CERTAIN SUBCLASSES OF P-VALENT STARLIKE FUNCTIONS

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Abstract: The object of the present paper is to introduce two interesting subclasses \( T^* (p, \alpha, \beta, \gamma) \) and \( C(p, \alpha, \beta, \gamma) \) of \( p \)-valent starlike functions in the open unit disc \( U = \{ z : |z| < 1 \} \), and prove various coefficient inequalities and distortion theorems for functions belonging to these subclasses. The radii of convexity for functions belonging to the classes \( T^* (p, \alpha, \beta, \gamma) \) and \( C(p, \alpha, \beta, \gamma) \) are also determined.

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Introduction

Let \( A(p) \) denote class of functions of the form:

\[
f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathbb{N} = \{1, 2, \ldots\})
\]

which are analytic and \( p \)-valent in the unit disc \( U = \{ z : |z| < 1 \} \). A function \( f(z) \in A(p) \) is called \( p \)-valent starlike of order \( \alpha \) if \( f(z) \) satisfies the conditions

\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha
\]

and

\[
\frac{2\pi}{0} \int \Re \left\{ \frac{zf'(z)}{f(z)} \right\} \, d\theta = 2\pi
\]

for \( 0 \leq \alpha < p, p \in \mathbb{N} \), and \( z \in U \). We denote by \( K(p, \alpha) \) the class of all \( p \)-valent convex functions of order \( \alpha \). We note that:

\[
f(z) \in K(p, \alpha) \quad \text{if and only if} \quad \frac{zf'(z)}{p} \in S(p, \alpha), 0 \leq \alpha < p.
\]

The class \( S(p, \alpha) \) was introduced by Patil and Thakare [3] and the class \( K(p, \alpha) \) was introduced by Owa [2].

Let \( T(p) \) denote the subclass of \( A(p) \) consisting of functions of the form:

\[
f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (a_{p+n} \geq 0, p \in \mathbb{N}).
\]

We denote by \( T^* (p, \alpha) \) and \( C(p, \alpha) \) the classes obtained by taking intersections, respectively, of the classes \( S(p, \alpha) \) and \( K(p, \alpha) \) with \( T(p) \); that is,

\[
T^* (p, \alpha) = S(p, \alpha) \cap T(p)
\]

and

\[
C(p, \alpha) = K(p, \alpha) \cap T(p).
\]
The classes $T^*(p, \alpha)$ and $C(p, \alpha)$ were introduced by Owa [2]. In particular, the classes $S^*(1, \alpha) = S^*(\alpha)$ and $(C1, \alpha) = C(\alpha)$ when $p = 1$ were studied by Silverman [4].

Let the function $g(z)$ be defined by
\[
g(z) = z^p - \sum_{n=1}^{\infty} b_{p+n} z^{p+n} \quad (b_{p+n} \geq 0, p \in \mathbb{N}). \tag{1.8}
\]

Then a function $f(z) \in T(p)$ is said to be in the class $T^*(p, \alpha, \beta, \gamma)$ if
\[
\left| \frac{z f'(z)}{g(z)} - p \right| < \gamma \quad (z \in U) \tag{1.9}
\]
for $g(z) \in T^*(p, \alpha)(0 \leq \alpha < p)$, where $0 \leq \beta < p$ and $0 < \gamma \leq 1$. If a function $f(z)$ belonging to the class $T(p)$ satisfies the condition (1.9) for $g(z) \in C(p, \alpha)(0 \leq \alpha < p)$, $0 \leq \beta < p$ and $0 < \gamma \leq 1$, we say that the function $f(z)$ is in the class $C(p, \alpha, \beta, \gamma)$.

**Coefficient Inequalities**

We begin by recalling the following lemmas from Owa [2].

**Lemma 1**

Let the function $g(z)$ defined by (1.8). Then $g(z)$ is in the class $T^*(p, \alpha)$ if and only if
\[
\sum_{n=1}^{\infty} (p + n - \alpha) b_{p+n} \leq (p - \alpha). \tag{2.1}
\]

**Lemma 2**

Let the function $g(z)$ defined by (1.8). Then $g(z)$ is in the class $C(p, \alpha)$ if and only if
\[
\sum_{n=1}^{\infty} (p + n)(p + n - \alpha) b_{p+n} \leq p(p - \alpha). \tag{2.2}
\]

Applying the above lemmas, we now prove our first result on the coefficient inequalities for functions belonging to the class $T^*(p, \alpha, \beta, \gamma)$, given by

