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PARTIAL SUMS OF CERTAIN CLASSES OF MEROMORPHIC FUNCTIONS

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Abstract: Using Al-Oboudi differential operator, we study some classes of meromorphic functions. A necessary and sufficient condition for belonging to these classes, coefficients, extremal functions and partial sums are studied.

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1. Introduction and definitions

Let $\sum_p$ denote the class of functions of the form

$$f(z)=\sum_{k=-p}^{\infty} a_{k+p-1}z^k=\frac{1}{z^p}+\frac{a_0}{z^{p-1}}+\frac{a_1}{z^{p-2}}+\ldots,\ a_{-1}=1,\quad p\in N=1,2,3.... \tag{1.1}$$

which are regular and multivalent in punctured disk U={z:0<|z|<1}.

Then a function f belonging to $\sum_p$ is said to be meromorphic starlike $\sum_p^*$ function if and only if it satisfies

$$\Re e\left\{-\left(\frac{zf'(z)}{f(z)}\right)\right\}>\alpha. \tag{1.2}$$

Also a function f belonging to $\sum_p$ is said to be meromorphic convex function $\sum_k$ if and only if it satisfies

$$\Re e\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>\alpha \tag{1.3}$$

Recently, Al-Oboudi [1] defined a differential operator of an analytic function f of the form

$$f(z)=z+\sum_{k=2}^{\infty} a_k z^k \tag{1.4}$$

as follows.

Let

$$D^0 f(z)=f(z),$$
$$D^1 f(z)=D_\lambda f(z)=(1-\lambda)f(z)+\lambda zf'(z)$$

and

$$D^n f(z)=D_\lambda\left(D^{n-1}f(z)\right)=(1-\lambda)D^{n-1}f(z)+\lambda\, z\left(D^{n-1}f(z)\right)'$$

for $\lambda\geq 0$ and $n\in N_0=\{0,1,2,3,\ldots\}$. When λ=1, we have Sãlãgean differential operator [4]. It can easily be seen that f is given by (1.4). Then

$$D^n f(z)=z+\sum_{k=2}^{\infty}[1+\lambda(k-1)]^n a_k z^k=f(z)*\left(z+\sum_{k=2}^{\infty}[1+\lambda(k-1)]^n z^k\right).$$

Applying this differential operator to p-valent meromorphic function in, $\sum_p$, we have

$$D f(z)=\frac{1}{z^p}+\sum_{k=-p+1}^{\infty}\left[\frac{1+(k-1)\lambda}{1-(p+1)\lambda}\right]a_{k+p-1}z^k$$

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and

$$D^n f(z)=\frac{1}{z^p}+\sum_{k=-p+1}^{\infty}\left[\frac{1+(k-1)\lambda}{1-(p+1)\lambda}\right]^n a_{k+p-1}z^k$$
$$=f(z)*\left(\frac{1}{z^p}+\sum_{k=-p+1}^{\infty}\left[\frac{1+(k-1)\lambda}{1-(p+1)\lambda}\right]^n z^k\right), \quad (1.5)$$

for all and $-p\le k<\infty$, $p\in N=\{1,2,3,\ldots\}$ and $n\in N_0=\{0,1,2,3,\ldots\}$.

Definition 1

A function f of the form

$$f(z)=\frac{1}{z^p}+\sum_{k=-p+1}^{\infty}a_k z^k, \quad (1.6)$$

belongs to $M(n\lambda,\alpha,p)$ if and only if

$$\Re e\left\{\frac{D^{n+1}f(z)}{D^n f(z)}-(p-1)\right\}<-\alpha \quad (z\in U) \quad (1.7)$$

where and $0\le\alpha<p,\lambda\ge\frac{2\alpha(1+p-\alpha)}{p-\alpha}>0$, and $n\in N_0$.

We can see that for different selections of n,λ and α, we have many different classes of p-valent meromorphic functions.

