Proc. Pakistan Acad. Sci. 43(2):95-98.2006

FERMIONS IN NUT-KERR-NEWMAN SPACE-TIME

Elias Uddin Biswas and Chandrani Nag

Department of Mathematics, Shah Jalal University of Science and Technology
Sylhet, Bangladesh

Received December 2005, accepted April 2006

Communicated by Prof. Dr. M. Iqbal Choudhary

Abstract: The aim of this paper is to build up the U(1)-gauge theory for fermions in the curved space-time such as NUT-Kerr-Newman space time. The NUT-Kerr-Newman space time which is not a black hole space-time but its common feature with the black hole space-time is that it has horizon.

***Keywords*:** Gauge theory, black hole space-time, horizon, NUT parameter, null complex tetrads.

Introduction

Carmeli [1}, Carmeli and Carmeli [2]and Carmeli and Malin [3] derived the Klein-Gordon, Weyl and Dirac-type equations on $R \times S^3$ space-time by simply going from the momentum to the angular momentum representation. Sen [4] obtained the most general Lagrangians for the Dirac, Weyl, and Majorana fermions. Sen's work offers an excellent description of fermions in the space-time $R \times S^3$. M. Dariescu *et al.* [5] developed a U(1)-gauge theory for massive fermionic fields minimally coupled to a curved space-time such as Kerr-Newman black hole space-time. C. Dariescu et al. [6] developed the tetradic Lorentz-gauge invariant formulation of the SU(2)×U(1) theory in $R \times S^3$ space-time. Biswas [7] also developed a U(1)-gauge theory for massive fermionic fields minimally coupled to a curved space-time such as Kerr-Newman-Kasuya space-time. To develop a U(1)-gauge theory for massive fermionic fields on curved space-time such as Kerr-Newman black hole space-time they used the Dirac-type equation and the U(1)-gauge invariant Lagrangian. In this paper, we study the U(1)-gauge theory for massive fermionic fields in a NUT-Kerr-Newman space-time. The NUT-Kerr-Newman space time is not a black hole space-time but it includes Kerr-Newman black hole space-time as a special case. It also includes NUT space-time as a special case which possesses very interesting properties. Although the NUT-Kerr-Newman space-time is not a black hole space-time; its common feature with the black hole space-time is that it has horizon.

E-mail: ebiswasmat@yahoo.com

The NUT-Kerr-Newman space-time

The NUT-Kerr-Newman space-time is described by the metric

$$\begin{aligned} ds^2 &= \left(r^2 + (n + h\cos\theta)^2\right)\left(\frac{dr^2}{r^2 - 2mr + e^2 + h^2 - n^2} + d\theta^2\right) \\ &- \left(\frac{r^2 + h^2\cos^2\theta - 2mr + e^2 - n^2}{r^2 + (n + h\cos\theta)^2}\right)dt^2 \\ &+ \left\{\frac{\left(r^2 + h^2 + n^2\right)^2\sin^2\theta - \left(h\sin^2\theta - 2n\cos\theta\right)^2\left(r^2 - 2mr + e^2 + h^2 - n^2\right)}{r^2 + (n + h\cos\theta)^2}\right\}d\varphi^2 \\ &+ \frac{2\left\{\left(h\sin^2\theta - 2n\cos\theta\right)\left(r^2 - 2mr + e^2 + h^2 - n^2\right) - h\sin^2\theta\left(r^2 + h^2 + n^2\right)\right\}}{r^2 + (n + h\cos\theta)^2}\,dt\,d\varphi \end{aligned} \qquad (1)$$

where m, h, e and n are the mass, angular momentum per unit mass, electric charge and NUT (magnetic mass) parameters respectively.