**Theorem 1**

Let the function $f(z)$ defined by (1.7) be in the class $T^*(p, \alpha, \beta, \gamma)$. Then
\[
\left| \frac{z f'(z) - pg(z)}{z f'(z) + (p - 2\beta)g(z)} \right| < \gamma \quad (z \in U). \tag{2.4}
\]

**Proof**

Since $f(z) \in T^*(p, \alpha, \beta, \gamma)$, there exists a function $g(z)$ belonging to the class $T^*(p, \alpha)$ such that
\[
\sum_{n=1}^{\infty} \left\{ (1 + \gamma)(p + n) a_{p+n} - \frac{(p - \alpha)(p(1 - \gamma) + 2\beta)}{(p + n - \alpha)} \right\} \leq 2\gamma(p - \beta). \tag{2.3}
\]

If follows from (2.4) that
\[
\Re \left\{ \frac{\sum_{n=1}^{\infty} (p + n)(p + n - \alpha) b_{p+n} - \sum_{n=1}^{\infty} b_{p+n} z^n}{2(p - \beta) + \sum_{n=1}^{\infty} (p + n)a_{p+n} + (p - 2\beta)b_{p+n} z^n} \right\} < \gamma(z \in U). \tag{2.5}
\]

Choose values of $z$ on the real axis so that $\frac{z f'(z)}{g(z)}$ is real. Thus, upon clearing the denominator in (2.5) and letting $z \to 1^-$ through real values, we have
\[
\sum_{n=1}^{\infty} \left\{ (p + n)(p + n - \alpha) b_{p+n} - p b_{p+n} \right\} \leq \gamma \left( 2(p - \beta) - \sum_{n=1}^{\infty} (p + n) a_{p+n} + (p - 2\beta) b_{p+n} \right) \tag{2.6}
\]
or, equivalently,
\[
\sum_{n=1}^{\infty} \left\{ (1 + \gamma)(p + n) a_{p+n} - [p(1 - \gamma) + 2\beta] b_{p+n} \right\} \leq 2\gamma(p - \beta). \tag{2.7}
\]

Note that, by using Lemma 1, $g(z) \in T^*(p, \alpha)$ implies
Making use of (2.8) in (2.7), we complete the proof of Theorem 1.

**Corollary 1**

Let the function \( f(z) \) defined by (1.7) be in the class \( T^*(p, \alpha, \beta, \gamma) \). Then

\[
a_{p+n} \leq \frac{2\gamma(p+n)(p+n-\alpha)(p-\beta)+(p-\alpha)(p(1-\gamma)+2\beta\gamma)}{(p+n)(p+n-\alpha)(1+\gamma)} \quad (n \geq 1)
\]  

(2.9)

The result (2.9) is sharp for a function of the form:

\[
f(z) = z^p - \frac{2\gamma(p+n)(p+n-\alpha)(p-\beta)+(p-\alpha)(p(1-\gamma)+2\beta\gamma)}{(p+n)(p+n-\alpha)(1+\gamma)} z^{p+n}
\]  

(2.10)

with respect to

\[
g(z) = z^p - \frac{(p-\alpha)}{(p+n-\alpha)} z^{p+n} \quad (n \geq 1).
\]  

(2.11)

**Remark 1**

(i) Letting \( p=1 \) in Theorem 1 and Corollary 1, we obtain the results proved by Srivastava and Owa [5, Theorem 1 and Corollary 1].

(ii) Letting \( p=1 \) and \( \alpha = 0 \) in Corollary 1, we obtain a result proved by Gupta [1, Theorem 3].

In a similar manner, Lemma 2 can be used to prove

**Theorem 2**

Let the function \( f(z) \) defined by (1.7) be in the class \( C(p, \alpha, \beta, \gamma) \). Then

\[
\sum_{n=1}^{\infty} \left(1+\gamma(p+n)\right)a_{p+n} - \frac{p(p-\alpha)(p(1-\gamma)+2\beta\gamma)}{(p+n)(p+n-\alpha)(1+\gamma)} \leq 2\gamma(p-\beta).
\]  

(2.12)

\[b_{p+n} \leq \frac{p-\alpha}{p+n-\alpha} \quad (n \geq 1).
\]  

(2.8)

**Corollary 2**

Let the function \( f(z) \) defined by (1.7) be in the class \( C(p, \alpha, \beta, \gamma) \). Then

\[
a_{p+n} \leq \frac{2\gamma(p+n)(p+n-\alpha)(p-\beta)+(p-\alpha)(p(1-\gamma)+2\beta\gamma)}{(p+n)(p+n-\alpha)(1+\gamma)} \quad (n \geq 1).
\]  