Definition 2

If $f(z)=\frac{1}{z^p}-\sum_{k=-p+1}^{\infty}a_k z^k,\ a_k\ge 0,$ (1.8)
then $f\in T(n,\lambda,\alpha,p)$ if and only if

$$\Re e\left\{\frac{D^{n+1}f(z)}{D^n f(z)}-(p+1)\right\}<-\alpha \quad (z\in U)$$

where $0\le\alpha<p,\lambda\ge\frac{2\alpha(1+p-\alpha)}{p-\alpha}>0$, and $n\in N_0$.

2. Inclusion and sufficient conditions

In the next theorem we prove that all functions in the class $M(n\lambda,\alpha,p)$ are p-valent meromorphic starlike of the order α. We need the following lemma.

Lemma 2.1 [2].

Let w be a non constant and regular function in U, $w(0)=0$. If $|w(z)|\le 1$ attains its maximum value on the circle $|z|<r<1$ at z_0 we have, $z_0w'(z)=kw(z_0)$ where k is real number and $k\ge 1$.

Theorem 2.1

Let $f(z)=f(z)=\sum_{k=-p}^{\infty}a_k z^k$ be in $M(n\lambda,\alpha,p)$, where $a_{-p}=1$. Then for all $n\in N_0$ and $0\le\alpha<p$ and $\lambda\ge 0$

$$M(n+1,\lambda,\alpha,p)\subseteq M(n,\lambda,\alpha,p).$$

Proof

Let $f\in M(n+1,\lambda,\alpha,p)$. Then

$$\left\{\frac{D^{n+1}f(z)}{D^n f(z)}-(p-1)\right\}=-\frac{-p-(2\alpha-p)w(z)}{1+w(z)}. \quad (2.1)$$

We have to show that (2.1) implies the inequality

$$\left|\frac{-\frac{D^n f(z)}{D^{n-1}f(z)}+2(p-1)+1}{\frac{D^n f(z)}{D^{n-1}(z)}-1}\right|<1. \quad (2.2)$$

Let us define w in U by

$$\left\{\frac{D^n f(z)}{D^{n-1}f(z)}-(p+1)\right\}=-\left\{\alpha+(p-\alpha)\frac{1-w(z)}{1+w(z)}\right\}. \quad (2.3)$$

$$\left\{\frac{D^n f(z)}{D^{n-1}f(z)}-(p+1)\right\}=-\frac{1+(2p-2\alpha+1)w(z)}{1+w(z)}. \quad (2.4)$$

Differentiating (2.4) logarithmically, we obtain

$$\left\{\frac{D^{n+1}f(z)}{D^n f(z)}-(p+1)\right\}=\frac{2\lambda(p-\alpha)zw'(z)}{(1+w(z))(1+(1+(2p-2\alpha+1)w(z)))}+\frac{D^n f(z)}{D^{n-1}f(z)}-(p+1)$$

$$=\left\{\frac{2\lambda(p-\alpha)zw'(z)}{(1+w(z))(1+(1+(2p-2\alpha+1)w(z)))}-\alpha-(p-\alpha)\frac{1-w(z)}{1+w(z)}\right\}. \quad (2.5)$$

We claim that |w(z)|<1 in U. For otherwise [by Lemma 2.1] there exists z_0 in U such that

$$z_0 w'(z) = kw(z_0) \quad (2.6)$$

where $|w(z_0)|=1$ and k≥1. From (2.5) and 2.6), we obtain

$$\left\{\frac{D^{n+1}f(z)}{D^n f(z)}-(p+1)\right\}=-\frac{-p+(2\alpha-p)w(z)}{1+w(z)}+\frac{2\lambda(p-\alpha)zw'(z)}{(1+w(z))(1+(2p-2\alpha+1)w(z))} \quad (2.7)$$

Thus

$$\Re e\left\{\frac{D^{n+1}f(z)}{D^n f(z)}-(p+1)\right\}\geq\left\{\frac{2\lambda(p-\alpha)k-4\alpha(1+(p-\alpha))}{(1+(2p-2\alpha+1)w(z_0))(1+w(z_0))}\right\}>0, \quad (2.8)$$

which contradicts (2.1). Hence $|w(z)|<1$ and from (2.3) $f\in M(n,\lambda,\alpha,p)$.