This is a solution to the Einstein-Maxwell equations with vector potential

$$A_\mu dx^\mu = \frac{er\left\{dt - (h\mathrm{Sin}^2\theta - 2n\mathrm{Cos}\theta)d\varphi\right\}}{r^2 + (n + h\mathrm{Cos}\theta)^2} \qquad (2)$$

The space-time given by (1) encompasses all the black hole space-times, which are asymptotically flat. Specially, the metric (1) includes:

(i) Kerr-Newman black hole space-time when $n = 0$.
(ii) Kerr black hole space-time for $n = e = 0$.
(iii) Reissner-Nordstrom black hole space-time if $n = h = 0$.
(iv) Schwarzschild black hole space-time when $n = e = h = 0$.
(v) NUT-Kerr space-time if $e = 0$.
(vi) NUT space-time with $e = h = 0$.

This metric can be transformed to Boyer coordinates under the proper coordinate transformation such as

$$\{x^{\mu}\}=\left\{p = n + h\cos\theta\ ;\ \sigma = -\frac{\varphi}{h}\ ;\ q = r\ ;\ \tau = t - \frac{(n^2 + h^2)}{h}\varphi\right\} \quad (3)$$

with the suitable adjustment of the parameter

$$X(p) = h^2 - (n-p)^2\ ;\ Y(q) = q^2 - 2mq + e^2 + h^2 - n^2 \quad (4)$$

Therefore, the metric (1) can be written as

$$ds^2 = \frac{p^2+q^2}{X}dp^2 + \frac{X}{p^2+q^2}(d\tau + q^2 d\sigma)^2 + \frac{p^2+q^2}{Y}dq^2 - \frac{Y}{p^2+q^2}(d\tau - p^2 d\sigma)^2 \quad (5)$$

This metric represents the NUT-Kerr-Newman space-time in Boyer coordinates, which has been studied in detail by Plebanski [8].

After a suitable choice of the null complex tetrads $\{\omega^a\}$ which consists of two complex conjugate null vectors $m, \overline{m}$ and two real null vectors k_1, k_2 ; $\{\omega^a\} = \{m, \overline{m}, k_1, k_2\}$:

$$\omega^1 = m = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{p^2+q^2}{X}}dp + i\sqrt{\frac{X}{p^2+q^2}}\left(d\tau + q^2 d\sigma\right)\right]$$

$$\omega^2 = \overline{m} = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{p^2+q^2}{X}}dp - i\sqrt{\frac{X}{p^2+q^2}}\left(d\tau + q^2 d\sigma\right)\right] \quad (6)$$

$$\omega^3 = k_1 = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{Y}{p^2+q^2}}\left(d\tau - p^2 d\sigma\right) - \sqrt{\frac{p^2+q^2}{Y}}dq\right]$$

$$\omega^4 = k_2 = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{Y}{p^2+q^2}}\left(d\tau - p^2 d\sigma\right) + \sqrt{\frac{p^2+q^2}{Y}}dq\right]$$

the metric (1) becomes in the simple form

$$ds^2 = 2(\omega^1\omega^2 - \omega^3\omega^4) = g_{ab}\omega^a\omega^b \quad (7)$$

with

$$\left(g_{ab}\right) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad (8)$$

Field equations

For the massive fermionic complex fields ψ the U(1)-gauge invariant Lagrangian is given by

$$L = \overline{\psi}\gamma^{\mu}D_{\mu}\psi + M\overline{\psi}\psi + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad (9)$$

where γ^{μ} is the generalized Dirac gamma metrics, the U(1)-gauge field-strength tensor is defined as

$$F^{\mu\nu} = g^{\mu\alpha}\partial_{\alpha}A^{\nu} - g^{\nu\alpha}\partial_{\alpha}A^{\mu} - \left(g^{\mu\alpha}\partial_{\alpha}g^{\nu\beta} - g^{\nu\alpha}\partial_{\alpha}g^{\mu\beta}\right)g_{\beta\sigma}A^{\sigma} \quad (10)$$

and the gauge-covariant derivative is defined as

$$D_{\mu}\psi = \nabla_{\mu}\psi + igA_{\mu}\psi \quad \text{and its h.c.} \quad (11)$$