(2.13)

The result (2.13) is sharp for a function of the form:

\[
f(z) = z^p - \frac{2\gamma(p+n)(p+n-\alpha)(p-\beta)+(p-\alpha)(p(1-\gamma)+2\beta\gamma)}{(p+n)(p+n-\alpha)(1+\gamma)} z^{p+n}
\]  

(2.14)

with respect to

\[
g(z) = z^p - \frac{(p-\alpha)}{(p+n-\alpha)} z^{p+n} \quad (n \geq 1).
\]  

(2.15)

**Distortion Theorems**

Applications of Lemma 1 and Lemma 2 lead to the following distortion theorems for functions belonging to the classes \( T^*(p, \alpha, \beta, \gamma) \) and \( C(p, \alpha, \beta, \gamma) \).

**Theorem 3**

Let the function \( f(z) \) defined by (1.7) be in the class \( T^*(p, \alpha, \beta, \gamma) \). Then

\[
|z|^p - A(p, \alpha, \beta, \gamma)|z|^{p+1} \leq |f(z)| \leq |z|^p + A(p, \alpha, \beta, \gamma)|z|^p
\]  

(3.1)

and

\[
p|z|^{p+1} - (p+1)A(p, \alpha, \beta, \gamma)|z|^{p+1} \leq |f(z)| \leq p|z|^{p+1} + (p+1)A(p, \alpha, \beta, \gamma)|z|^{p+1}
\]  

(3.2)

for \( z \in U \), where

\[
A(p, \alpha, \beta, \gamma) = \frac{(p-\alpha)p(1+\gamma)+2\beta(p-\beta)}{(p+1)(p+1-\alpha)(1+\gamma)}.
\]  

(3.3)

The results (3.1) and (3.2) are sharp.

**Proof**

For \( f(z) \in T^*(p, \alpha, \beta, \gamma) \), (2.7) implies
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\[(p+1)(1+\gamma)\sum a_{p+n}-(p(1+\gamma)+2\beta\gamma)\sum b_{p+n} \leq 2\gamma(p-\beta).\]  (3.4)

For \(g(z) \in T^*(p, \alpha)\), Lemma 1 implies

\[\sum_{n=1}^{\infty} b_{p+n} \leq \frac{p-\alpha}{p+1-\alpha}.\]  (3.5)

so that (3.4) reduces to

\[\sum_{n=1}^{\infty} a_{p+n} \leq \frac{(p-\alpha)p(1+\gamma)+2\gamma(p-\beta)}{(p+1)(p+1-\alpha)(1+\gamma)} = A(p, \alpha, \beta, \gamma).\]  (3.6)

Consequently,

\[|f(z)| \geq |z|^p - \sum_{n=1}^{\infty} a_{p+n}|z|^{p+n}\]
\[\geq |z|^p - |z|^{p+1} \sum_{n=1}^{\infty} a_{p+n}\]
\[\geq |z|^p - A(p, \alpha, \beta, \gamma)|z|^{p+1}\]  (3.7)

and

\[|f(z)| \leq |z|^p + |z|^{p+1} \sum_{n=1}^{\infty} a_{p+n}\]
\[\leq |z|^p + A(p, \alpha, \beta, \gamma)|z|^{p+1}.\]  (3.8)

Furthermore, we note from (2.7) that

\[(1+\gamma)\sum (p+n)a_{p+n}-(p(1+\gamma)+2\beta\gamma)\sum b_{p+n} \leq 2\gamma(p-\beta).\]  (3.9)

which, in view of (3.5), becomes

\[\sum (p+n)a_{p+n} \leq \frac{(p-\alpha)p(1+\gamma)+2\gamma(p-\beta)}{(p+1)(p+1-\alpha)(1+\gamma)} = (p+1)A(p, \alpha, \beta, \gamma).\]  (3.10)

Thus, we have

\[|f'(z)| \geq p|z|^{p-1} - |z|^p \sum_{n=1}^{\infty} (p+n)a_{p+n}\]
\[\geq p|z|^{p-1} - (p+1)A(p, \alpha, \beta, \gamma)|z|^p\]  (3.11)

and

\[|f'(z)| \leq p|z|^{p-1} + |z|^p \sum_{n=1}^{\infty} (p+n)a_{p+n}\]
\[\leq p|z|^{p-1} + (p+1)A(p, \alpha, \beta, \gamma)|z|^p.\]  (3.12)

Finally, we can prove that the bounds in (3.1) and (3.2) are sharp by taking the function

\[f(z) = z^p - A(p, \alpha, \beta, \gamma)z^{p+1}\]  (3.13)

with respect to

\[g(z) = z^p - \frac{(p-\alpha)}{(p+1-\alpha)}z^{p+1}.\]  (3.14)

This completes the proof of Theorem 3.