Corollary (1)

All $f\in M(n,\lambda,\alpha,p)$ belong to $\Sigma^*(\beta)$ for all n, where $\beta=\dfrac{p-\alpha+\lambda}{\lambda}$.

In the next theorem we derive some properties of the operator D^n and give an application of the following Miller-Mocanu lemma [3].

Lemma 2.2 [3]

Let $\Phi(u,v)$ be a complex valued function, $\Phi: D\to C, D\subset C\times C$ (C is a complex plane), and let $u=u_1+iu_2$ and $v=v_1+iv_2$. Suppose that the function $\Phi(u,v)$ satisfies the conditions

(i) $\Phi(u,v)$ is continuous in D;
(ii) $(1,0)\in D$, $\Re e\{\Phi(u,v)\}>0$;
(iii) $\Re e\{(\Phi(iu_2,v_1)\}\leq 0$; for all $(iu_2, v_1)\in D$ such that $v_1\leq\dfrac{-(1+u_2^2)}{2}$. If $h(z)=1+c_n z^n+\ldots$, is an analytic function in U and $\Phi(h, zh';z)\in D$ for all $z\in U$, then $\Re e h(z)>0$.

Theorem 2.2

If $f\in M(n+1,\lambda,\alpha,p)$, then $f\in M(n,\lambda,\beta,p)$ for β where

$$\beta=\frac{2(\alpha-p-1)+\sqrt{4(\alpha-p)^2-12(\alpha-p)+17}}{4}.$$

Proof

Let us define by

$$\left\{\frac{D^n f(z)}{D^{n-1}f(z)}-(p+1)\right\}=-\{\beta+(p-\beta)h(z)\}, \quad (2.9)$$

with, $\beta=\dfrac{2(\alpha-p-1)+\sqrt{4(\alpha-p)^2-12(\alpha-p)+17}}{4}$.

Then $h(z)=1+p_1z+p_2z^2+\ldots$ is analytic in the open disk U.

Differentiating (2.9) logarithmically, we obtain

$$\frac{\lambda z(D^n f(z))'}{D^n f(z)}=\frac{(z\lambda D^{n-1}f(z))'}{D^{n-1}f(z)}+\frac{-(p-\beta)\lambda zh'(z)}{(1+p-\beta)-(p-\beta)h(z)}.$$

Thus

$$\frac{D^{n+1}f(z)}{D^n f(z)}=\frac{D^n f(z)}{D^{n-1}f(z)}-\frac{(p-\beta)\lambda zh'(z)}{(1+p-\beta)-(p-\beta)h(z)}$$

and

$$\Re e\left\{-\frac{D^n f(z)}{D^{n-1} f(z)}+(p+1)-\beta\right\}$$

$$=\Re e\left\{-\frac{D^{n+1} f(z)}{D^n f(z)}+(p+1)-\beta+\frac{(p-\beta)\lambda z h'(z)}{[(\beta-p-1)+(p-\beta)h(z)]}\right\}$$

$$=\Re e\left\{(\alpha-\beta)+(p-\alpha)h(z)+\frac{(p-\beta)\lambda z h'(z)}{(\beta-1-p)+(p-\beta)h(z)}\right\}>0.$$

Let us define a function $\Phi(u,v)$ by

$$\Phi(u,v)=(\alpha-\beta)+(p-\alpha)u-\frac{(p-\beta)\lambda v}{[(1+p-\beta)-(p-\beta)u]}.$$

Then we set

(i) $\Phi(u,v)$ is continuous in $D=\left\{C-\frac{p-\beta+1}{(p-\beta+1)-1}\right\}$

(ii) $(0,1)\in$ D, $\Re e\{\Phi(0,1)\}=p-\alpha>0$

(iii) For all $(iu_2,v_1)\in$ D such that $v_1\leq\frac{-(1+u_2{}^2)}{2}$

$$\Re e\{\Phi(iu_2,v_1)\}=\Re e\left\{(\alpha-\beta)+(p-\alpha)iu_2+\frac{(p-\beta)\lambda v_1}{[(\beta-p-1)+(p-\beta)iu_2]}\right\}$$

$$\leq(\alpha-\beta)-\frac{(\beta-p-1)(p-\beta)\lambda(1+u_2{}^2)}{2\{(\beta-p-1)^2+(p-\beta)^2u_2{}^2\}}\leq 0.$$

