}}+\frac{\partial^2\bar{u}}{\partial\bar{y}^2}\\ +\varepsilon\left[\frac{\partial}{\partial\bar{x}}\left(\bar{u}\frac{\partial^2\bar{u}}{\partial\bar{y}^2}\right)+\frac{\partial\bar{u}}{\partial\bar{y}}\frac{\partial^2\bar{v}}{\partial\bar{y}^2}+\frac{\partial^3\bar{u}}{\partial\bar{t}\partial\bar{y}^2}+\bar{v}\frac{\partial^3\bar{u}}{\partial\bar{y}^3}\right]\end{array}\right], \quad (8)$$

$$0=\bar{p}_{\bar{y}} \quad \text{i.e.,} \quad .\bar{p}=\bar{p}(\bar{x},\bar{t}), \quad (9)$$

where

$$\varepsilon=\frac{\alpha_1 R}{\rho L^2}. \quad (10)$$

The associated boundary conditions are

$$\begin{aligned}&\bar{u}=\bar{v}=0 \quad \text{at} \quad \bar{y}=0,\\ &\bar{u}\to\bar{U} \quad \text{as} \quad \bar{y}\to\infty.\end{aligned} \quad (11)$$

The velocity components in stream function $\bar{\psi}$ can be written as

$$\bar{u}=\frac{\partial\bar{\psi}}{\partial\bar{y}},\quad \bar{v}=\frac{\partial\bar{\psi}}{\partial\bar{x}} \quad (12)$$

In view of equation (12), equation (8) and boundary conditions (11) become

$$\frac{\partial^2\bar{\psi}}{\partial\bar{t}\partial\bar{y}}+\frac{\partial\bar{\psi}}{\partial\bar{y}}\frac{\partial^2\bar{\psi}}{\partial\bar{x}\partial\bar{y}}-\frac{\partial\bar{\psi}}{\partial\bar{x}}\frac{\partial^2\bar{\psi}}{\partial\bar{y}^2}=\bar{U}\frac{\partial\bar{U}}{\partial\bar{x}}+\frac{\partial\bar{U}}{\partial\bar{t}}+\frac{\partial^3\bar{\psi}}{\partial\bar{y}^3}$$

$$+\varepsilon\left[\begin{array}{c}\frac{\partial}{\partial\bar{x}}\left(\frac{\partial\bar{\psi}}{\partial\bar{y}}\frac{\partial^3\bar{\psi}}{\partial\bar{y}^3}\right)-\frac{\partial^2\bar{\psi}}{\partial\bar{y}^2}\frac{\partial^3\bar{\psi}}{\partial\bar{x}\partial\bar{y}^2}\\ +\frac{\partial^4\bar{\psi}}{\partial\bar{t}\partial\bar{y}^3}-\frac{\partial\bar{\psi}}{\partial\bar{x}}\frac{\partial^4\bar{\psi}}{\partial\bar{y}^4}\end{array}\right], \quad (13)$$

$$\frac{\partial\bar{\psi}}{\partial\bar{x}} = \frac{\partial\bar{\psi}}{\partial\bar{y}} = 0 \ \text{ at } \ \bar{y} = 0, \frac{\partial\bar{\psi}}{\partial\bar{y}} \to \bar{U} \ \text{ as } \ \bar{y} \to \infty. \tag{14}$$

We now study the boundary layer flow of an incompressible second grade fluid past a wedge placed symmetrically with respect to flow direction. Included as special cases are the flow past a flat plate and the flow near a stagnation point. We observe that the resulting equation of motion (13) is highly nonlinear as compared to the case of viscous fluid. As a result, it seems to be impossible to obtain the general solution in closed form for arbitrary values of all the parameters arising in this nonlinear equation. Even in the case of viscous fluid, numerical solutions are given. Therefore, we seek the solution of the problem as a power series expansion in small parameter ε. According to this $\bar{\psi}$ can be expressed as Beard and Walters [7],

$$\bar{\psi} = \bar{\psi}_0(\bar{x},\bar{y}) + \varepsilon\bar{\psi}_1(\bar{x},\bar{y}) + \ldots + \varepsilon^n\bar{\psi}_n(\bar{x},\bar{y}) + \ldots \tag{15}$$

Substituting equation (15) in equations (13) and (14) and equating powers of ε, we obtain the following equations:

