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LEVERRIER-FADDEEV'S ALGORITHM APPLIED TO SPACETIMES OF CLASS ONE

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Abstract: We explain that the Leverrier-Takeno´s method to construct the characteristic polynomial of an arbitrary matrix $\underset{\sim}{A}$, plus the Cayley-Hamilton theorem, it is equivalent to the Faddeev´s process to obtain $\underset{\sim}{A}^{-1}$. We apply this algorithm to the second fundamental form of a spacetime embedded into flat 5-space.

***Keywords**:* Cayley-Hamilton theorem, Faddeev algorithm, Leverrier-Takeno method, Riemannian 4-spaces of class one.

Introduction

For any matrix $\underset{\sim}{A}_{n\times n} = (A^i_{\ j})$ its characteristic equation:

$$\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n = 0 \qquad (1)$$

can be obtained, through several procedures [1-5], directly from the condition $\det(A^i_{\ j}) - \lambda\delta^i_{\ j}) = 0$. The approach of Leverrier-Takeno [1,6-9] is a simple and interesting technique to construct (1) based in the traces of the powers $\underset{\sim}{A}^r$, r=1, …,n.

On the other hand, it is well known that an arbitrary matrix **A** satisfies its characteristic polynomial:

$$\underset{\sim}{A}^n + a_1 \underset{\sim}{A}^{n-1} + \cdots + a_{n-1} \underset{\sim}{A} + a_n \underset{\sim}{I} = \underset{\sim}{0}, \qquad (2)$$

which is the Cayley-Hamilton identity. If $\underset{\sim}{A}$ is non-singular (that is, det $\neq 0$), then from (2) we obtain its inverse matrix:

$$\underset{\sim}{A}^{-1} = -\frac{1}{a_n}\left(\underset{\sim}{A}^{n-1} + a_1 \underset{\sim}{A}^{n-2} + \cdots + a_{n-1} \underset{\sim}{I}\right), \qquad (3)$$

where $a_n \neq 0$ because $a_n = (-1)^n \det \underset{\sim}{A}$.

Faddeev [10-13] proposed an algorithm to determine $\underset{\sim}{A}^{-1}$ in terms of $\underset{\sim}{A}^r$ and their traces. Here we exhibit that (3) coincides with the Faddeev´s result if we employ the Leverrier-Takeno´s formulae for a_j. After, we apply this analysis to second fundamental form **b** governing the extrinsic geometry of Riemannian 4-spaces of class one (that is, 4-spaces embedded into (E_5) [14-18]. The corresponding eq. (3) leads to an original expression for its inverse matrix as function of **b** and the double dual of Riemann tensor (projection of it onto the Levi-Civita tensor) [18,19]. Normally the Leverrier-Takeno-Faddeev technique is considered useful only in problems of numerical analysis, but the aim of our work is to show the importance of this algorithm in geometrical theories as general relativity, with potential applications to several physical fields.

Leverrier-Takeno and Faddeev methods

If we define the quantities

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$$a_0 = 1, \quad s_r = tr\,\underset{\sim}{A}^r, \quad r = 1,2,\dots,n \qquad (4)$$

then the process of Leverrier-Takeno[1,6-9] implies (1) wherein the a_i are determined with the recurrence expression

$$r a_r + s_1 a_{r-1} + s_2 a_{r-2} + \cdots + s_{r-1} a_1 + s_r = 0,$$
$$r = 1,2,\dots,n \qquad (5)$$

Therefore

$$a_1 = -s_1, \qquad 2!\, a_2 = (s_1)^2 - s_2,$$
$$3!\, a_3 = -(s_1)^3 + 3 s_1 s_2 - 2 s_3, \qquad (6)$$

$$4!\, a_4 = (s_1)^4 - 6 (s_1)^2 s_2 + 8 s_1 s_3 + 3 (s_2)^2 - 6 s_4,$$
etc.

In particular det $\underset{\sim}{A} = (-1)^n a_n$, that is, the determinant of any square matrix only depends on the traces s_r, which means that $\underset{\sim}{A}$ and its transpose have the same determinant.

The Faddeev procedure [10-13] to obtain $\underset{\sim}{A}^{-1}$ is a sequence of algebraic computations on the powers $\underset{\sim}{A}^r$ and their traces. In fact, his algorithm is given by the instructions

$$\underset{\sim}{A}_1 = \underset{\sim}{A}, \quad q_1 = tr\,\underset{\sim}{A}_1, \quad \underset{\sim}{B}_1 = \underset{\sim}{A}_1 - q_1 \underset{\sim}{I},$$
$$_2 = \underset{\sim}{B}_1 \underset{\sim}{A}, \quad q_2 = \frac{1}{2} tr\,\underset{\sim}{A}_2, \underset{\sim}{B}_2 = \underset{\sim}{A}_2 - q_2 \underset{\sim}{I},$$
$$_{n-1} = \underset{\sim}{B}_{n-2}, \quad q_{n-1} = \frac{1}{n-1} tr\,\underset{\sim}{A}_{n-1}, \underset{\sim}{B}_{n-1}$$
$$= \underset{\sim}{A}_{n-1} \quad , \qquad (7)$$

$$_n = \underset{\sim}{B}_{n-1} \underset{\sim}{A}, \quad q_n = \frac{1}{n} tr\,\underset{\sim}{A}_n$$