The function $\Phi(u,v)$ satisfies the conditions of Lemma 2.2. Consequently, we obtain

$$\Re e\left\{\frac{D^n f(z)}{D^{n-1} f(z)}-(p-1)\right\}<\frac{2(\alpha-p-1)+\sqrt{4(\alpha-p)^2-12(\alpha-p)+17}}{4}$$

Hence the theorem.

We now derive a sufficient condition for a function f to be in $M(n,\lambda,\alpha,p)$.

Theorem 2.3

If $f\in\sum_p$ in U and defined by (1.6) and

$$\sum_{k=1}^{\infty}[1+(k-p-1)\lambda]^n[\lambda(k-p-1)-(p-\alpha)(1-(p+1)\lambda)]|a_{k-p}|\leq(p-\alpha)(1-\lambda(p+1))^{n+1}, \quad (2.10)$$

then $f\in M(n,\lambda,\beta,p)$.

Proof

Suppose (2.10) holds for all admissible values of n, λ, p and α. Using the hypotheses, a simple calculation shows that for all $z\in$ U we have

$$\left|2(p-\alpha)+1-\frac{D^{n+1} f(z)}{D^n f(z)}\right|-\left|\frac{D^{n+1} f(z)}{D^n f(z)}-1\right|\leq\left|-2\frac{D^{n+1} f(z)}{D^n f(z)}+2(p-\alpha)+2\right|$$

$$=\left|(\alpha-p)+\sum_{k=1}^{\infty}\frac{[1+(k-p-1)\lambda]^n[\lambda(k-p-1)-(1-(p+1)\lambda)(p-\alpha)]}{(1-(p+1)\lambda)^{n+1}}a_{k-p}z^k\right|$$

$$\leq-(\alpha-p)+\frac{\sum_{k=1}^{\infty}[1+(k-p-1)\lambda]^n[\lambda(k-p-1)-(p-\alpha)]}{(1-(p+1)\lambda)^{n+1}}|a_{k-p}z^k|\leq 0,$$

by hypothesis. According to the maximum modulus theorem we get $f\in M(n,\lambda,\beta,p)$.

When $f\in T(n,\lambda,\alpha,p)$, then the above sufficient condition is also necessary and we have the following.

Theorem 2.4

Let f be defined by (1.8). Then $f\in T(n,\lambda,\alpha,p)$ if and only if

$$\sum_{k=1}^{\infty}[1+(k-p-1)\lambda]^n\frac{[\lambda(k-p-1)-(p-\alpha)(1-(p+1)\lambda)]}{(1-(p+1)\lambda)^{n+1}}|a_{k-p}|\leq(p-\alpha) \quad (2.11)$$

Proof

Here, we only need to prove the "if part". Let $f \in T(n,\lambda,\alpha,p)$. Then

$$\left|\frac{-\frac{D^{n+1}f(z)}{D^n f(z)}+2(p-\alpha)+1}{\frac{D^{n+1}f(z)}{D^n f(z)}-1}\right|. \qquad (2.12)$$

Using the fact that $\Re e\{z\} \le |z|$ for all z, we have the following

$$\Re e\left\{\frac{\frac{(p-\alpha)}{z^p}-\sum_{k=-p+1}^{\infty}\frac{[2(p-\alpha)(1-(p+1)\lambda)-(k-1)\lambda]}{(1-(p+1)\lambda)^{n+1}}(1+(k-1)\lambda)^n a_k z^k}{-\sum_{k=-p+1}^{\infty}\left[\frac{1+(k-1)\lambda}{1-(p+1)\lambda}\right]^n[(k-1)\lambda]a_k z^k}\right\}<1.$$

Let us choose values of z on the real axis so that $\frac{D^{n+1}f(z)}{D^n f(z)}$ is real. Upon clearing the denominator in (2.12) and letting $z \to 1^-$ through the real values, we obtain the required condition.