Zeroeth order system

$$\frac{\partial^2\bar{\psi}_0}{\partial\bar{t}\partial\bar{y}} + \frac{\partial\bar{\psi}_0}{\partial\bar{y}}\frac{\partial^2\bar{\psi}_0}{\partial\bar{x}\partial\bar{y}} - \frac{\partial\bar{\psi}_0}{\partial\bar{x}}\frac{\partial^2\bar{\psi}_0}{\partial\bar{y}^2} = \bar{U}\frac{\partial\bar{U}}{\partial\bar{x}} + \frac{\partial\bar{U}}{\partial\bar{t}} + \frac{\partial^3\bar{\psi}_0}{\partial\bar{y}^3}, \tag{16}$$

$$\frac{\partial\bar{\psi}_0}{\partial\bar{x}} = \frac{\partial\bar{\psi}_0}{\partial\bar{y}} = 0 \ \text{ at } \ \bar{y} = 0, \frac{\partial\bar{\psi}_0}{\partial\bar{y}} \to \bar{U} \ \text{ as } \ \bar{y} \to \infty. \tag{17}$$

First order system

$$\frac{\partial^2\bar{\psi}_1}{\partial\bar{t}\partial\bar{y}} + \frac{\partial\bar{\psi}_0}{\partial\bar{y}}\frac{\partial^2\bar{\psi}_1}{\partial\bar{x}\partial\bar{y}} + \frac{\partial^2\bar{\psi}_0}{\partial\bar{x}\partial\bar{y}}\frac{\partial\bar{\psi}_1}{\partial\bar{y}} - \frac{\partial\bar{\psi}_0}{\partial\bar{x}}\frac{\partial^2\bar{\psi}_1}{\partial\bar{y}^2} - \frac{\partial^2\bar{\psi}_0}{\partial\bar{y}^2}\frac{\partial\bar{\psi}_1}{\partial\bar{x}}$$
$$= \frac{\partial^3\bar{\psi}_1}{\partial\bar{y}^3} + \left[\frac{\partial}{\partial\bar{x}}\left(\frac{\partial\bar{\psi}_0}{\partial\bar{y}}\frac{\partial^3\bar{\psi}_0}{\partial\bar{y}^3}\right) - \frac{\partial^2\bar{\psi}_0}{\partial\bar{y}^2}\frac{\partial^3\bar{\psi}}{\partial\bar{x}\partial\bar{y}^2} + \frac{\partial^4\bar{\psi}_0}{\partial\bar{t}\partial\bar{y}^3} - \frac{\partial\bar{\psi}_0}{\partial\bar{x}}\frac{\partial^4\bar{\psi}_0}{\partial\bar{y}^4}\right], \tag{18}$$

$$\frac{\partial\bar{\psi}_1}{\partial\bar{x}} = \frac{\partial\bar{\psi}_1}{\partial\bar{y}} = 0 \ \text{ at } \ \bar{y} = 0, \frac{\partial\bar{\psi}_1}{\partial\bar{y}} \to \bar{U} \ \text{ as } \ \bar{y} \to \infty. \tag{19}$$

Solution of the Problem

Introducing the similarity transformations

$$\bar{U} = \frac{\bar{x}}{\bar{t}}, \quad \eta = \frac{\bar{y}}{\sqrt{\bar{t}}},$$
$$\bar{\psi}_i = \frac{\bar{x}}{\sqrt{\bar{t}}} f_i(\eta) \ \ i = 0, 1, \tag{20}$$

where *f* is non-dimensionalized stream function. We get the following systems of zeroeth order and first order respectively:

Zeroeth Order System

$$f_0''' + (f_0 + \frac{\eta}{2})f_0'' + f_0'(1 - f_0') = 0, \tag{21}$$

$$f_0(0) = f_0'(0) = 0, \ \ f_0'(\eta) \to 1 \ \text{ as } \ \eta \to \infty. \tag{22}$$

First Order System

$$f_1''' = \frac{1}{\bar{t}}\left[(2f_0' - f_0'')f_0'' - 2f_0''' - (\frac{\eta}{2} + f_0)f_0''''\right] - (\frac{\eta}{2} + f_0)f_1'' - f_0''f_1 + (2f_0' - 1)f_1', \tag{23}$$

$$f_1(0) = f_1'(0) = 0, f_1'(\eta) \to 0 \ \text{ as } \ \eta \to \infty. \tag{24}$$

Zeroeth Order Solution

We write equation (21) in the set of first-order system by taking (8)

$$f_0 = F, \quad f_0' = P, \quad f_0'' = Q. \tag{25}$$

With the help of equation (25), the equations (21) and (22) take the form

Table 1. Values of f_0 (η) and f_0' (η).