**Corollary 3**

Let the function \(f(z)\) defined by (1.7) be in the class \(T^*(p, \alpha, \beta, \gamma)\). Then the unit disc \(U\) is mapped onto a domain that contains the disc \(|w|< r_1\), where

\[r_1 = \frac{2p+1-\alpha+\gamma-\alpha\gamma+2\beta\gamma}{(p+1)(p+1-\alpha)(1+\gamma)}.\]  (3.15)

The result is sharp with the extremal function defined by (3.13).

**Remark 2**

(i) Letting \(p=1\) in Theorem 3, we obtain a result proved by Srivastava and Owa [5, Theorem 3].

(ii) Letting \(p=1\) and \(\alpha = 0\) in Theorem 3, we obtain a result proved by Gupta [1, Theorem 4].

**Theorem 4**

Let the function \(f(z)\) defined by (1.7) be in the class \(C(p, \alpha, \beta, \gamma)\). Then

\[|f^p - B(p, \alpha, \beta, \gamma)|f|^{p+1} \leq |f(z)| \leq |f^p + B(p, \alpha, \beta, \gamma)|f|^{p+1}\]  (3.16)

and
\[ p|f^{p+1} - (p+1)B(p, \alpha, \beta, \gamma)| \leq \frac{p(p-\alpha)}{(p+1)(p+1-\alpha)} \leq \frac{p|f^{p+1} + (p+1)B(p, \alpha, \beta, \gamma)|}{1 - (p+1)(p+1-\alpha)(1+\gamma)^2} \]

for \( z \in U \), where

\[ B(p, \alpha, \beta, \gamma) = \frac{p(p-\alpha)(p+1+\gamma)}{(p+1)(p+1-\alpha)(1+\gamma)} + 2\gamma(p-\beta)(p+1)(p+1-\alpha). \]  

The results (3.16) and (3.17) are sharp.

**Proof**

By using Lemma 2, we have

\[ \sum_{n=1}^{\infty} b_{p+n} \leq \frac{p(p-\alpha)}{(p+1)(p+1-\alpha)}, \]  

since \( g(z) \in \mathcal{C}(p, \alpha) \). The assertions (3.16) and (3.17) of Theorem 4 follow if we apply (3.19) to (2.7).

The bounds in (3.16) and (3.17) are attained by the function

\[ f(z) = z^p - B(p, \alpha, \beta, \gamma)|z|^{p+1} \]  

with respect to

\[ g(z) = z^p - \frac{p(p-\alpha)}{(p+1)(p+1-\alpha)}z^{p+1}. \]

This evidently completes the proof of Theorem 4.

**Corollary 4**

Let the function \( f(z) \) defined by (1.7) be in the class \( \mathcal{C}(p, \alpha, \beta, \gamma) \). Then the unit disc \( U \) is mapped onto a domain that contains the disc \( |w| < r_3 \), where

\[ r_3 = \frac{1}{(p+1)^2(p+1-\alpha)(1+\gamma)} \frac{1}{(p+1)(p+1-\alpha)(1+\gamma)^2}. \]

\[ [(p+1)(1+\gamma) - 2\gamma(p-\beta) - p(p-\alpha)(p+1+\gamma) + 2\beta\gamma]. \]

Convexity of functions in \( \mathbb{T}^* (p, \alpha, \beta, \gamma) \) and \( \mathcal{C}(p, \alpha, \beta, \gamma) \)

In view of Lemma 1, we know that the function \( f(z) \) defined by (1.7) is \( p \)-valent starlike in the unit disc \( U \) if and only if

\[ \sum_{n=1}^{\infty} (p+n)a_{p+n} \leq p. \]  