Here $\nabla_{\mu}\psi$ be the Levi-Civita covariant derivative and g be the gauge coupling constant. Under these assumptions, the Dirac-type equation is obtained in the covariant expression

$$\gamma^{\mu}\left(\partial_{\mu} + igA_{\mu}\right)\psi - \frac{1}{4}\Gamma_{\alpha\beta\mu}\gamma^{\mu}\gamma^{\alpha}\gamma^{\beta}\psi + M\psi = 0 \quad (12)$$

The Dirac-type equation governs the particle, in curved space-time, and the Maxwell equations with sources can be expressed in the standard form

$$\frac{1}{\sqrt{-g}}\partial_{\mu}\left[\sqrt{-g}F^{\nu\mu}\right]=J^{\nu} \tag{13}$$

which will be generalized for the case of a null tetradic base $\{e_a\}, a=1,4$. To build up a U(1)-gauge theory of a massive fermionic complex field in the curved space-time described by the metric (7) we use the U(1)-gauge invariant Lagrangian. The general expression for the covariant derivative (11) becomes,

$$D_a\psi=\nabla_a\psi+igA_a\psi \quad \text{and its h.c.} \tag{14}$$

and the Lagrangian (9) as

$$\mathrm{L}=\bar{\psi}\gamma^{a}D_a\psi+M\bar{\psi}\psi+\frac{1}{4}F_{ab}F^{ab} \tag{15}$$

The electromagnetic tensor F^{ab} can be expressed in the base of coordinate (p,q,σ,τ)

$$F^{ab}=\omega_{\mu}^{a}\omega_{\nu}^{b}F^{\mu\nu} \tag{16}$$

Therefore, the essential components of $F^{\mu\nu}$ are given below

$$\begin{aligned}F^{12}=&\frac{X}{p^2+q^2}A^{2}{}_{,p}-\frac{p^2+q^2}{q^4X-p^4Y}A^{1}{}_{,\sigma}-\frac{p^2+q^2}{q^2X+p^2Y}A^{1}{}_{,\tau}\\&-\left[\frac{X\left(q^4X-p^4Y\right)}{\left(p^2+q^2\right)^2}\left(\frac{p^2+q^2}{q^4X-p^4Y}\right)_{,p}+\frac{X\left(q^2X+p^2Y\right)}{\left(p^2+q^2\right)^2}\left(\frac{p^2+q^2}{q^2X+p^2Y}\right)_{,p}\right]A^2\\&-\left[\frac{X(X-Y)}{\left(p^2+q^2\right)^2}\left(\frac{p^2+q^2}{q^2X+p^2Y}\right)_{,p}+\frac{X\left(q^2X+p^2Y\right)}{\left(p^2+q^2\right)^2}\left(\frac{p^2+q^2}{q^4X-p^4Y}\right)_{,p}\right]A^4\end{aligned} \tag{17.1}$$

$$F^{13}=\frac{X}{p^2+q^2}A^{3}{}_{,p}-\frac{Y}{p^2+q^2}A_{,q}{}^{1}-\frac{X}{Y}\left(\frac{Y}{p^2+q^2}\right)_{,p}A^3+\frac{Y}{X}\left(\frac{X}{p^2+q^2}\right)_{,q}A^1 \tag{17.2}$$

$$\begin{aligned}F^{14}=&\frac{X}{p^2+q^2}A^{4}{}_{,p}-\frac{p^2+q^2}{q^2X+p^2Y}A^{1}{}_{,\sigma}-\frac{p^2+q^2}{X-Y}A^{1}{}_{,\tau}\\&-\left[\frac{X\left(q^4X-p^4Y\right)}{\left(p^2+q^2\right)^2}\left(\frac{p^2+q^2}{q^2X-p^2Y}\right)_{,p}+\frac{X\left(q^2X+p^2Y\right)}{\left(p^2+q^2\right)^2}\left(\frac{p^2+q^2}{X-Y}\right)_{,p}\right]A^2\\&-\left[\frac{X(X-Y)}{\left(p^2+q^2\right)^2}\left(\frac{p^2+q^2}{X-Y}\right)_{,p}+\frac{X\left(q^2X+p^2Y\right)}{\left(p^2+q^2\right)^2}\left(\frac{p^2+q^2}{q^2X+p^2Y}\right)_{,p}\right]A^4\end{aligned} \tag{17.3}$$