η	$f_0(\eta)$	η	$f_0'(\eta)$	η	$f_0(\eta)$	η	$f_0'(\eta)$
0.0	0.00000	2.8	2.105410	0.0	0.00000	2.8	0.998912
0.1	0.004560	2.9	2.205321	0.1	0.091125	2.9	0.999295
0.2	0.18189	3.0	2.305265	0.2	0.181162	3.0	0.999550
0.3	0.040716	3.1	2.405229	0.3	0.268892	3.1	0.999717
0.4	0.071855	3.2	2.505206	0.4	0.353232	3.2	0.999825
0.5	0.111219	3.3	2.605193	0.5	0.433251	3.3	0.999893
0.6	0.158336	3.4	2.705184	0.6	0.508193	3.4	0.999936
0.7	0.212670	3.5	2.805179	0.7	0.577481	3.5	0.999962
0.8	0.273633	3.6	2.905176	0.8	0.640725	3.6	0.999978
0.9	0.340609	3.7	3.005174	0.9	0.697713	3.7	0.999987
1.0	0.412968	3.8	3.105173	1.0	0.748404	3.8	0.999992
1.1	0.490086	3.9	3.205173	1.1	0.792908	3.9	0.999995
1.2	0.571356	4.0	3.305172	1.2	0.831470	4.0	0.999997
1.3	0.656198	4.1	3.405172	1.3	0.864440	4.1	0.999998
1.4	0.744076	4.2	3.505172	1.4	0.892252	4.2	0.999999
1.5	0.834497	4.3	3.605172	1.5	0.915394	4.3	0.999999
1.6	0.927020	4.4	3.705172	1.6	0.934386	4.4	0.999999
1.7	1.021257	4.5	3.805172	1.7	0.949755	4.5	0.999999
1.8	1.116871	4.6	3.905172	1.8	0.962017	4.6	0.999999
1.9	1.213576	4.7	4.005172	1.9	0.971661	4.7	0.999999
2.0	1.311134	4.8	4.105172	2.0	0.979137	4.8	0.999999
2.1	1.409347	4.9	4.205172	2.1	0.984847	4.9	0.999999
2.2	1.508058	5.0	4.305171	2.2	0.989145	5.0	0.999999
2.3	1.607141	5.1	4.405171	2.3	0.992331	5.1	0.999999
2.4	1.706497	5.2	4.505171	2.4	0.994659	5.2	0.999999

$$Q' = -(F + \frac{\eta}{2})Q - P(1-P), \tag{26}$$

$$F(0) = P(0) = 0, \quad P(\eta) \to 1 \quad \text{as} \quad \eta \to \infty. \tag{27}$$

We solve equation (26) by using fourth-order Runge-Kutta method. The values of f_0, f'_0, for different values of η are given in Table 1 and shown in Figure 1.

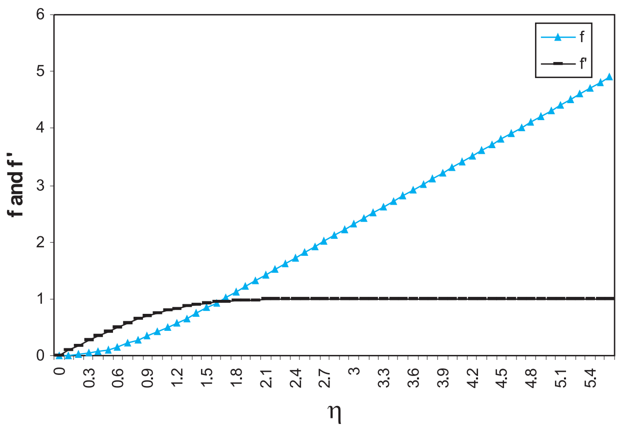

Figure 1. Plot of zeroeth order solution.

We see in Fig. 1 that the value of $f_0(\eta)$ increases with the increase in the value of η and the value of the velocity component $f'_0(\eta)$ starts from zero and attains the value of 1 which is the free stream velocity.

First Order Solution

For the first order system we take

$$f_1' = P_1, \quad f_1'' = Q_1, \quad f_0' = P,$$
$$f_0'' = Q, \quad f_0''' = R, \quad f_0'''' = S. \tag{28}$$

With the help of equation (28), the equations (23) and (24) become

$$Q_1' = -\frac{1}{\bar{t}}\left[(2P-Q)Q - 2R - (\frac{\eta}{2}+F)S\right] - (\frac{\eta}{2}+F)Q_1 - QF_1 + (2P-1)P_1, \tag{29}$$

$$F_1(0) = 0, \quad Q_1(0) = 0, \quad P_1 \to 0 \quad \text{as} \quad \eta \to \infty. \tag{30}$$

We take $\bar{t} = 1$ and obtain the following values of f_1 and f'_1 (Table 2) for different values of η by using fourth-order Runge-Kutta method.