For \( f(z) \in \mathbb{T}^* (p, \alpha, \beta, \gamma) \), we find from (2.7) and (3.5) that

\[ \sum_{n=1}^{\infty} (p+n)a_{p+n} \leq (p+1)A(p, \alpha, \beta, \gamma) \leq p. \]  

where \( A(p, \alpha, \beta, \gamma) \) is defined by (3.3). Furthermore, for \( f(z) \in \mathcal{C}(p, \alpha, \beta, \gamma) \), we

\[ \sum_{n=1}^{\infty} (p+n)a_{p+n} \leq (p+1)B(p, \alpha, \beta, \gamma) \leq p. \]  

where \( B(p, \alpha, \beta, \gamma) \) is defined by (3.18). Thus we observe that \( \mathbb{T}^* (p, \alpha, \beta, \gamma) \) and \( \mathcal{C}(p, \alpha, \beta, \gamma) \) are subclasses of \( p \)-valent starlike functions. Naturally, therefore, we are interested in finding the radii of convexity for functions in \( \mathbb{T}^* (p, \alpha, \beta, \gamma) \) and \( \mathcal{C}(p, \alpha, \beta, \gamma) \). We first state:

**Theorem 5**

Let the function \( f(z) \) defined by (1.7) be in the class \( \mathbb{T}^* (p, \alpha, \beta, \gamma) \). Then \( f(z) \) is \( p \)-valent convex in the \( |z| < r_3 \), where

\[ r_3 = \inf_{n \geq 1} \left[ \frac{p^2}{(p+1)(p+n)A(p, \alpha, \beta, \gamma)} \right] \]

\[ \frac{1}{(p+1)^2A(p, \alpha, \beta, \gamma)}, \]  

where \( A(p, \alpha, \beta, \gamma) \) is given by (3.3). The result is sharp.
Proof

If suffices to prove

\[ \left| 1 + \frac{zf''(z)}{f'(z)} - p \right| \leq p \quad (|z| < r_3). \tag{4.5} \]

Indeed we have

\[ \left| 1 + \frac{zf''(z)}{f'(z)} - p \right| = \sum_{n=1}^{\infty} n(p + n) a_{p+n} z^n \]

\[ \leq \sum_{n=1}^{\infty} n(p + n) a_{p+n} |z|^n \]

\[ \leq p \sum_{n=1}^{\infty} (p + n) a_{p+n} |z|^n \]

Hence (4.5) holds true if

\[ \sum_{n=1}^{\infty} n(p + n) a_{p+n} |z|^n \leq p^2 - \sum_{n=1}^{\infty} p(p + n) a_{p+n} |z|^n , \tag{4.6} \]

that is, if

\[ \sum_{n=1}^{\infty} (p + n)^2 a_{p+n} |z|^n \leq p^2 . \tag{4.7} \]

With the aid of (3.10), (4.8) is true if

\[ (p + n)|z|^n \leq \frac{p^2}{(p + 1)(p + n) A(p, \alpha, \beta, \gamma)} (n \geq 1). \tag{4.9} \]

It follows from (4.9) that

\[ |z| \leq \left[ \frac{p^2}{(p + 1)(p + n) A(p, \alpha, \beta, \gamma)} \right]^{1/n} (n \geq 1). \tag{4.10} \]

Finally, since \((p + n)^{-n} \) is an increasing function for integers \( n \geq 1, p \in \mathbb{N} \), we have (4.5) for \( |z| < r_3 \), where \( r_3 \) is given by (4.4).

In order to complete the proof of Theorem 5, we note that the result is sharp for the function \( f(z) \in T^n (p, \alpha, \beta, \gamma) \) of the form:

\[ f(z) = z^p - \frac{(p + 1) A(p, \alpha, \beta, \gamma)}{(p + n)} z^{p+n} \tag{4.11} \]

for some integer \( n \geq 1 \).

Similarly, we can prove Theorem 6.

Theorem 6

Let the function \( f(z) \) defined by (1.7) be in the class \( C (p, \alpha, \beta, \gamma) \). Then \( f(z) \) is \( p \)-valent convex in the disc \( |z| < r_4 \), where

\[ r_4 = \inf_{n \geq 1} \left[ \frac{p^2}{(p + 1)(p + n) B(p, \alpha, \beta, \gamma)} \right]^{1/n} \tag{4.12} \]

where \( B(p, \alpha, \beta, \gamma) \) being given by (3.18). The result is sharp for the function \( f(z) \in C (p, \alpha, \beta, \gamma) \) of the form:

\[ f(z) = z^p - \frac{(p + 1) B(p, \alpha, \beta, \gamma)}{(p + n)} z^{p+n} \tag{4.13} \]

for some integer \( n \geq 1 \).

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References