The function

$$f_k(z)=\frac{1}{z^p}-\frac{(p-\alpha)(1-(p+1)\lambda)^{n+1}}{[1+(k-p-1)\lambda]^n[\lambda(k-p-1)-(p-\alpha)(1-(p+1)\lambda)]}z^{k-p}, k\ge 1. \qquad (2.13)$$

is an extremal function for the theorem.

Corollary (2)

If $f \in T(n,\lambda,\alpha,p)$, then

$$a_k \le \frac{(p-\alpha)(1-(p+1)\lambda)^{n+1}}{[1+(k-1)\lambda]^n[\lambda(k-1)-(p-\alpha)(1+(p+1)\lambda)]},$$

for each $k = -p+1, -p+2, -p+3, \ldots$.

The equality holds for the function given by (2.13).

Corollary (3)

$T(n+1,\lambda,\alpha,p) \subseteq T(n,\lambda,\alpha,p)$, for all $n \subseteq N$.

3. Closure theorems

Theorem 3.1

The class $T(n,\lambda,\alpha,p)$ is closed under convex combinations.

Proof

Let $f, g \in T(n,\lambda,\alpha,p)$ and let f is given by (1.8) and

$$g(z)=\frac{1}{z^p}-\sum_{k=-p+1}^{\infty}b_k z^k, b_k \ge 0$$

For $0 \le \delta \le 1$, it is sufficient that the function h defined by

$$h(z) = (1-\delta) f(z) + \delta g(z), (z \in U)$$

belongs to $T(n,\lambda,\alpha,p)$. Since

$$h(z)=\frac{1}{z^p}-\sum_{k=-p+1}^{\infty}[(1-\delta)a_k+\delta b_k]z^k$$

applying the Theorem 2.4, we get

$$\begin{aligned}&(1-\delta)\sum_{k=1}^{\infty}[1+(k-p-1)\lambda]^n[\lambda(k-p-1)-(1-(p+1)\lambda)(p-\alpha)]a_{k-p}\\&+\delta\sum_{k=-p+1}^{\infty}[1+(k-p-1)\lambda]^n[\lambda(k-p-1)-(p-\alpha)(1-(p+1)\lambda)]b_{k-p}\\&\le[(1-\delta)(p-\alpha)+\delta(p-\delta)](1-(p+1)\lambda)^{n+1}\\&=(p-\delta)(1-(p+1)\lambda)^{n+1},\end{aligned}$$

implies $h \in T(n,\lambda,\alpha,p)$.

From Theorem 3.1 it follows that the closed convex hull of $T(n,\lambda,\alpha,p)$ is the same as $T(n,\lambda,\alpha,p)$. Now we determine the extreme points of $T(n,\lambda,\alpha,p)$.

Theorem 3.2

Let $f_{-1}(z)=\frac{1}{z^p}, f_k(z)=\frac{1}{z^p}-\frac{(p-\alpha)(1-(p+1)\lambda)^{n+1}}{[1+\lambda(k-p-1)]^n[\lambda(k-p-1)-(p-\alpha)(1-(p+1)\lambda)]}z^{k-p}$,

where $k=1,2,3,\ldots, z\in U$ *and* $n\in N_0$. Then $f\in T(n,\lambda,\alpha,p)$ if and only if it can be expressed as

$$f(z)=\sum_{k=-p}^{\infty}\sigma_k z^k$$

where $\sigma_k\geq 0$, and $\sum_{k=-p}^{\infty}\sigma_k=1$.