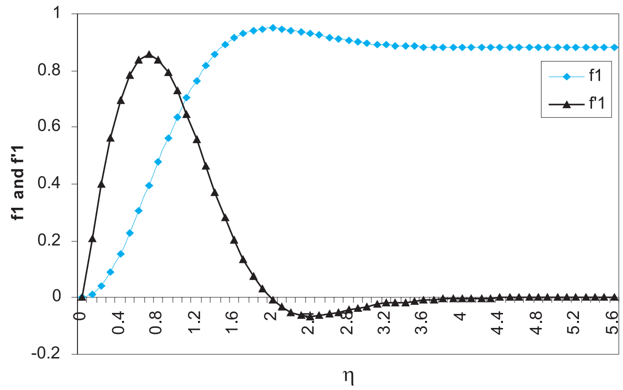

Figure 2. Plot of first order solution.

Similarly we can find other values by giving constant value to $\bar{t}$. If $\bar{t}$=0.5, then we get complete set of values given in Table 3. The plot for $\bar{t}$=0.5 is given in Fig. 2. It is clear from Fig. 2 that the value of f_1 increases from 0 to 0.948 and then decreases to attain the constant value 0.88, and the value of f'_1 increases from 0 to 0.85 and then decreases upto -0.065. Then it again increases to attain the constant value 0.

Table 2. Values of f_1 and f'_1 for $\bar{t}$ = 1.

η	f_1	f_1'
0	0	0
1.0	.318451	.364675
1.5	.44571	.141241
2.0	.47442	− .00363
3.0	.448599	− .018749
4.0	.441041	− .001431
5.0	.440648	− .000029
5.6	.440643	0

In conclusion, non-similar solutions are obtained in this study for the boundary layer flow of a homogeneous incompressible fluid of second grade past a wedge placed symmetrically with respect to the flow direction in unsteady case.

Table 3. Values of f_1 (η) and f'_1 (η) for t = 0.5.

η	$f_0(\eta)$	η	$f'_0(\eta)$	η	$f_0(\eta)$	η	$f'_0(\eta)$
0.0	0.000000	2.8	0.906044	0.0	0.000000	2.8	- 0.050988
0.1	0.010503	2.9	0.901282	0.1	0.208275	2.9	- 0.044226
0.2	0.041092	3.0	0.897198	0.2	0.399687	3.0	- 0.037497
0.3	0.089529	3.1	0.893770	0.3	0.563785	3.1	- 0.031150
0.4	0.152707	3.2	0.890948	0.4	0.693633	3.2	- 0.025399
0.5	0.226992	3.3	0.888667	0.5	0.785600	3.3	- 0.020356
0.6	0.308541	3.4	0.886853	0.6	0.839008	3.4	- 0.016052
0.7	0.393569	3.5	0.885433	0.7	0.855672	3.5	- 0.0.012466
0.8	0.478576	3.6	0.884338	0.8	0.839364	3.6	- 0.009542
0.9	0.560515	3.7	0.883505	0.9	0.795255	3.7	- 0.007202
1.0	0.636901	3.8	0.882881	1.0	0.729351	3.8	- 0.005364
1.1	0.705871	3.9	0.882419	1.1	0.647977	3.9	- 0.003943
1.2	0.766188	4.0	0.882081	1.2	0.557323	4.0	- 0.002862
1.3	0.817216	4.1	0.881838	1.3	0.463075	4.1	- 0.002052
1.4	0.858848	4.2	0.881664	1.4	0.370140	4.2	- 0.001453
1.5	0.891420	4.3	0.881542	1.5	0.282482	4.3	- 0.001017
1.6	0.915618	4.4	0.881457	1.6	0.203040	4.4	- 0.000703
1.7	0.932366	4.5	0.881398	1.7	0.133742	4.5	- 0.000480
1.8	0.942736	4.6	0.881359	1.8	0.075578	4.6	- 0.000324
1.9	0.947858	4.7	0.881332	1.9	0.028733	4.7	- 0.000216
2.0	0.948843	4.8	0.881314	2.0	- 0.007260	4.8	- 0.000142
2.1	0.946735	4.9	0.881303	2.1	- 0.033346	4.9	- 0.000092
2.2	0.942461	5.0	0.881296	2.2	- 0.050783	5.0	- 0.000058
2.3	0.936817	5.1	0.881291	2.3	- 0.060992	5.1	- 0.000036
2.4	0.930454	5.2	0.881288	2.4	- 0.065427	5.2	- 0.000021

We have first derived the boundary layer equations in unsteady form and then these equations have been transformed into ordinary differential equations by using similarity transformations. Finally, the solution of these ordinary differential equations is presented numerically.

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