$$\begin{aligned}F^{23}=&-\frac{Y}{p^2+q^2}A^{2}{}_{,q}+\frac{p^2+q^2}{q^4X-p^4Y}A^{3}{}_{,\sigma}+\frac{p^2+q^2}{q^2X+p^2Y}A^{3}{}_{,\tau}\\&+\left[\frac{Y\left(q^4X-p^4Y\right)}{\left(p^2+q^2\right)^2}\left(\frac{p^2+q^2}{q^4X-p^4Y}\right)_{,q}+\frac{Y\left(q^2X+p^2Y\right)}{\left(p^2+q^2\right)^2}\left(\frac{p^2+q^2}{q^2X+p^2Y}\right)_{,q}\right]A^2\\&+\left[\frac{Y(X-Y)}{\left(p^2+q^2\right)^2}\left(\frac{p^2+q^2}{q^2X+p^2Y}\right)_{,q}+\frac{Y\left(q^2X+p^2Y\right)}{\left(p^2+q^2\right)^2}\left(\frac{p^2+q^2}{q^4X-p^4Y}\right)_{,q}\right]A^4\end{aligned} \tag{17.4}$$

$$F^{24}=\frac{p^2+q^2}{q^4X-p^4Y}A^{4}{}_{,\sigma}+\frac{p^2+q^2}{q^2X+p^2Y}A^{4}{}_{,\tau}-\frac{p^2+q^2}{q^2X+p^2Y}A^{2}{}_{,\sigma}-\frac{p^2+q^2}{X-Y}A^{2}{}_{,\tau} \tag{17.5}$$

$$\begin{aligned}F^{34}=&\frac{Y}{p^2+q^2}A^{4}{}_{,q}-\frac{p^2+q^2}{q^2X+p^2Y}A^{3}{}_{,\sigma}-\frac{p^2+q^2}{X-Y}A^{3}{}_{,\tau}\\&-\left[\frac{Y\left(q^4X-p^4Y\right)}{\left(p^2+q^2\right)^2}\left(\frac{p^2+q^2}{q^2X-p^2Y}\right)_{,q}+\frac{Y\left(q^2X+p^2Y\right)}{\left(p^2+q^2\right)^2}\left(\frac{p^2+q^2}{X-Y}\right)_{,q}\right]A^2\\&-\left[\frac{Y(X-Y)}{\left(p^2+q^2\right)^2}\left(\frac{p^2+q^2}{X-Y}\right)_{,q}+\frac{Y\left(q^2X+p^2Y\right)}{\left(p^2+q^2\right)^2}\left(\frac{p^2+q^2}{q^2X+p^2Y}\right)_{,q}\right]A^4\end{aligned} \tag{17.6}$$

The components of $F^{\mu\nu}$ allow us to put the Maxwell equations (13) in the expression

$$e_aF^{ba}=J^b \tag{18}$$

Finally the Dirac-type equation is derived for the spinorial massive complex field ψ coupled to the NUT-Kerr-Newman-Kasuya space-time. Using the U(1) gauge invariant Lagarangian (15) the Dirac-type equation is obtained in the general form

$$\gamma^{a}\left(\partial_a+igA_a\right)\psi-\frac{1}{4}\Gamma_{bca}\gamma^{a}\gamma^{b}\gamma^{c}\psi+M\psi=0 \tag{19}$$