Proof

Suppose that

$$f(z)=\sum_{k=-p}^{\infty}\sigma_k f_k(z)=\frac{1}{z^p}-\sum_{k=1}^{\infty}\sigma_k\left(\frac{(p-\alpha)(1-(p+1)\lambda)^{n+1}}{[1+\lambda(k-p-1)]^n[\lambda(k-p-1)-(p-\alpha)(1-(p+1)\lambda)]}\right)z^{k-p}$$

Since

$$\sum_{k=-p}^{\infty}\frac{[1+(k-1)\lambda]^n[\lambda(k-1)-(p-\alpha)(1-(p+1)\lambda)]}{(p-\alpha)(1-(p+1)\lambda)^{n+1}}\sigma_k\left(\frac{(p-\alpha)(1-(p+1)\lambda)^{n+1}}{[1+(k-1)\lambda]^n[\lambda(k-1)-(p-\alpha)(1-(p+1)\lambda)]}\right)$$

$$=\sum_{k=-p}^{\infty}\sigma_k-1-\sigma_{-p}\leq 1,$$

it follows from the Theorem 2.4 that $f\in T(n,\lambda,\alpha,p)$.

Conversely suppose that

$$f(z)=\frac{1}{z^p}-\sum_{k=-p+1}^{\infty}a_k z^k\in T(n,\lambda,\alpha,p),$$

and since

$$a_k\leq\frac{(p-\alpha)(1-(p+1)\lambda)^{n+1}}{[1+(k-1)\lambda]^n[\lambda(k-1)-(p-\alpha)(1-(p+1)\lambda)]}, k=-p+1,-p+2,\ldots,$$

we set

$$\sigma_k=\frac{[1+\lambda(k-1)]^n[\lambda(k-1)-(p-\alpha)(1-(p+1)\lambda)]}{(p-\alpha)(1-(p+1)\lambda)^{n+1}}a_k \ \textit{for}\ k=-p+1,-p+2,\ldots$$

and

$$\sigma_{-p}=1-\sum_{k=-p+1}^{\infty}\sigma_k.$$

From Theorem 2.4, we have

$$\sum_{k=-p}^{\infty}\sigma_k\leq 1,\quad \textit{and}\ \sigma_{-p}\geq 0.$$

It follows that

$$f(z)=\sum_{k=-p}^{\infty}\sigma_k f_k(z).$$

Corollary (4)

The extreme points of $T(n,\lambda,\alpha,p)$ are the function f_k, $k=1,2,3,\ldots$ where f_k is given by (2.3).

4. Partial sums of p(n,λ,α,p) and t(n,λ,α,p)

Following the earlier works by Silverman [5] on the partial sums of analytic functions, we study the ratio of a function of the form (1.8) to its sequence of partial sums of the form

$$f_n(z)=\frac{1}{z^p}-\sum_{k=-p+1}^{-p+n}a_k z^k, \qquad (4.1)$$

where the coefficients of f satisfy the condition (2.1). We also determine sharp lower bounds for

$$\Re e\left\{\frac{f(z)}{f_n(z)}\right\}, \Re e\left\{\frac{f_n(z)}{f(z)}\right\}, \Re e\left\{\frac{f'(z)}{f_n'(z)}\right\} \ \textit{and}\ \Re e\left\{\frac{f_n'(z)}{f'(z)}\right\}.$$

It is seen that this study not only gives, as a particular case, the results of Silverman [6] but also gives rise to several new results.

Theorem 4.1

If f is of the form (1.8) and satisfies the condition (2.11), and

$$\frac{[1+(m-p)\lambda]^n}{(1-(p+1)\lambda)^{n+1}}[\lambda(m-p)-(p-\alpha)(1-(p+1)\lambda)]\geq$$

$$\begin{cases} (p-\alpha) & m=1,2,\ldots k \\ \dfrac{[1+\lambda(m-p)]^n}{(1-(p+1)\lambda)^{n+1}}[\lambda(m-p)+(p-\alpha)(1-(p+1)\lambda)] & m=k+1,k+2,k+3,\ldots \end{cases},$$

then for $(z\in U)$

$$\Re e\left(\frac{f(z)}{f_k(z)}\right)\geq\frac{[1+\lambda(k-p)]^n[\lambda(k-p)-(p-\alpha)(1-(p+1)\lambda)]-(p-\alpha)(1-(p+1)\lambda)^{n+1}}{[1+(k-p)\lambda]^n[\lambda(k-p)-(p-\alpha)(1-(p+1)\lambda)]}. \quad (4.2)$$

$$\Re e\left\{\frac{f_k(z)}{f(z)}\right\}\geq\frac{[1+\lambda(p-\alpha)]^n[\lambda(k-p)-(p-\alpha)(1-(p+1)\lambda)]}{(p-\alpha)(1-(p+1)\lambda)^{n+1}+[1+\lambda(k-p)]^n[\lambda(k-p)-(p-\alpha)(1-(p+1)\lambda)]} \quad (4.3)$$

The results (4.2) and (4.3) are sharp for a function given by (2.13).