Then

$$\underset{\sim}{A}^{-1} = \frac{1}{q_n} \underset{\sim}{B}_{n-1} \qquad (8)$$

For example, if we apply (7) for $n = 4$, then it is easy to see that the corresponding q_r imply (6) with $q_j = -a_j$, and besides (8) reproduces (3). By mathematical induction one can prove that (7) and (8) are equivalent to (3), (4) and (5), showing thus that the Faddeev technique has its origin in the Leverrier-Takeno method plus Cayley-Hamilton theorem.

Spacetimes of class one

A R_4 can be embedded into E5 (that is, the 4-space has class one) if and only if there exists the second fundamental form $bac = b_{ca}$ satisfying the Gauss-Codazzi equations [14-18,20]

$$R_{acij} = å(b_{ai} b_{cj} - b_{aj} b_{ci}), \qquad (9)$$

$$b_{ij;c} = b_{ic;j}, \qquad (10)$$

where $å = \underline{+}1$, R_{acij} is the Riemann tensor and *;r* means covariant derivative. It is well-known [21] that whenever $det\,(b^i{}_j)$ is different to zero then (9) implies (10). In other words, when a non-singular matrix **b** satisfies the Gauss equation, the Codazzi equation is verified automatically. However, in general the construction of **b** for a given spacetime should involve the study of both (9) and (10) together.

Employing (9) it is not difficult to deduce the following result [14,15,22].

$$-24\, det\,(b^i{}_j) = K_2\, {}^*R^{*\,ijac} R_{ijac}, \qquad (11)$$

where K_2 is a Lanczos invariant [19,23-25] defined in terms of the double dual of curvature tensor

$${}^*R^{*\,ij}{}_{ac} = \frac{1}{4} \eta^{ijrt} R_{rt}{}^{mn} \eta_{mnac}, \qquad (12)$$

with η_{ijrt} denoting the Levi-Civita symbol, then the Bianchi identities [20] adopt the compact form [18,19]

$$ {}^{*}R^{*\,i}{}_{jac\,;\,i} = 0, \qquad (13)$$

The present work deals with the case $K_2 \neq 0$, thus, according to (11) this implies that the inverse matrix $b^{-1}{}_{ij}$ exists. In this situation we employ (2), (6), (9) and (11) to obtain the characteristic polynomial of **b** [15,16]

$$\mathbf{b}^4 - b\mathbf{b}^3 - \frac{\varepsilon}{2}R\mathbf{b}^2 - p\mathbf{b} - \frac{K_2}{24}\mathbf{I} = \mathbf{0}, \qquad (14)$$

where $R = R^{ij}{}_{ji}$ is the scalar curvature and

$$b = tr\ \mathbf{b}, \quad p = \frac{\varepsilon}{3} b^{ac} G_{ac},$$

$$G_{ij} = {}^{*}R^{*a}{}_{ija} = \text{Einstein tensor}, \qquad (15)$$

Then with (3) and (14) we construct the inverse matrix of **b**

$$\frac{K_2}{24} b^{-1}{}_{ij} = \varepsilon b_{ir} G^{r}{}_{j} - p\, g_{ij}, \qquad (16)$$

In a given spacetime we know the metric tensor g_{ij}, G_{ac} and K_2; if besides we have **b** and ε, then (15) determines p. Thus (16) gives us $\mathbf{b}^{-1}$ and

$$b^{-1r}{}_{r} = tr\ \mathbf{b}^{-1} = -\frac{24p}{K_2}. \qquad (17)$$

On the other hand, the double dual (12) admits the expansion [16,19,23]

$${}^{*}R^{*}_{ijac} = -R_{ijac} + G_{ic}g_{ja} + G_{ja}g_{ic} - G_{ia}g_{jc} - G_{jc}g_{ia} + \frac{R}{2}\left(g_{ic}g_{ja} - g_{ia}g_{jc}\right), \qquad (18)$$

Therefore (9), (16), (17) and (18) imply the original relation

$$K_2\ b^{-1}{}_{ia} = 8\mathring{a}\ {}^{*}R^{*}{}_{ijac}\, b^{jc}, \qquad (19)$$

which means that $\mathbf{b}^{-1}$ essentially is the projection of **b** onto double dual tensor. The Codazzi equation (10) is a differential condition on **b**. However, we searched in the literature some differential restriction on $\mathbf{b}^{-1}$**,** but without success in this quest. The importance of (19) is that it, (10) and (13) generate such differential requisite

$$(K_2\, b^{-1i}{}_{a})_{;i} = 0, \qquad (20)$$

which is other original contribution of this work.

The deduction of (19) and (20) shows the usefulness of Leverrier-Faddeev-Takeno expressions in the study of 4-spaces of class one. However, our analysis also is applicable to geometric control theory, quantum information processing, field theory, quantum groups and chaos theory (see [26-32]).

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