Hence, by working out the above, the Dirac-type equation for the metric (5) can be expressed in the form

$$\begin{aligned}&\gamma^{a}\left(\partial_a+igA_a\right)\psi+M\psi-\frac{1}{4\sqrt{2\left(p^2+q^2\right)^3}}\times\\&\left\{\left[-\frac{\left(p^2+q^2\right)\left(\frac{\partial X}{\partial p}\right)-2pX}{\sqrt{X}}\left(\gamma^1+\gamma^2\right)+\frac{\left(p^2+q^2\right)\left(\frac{\partial Y}{\partial q}\right)-2qY}{\sqrt{Y}}\left(\gamma^3-\gamma^4\right)\right]\right.\\&\left.-8i\left[p\sqrt{Y}\gamma^1\gamma^2\left(\gamma^3+\gamma^4\right)-q\sqrt{X}\left(\gamma^1-\gamma^2\right)\gamma^3\gamma^4\right]\right\}\psi=0\end{aligned} \tag{20}$$

Using (3) and (4) into the equation (20), we obtain the Dirac-type equation for the metric (1) in the following form

$$
\begin{aligned}
&\gamma^a(\partial_a + igA_a)\psi + M\psi \\
&- \left[\frac{\{(r^2 + h^2 + n^2)h\cos\theta + nh^2(1 + \cos^2\theta)\}(\gamma^1 + \gamma^2)}{2h\sin\theta\sqrt{2\{r^2 + (n + h\cos\theta)^2\}^3}} \right]\psi \\
&- \left[\frac{\{(n + h\cos\theta)^2(r - m) + r(rm - e^2 - h^2 + n^2)\}(\gamma^3 - \gamma^4)}{2\sqrt{2\{r^2 + (n + h\cos\theta)^2\}^3}(r^2 - 2mr + e^2 + h^2 - n^2)} \right]\psi \\
&+ \left[\frac{2i\{(n + h\cos\theta)(\sqrt{r^2 - 2mr + e^2 + h^2 - n^2})\}(\gamma^3 + \gamma^4)\gamma^1\gamma^2}{\sqrt{2\{r^2 + (n + h\cos\theta)^2\}^3}} \right]\psi \\
&- \left[\frac{2irh\sin\theta(\gamma^1 - \gamma^2)\gamma^3\gamma^4}{\sqrt{2\{r^2 + (n + h\cos\theta)^2\}^3}} \right]\psi = 0
\end{aligned}
\tag{21}
$$

In conclusion, the result obtained in this paper apply for the NUT space-time when $e = h = 0$ and for the Kerr-Newman black hole space-time when n = 0. Under this observation, we like to claim that this study not only encompasses the known result of Dariescu *et al.* [5] in the context of Kerr-Newman black hole, but also provides the similar result for the NUT space-time. So it is interesting to note that we get the U(1)-gauge theory of fermions not only in the Kerr-Newman black hole space-time[5], but also in other space-times such as Kerr-Newman-Kasuya space-time[7] and NUT-Kerr-Newman space-time which are not black hole space-times but they have the common feature with the black hole space-time that they have horizons.

References

1. **Carmeli, M.** 1985. Field theory on $R \times S^3$ Topology. I: The Klein-Gordon and Schrödinger Equation. *Found. Phys*. 15:175-184.
2. **Carmeli, M. and Carmeli, S**. 1985. Field Theory on $R \times S^3$ Topology. II: The Weyl Equation[1]. *Found. Phys*. 15:185-191.
3. **Carmeli, M. and Malin, S**. 1985. Field Theory on Topology. III: The Dirac Equation. *Found. Phys*. 15:1019-1029.
4. **Sen, D**. 1986. Fermions in the space-time . *J. Math. Phys*. 27:472-482.
5. **Dariescu, M., Dariescu C. and Gottlieb I.** 1995. Fermions in a Kerr-Newman space-time, *Found. Phys.* 25:1523-1528.
6. **Dariescu C. and Dariescu, M**. 1994. Gauge Theory of Bosonic and Fermionic Fields in space-time. *Found. Phys*. 24:1577-1582.
7. **Biswas, E.U**. 2005. Fermions in Kerr-Newman-Kasuya space-time. *Proc. Pakistan Acad. Sci*. 42:239-242.
8. **Plebanski, J.F**. 1975. A class of solutions of Einstein-Maxwell equations. *Ann. Phys*. 90:196-255.