Proof

Let us define the function w by

$$\frac{1+w(z)}{1-w(z)}=\frac{[1+\lambda(k-p)]^n[\lambda(k-p)-(p-\alpha)(1-(p+1)\lambda)]}{(p-\alpha)(1-(p+1)\lambda)^{n+1}}$$

$$\times\left[\frac{f(z)}{f_k(z)}-\frac{[1+\lambda(k-p)]^n[\lambda(k-p)-(p-\alpha)(1-(p+1)\lambda)]-(1-(p+1)\lambda)^{n+1}(p-\alpha)}{[1+\lambda(k-p)]^n[\lambda(k-p)-(p-\alpha)(1-(p+1)\lambda)]}\right]$$

$$=\frac{1+\sum_{j=1}^{k}a_{j-p}z^j+\left(\frac{[1+\lambda(k-p)]^n[\lambda(k-p)-(p-\alpha)(1-(p+1)\lambda)]}{(p-\alpha)(1-(p+1)\lambda)^{n+1}}\right)\sum_{j=k+1}^{\infty}a_{j-p}z^j}{1+\sum_{j=1}^{k}a_{j-p}z^j}. \quad (4.4)$$

It is sufficient to show that $|w(z)|\leq 1$. From (4.4) we can write

$$w(z)=\frac{\left(\frac{[1+\lambda(k-p)]^n[\lambda(k-p)-(p-\alpha)(1-(p+1)\lambda)]}{(p-\alpha)(1-(p+1)\lambda)^{n+1}}\right)\sum_{j=k+1}^{\infty}a_{j-p}z^j}{2+2\sum_{j=1}^{k}a_{j-p}z^j+\left(\frac{[1+\lambda(k-p)]^n[\lambda(k-p)-(p-\alpha)(1-(p+1)\lambda)]}{(p-\alpha)(1-(p+1)\lambda)^{n+1}}\right)\sum_{j=k+1}^{\infty}a_{j-p}z^j}.$$

and

$$|w(z)|\leq\frac{\left(\frac{[1+\lambda(k-p)]^n[\lambda(k-p)-(p-\alpha)(1-(p+1)\lambda)]}{(p-\alpha)(1-(p+1)\lambda)^{n+1}}\right)\sum_{j=k+1}^{\infty}|a_{j-p}|}{2-2\sum_{j=1}^{k}|a_{j-p}|-\left(\frac{[1+\lambda(k-p)]^n[\lambda(k-p)-(p-\alpha)(1-(p+1)\lambda)]}{(p-\alpha)(1-(p+1)\lambda)^{n+1}}\right)\sum_{j=k+1}^{\infty}|a_{j-p}|}.$$

Now $|w(z)|\leq 1$ if

$$2\left(\frac{[1+\lambda(k-p)]^n[\lambda(k-p)-(p-\alpha)(1-(p+1)\lambda)]}{(p-\alpha)(1-(p+1)\lambda)^{n+1}}\right)\sum_{j=k+1}^{\infty}|a_{j-p}|\leq 2-2\sum_{j=1}^{k}|a_{j-p}|,$$

or equivalently

$$\sum_{j=1}^{k}|a_{j-p}|+\sum_{j=k+1}^{\infty}\left(\frac{[1+\lambda(k-p)]^n[\lambda(k-p)-(p-\alpha)(1-(p+1)\lambda)]}{(\alpha-p)(1-(p+1)\lambda)^n}\right)|a_{j-p}|\leq 1. \quad (4.5)$$

From the condition of (2.10), it is sufficient to show that

$$\sum_{j=1}^{\infty}|a_{j-k}|+\sum_{j=k+1}^{\infty}\frac{[1+\lambda(k-p)]^n[\lambda(k-p)-(p-\alpha)(1-(p+1)\lambda)]}{(p-\alpha)(1-(p+1)\lambda)^{n+1}}|a_{j-p}|$$

$$\le \sum_{j=1}^{\infty}\frac{[1+\lambda(j-p-1)]^n[\lambda(j-p-1)-(p-\alpha)(1-(p+1)\lambda)]}{(p-\alpha)(1-(p+1)\lambda)^{n+1}}|a_{j-p}|.$$

which is equivalent to

$$\sum_{j=1}^{k}\frac{[1+\lambda(j-p-1)]^n[\lambda(j-p-1)-(p-\alpha)(1-(p+1)\lambda)]-(p-\alpha)(1-(p+1)\lambda)^{n+1}}{(p-\alpha)(1-(p+1)\lambda)^{n+1}}|a_{j-p}|+$$

$$\sum_{j=k+1}^{\infty}\frac{[1+\lambda(j-p-1)]^n[\lambda(j-p-1)-(p-\alpha)(1-(p+1)\lambda)]}{(p-\alpha)(1-(p+1)\lambda)^{n+1}}|a_{j-p}|$$

$$-\sum_{j=k+1}^{\infty}\frac{[1+\lambda(k-p)]^n[\lambda(k-p)-(p-\alpha)(1-(p+1)\lambda)]}{(p-\alpha)(1-(p+1)\lambda)^{n+1}}|a_{j-p}|\ge 0. \quad (4.6)$$

To see that the function given by (2.13) shows that this result is sharp, let

$$f(z)=\frac{1}{z^p}+\frac{(p-\alpha)(1-(p+1)\lambda)^{n+1}}{[1+\lambda(k-p)]^n[\lambda(k-p)-(p-\alpha)(1-(p+1)\lambda)]}z^{k-p}, k=1,2,....$$

$$=\frac{1}{z^p}-\frac{(\alpha-p)(1-(p+1)\lambda)^{n+1}}{[1+\lambda(k-p)]^n[\lambda(k-p)-(p-\alpha)(1-(p+1)\lambda)]}z^{k-p}$$

$$\frac{f(z)}{f_k(z)}=\frac{[1+\lambda(k-p)]^n[\lambda(k-p)-(p-\alpha)(1-(p+1)\lambda)]-(p-\alpha)(1-(p+1)\lambda)^{n+1}}{[1+\lambda(k-p)]^n[\lambda(k-p)-(p-\alpha)(1-(p+1)\lambda)]},$$

when $r\to 1^-$.

To prove the second part of this theorem, we write

$$\frac{1+w(z)}{1-w(z)}=\frac{(p-\alpha)(1-(p+1)\lambda)^{n+1}+[1+\lambda(k-p)]^n[\lambda(k-p)-(p-\alpha)(1-(p+1)\lambda)]}{(p-\alpha)(1-(p+1)\lambda)^{n+1}}$$

$$\times\left[\frac{f_k(z)}{f(z)}-\frac{[1+\lambda(k-p)]^n[\lambda(k-p)-(p-\alpha)(1-(p+1)\lambda)]}{(p-\alpha)(1-(p+1)\lambda)^{n+1}+[1+\lambda(k-p)]^n[\lambda(k-p)-(p-\alpha)(1-(p+1)\lambda)]}\right]$$

$$=\frac{1+\sum_{j=-p+1}^{k-p}a_jz^{j+p}-\left(\frac{[1+\lambda(k-p)]^n[\lambda(k-p)-(p-\alpha)(1-(p+1)\lambda)]}{(p-\alpha)(1-(p+1)\lambda)^{n+1}}\right)\sum_{j=k-p+1}^{\infty}a_jz^{j+p}}{1+\sum_{j=-p+1}^{k-p}a_jz^{j+p}},$$

where

$$|w(z)|\le\frac{\left(\frac{[1+\lambda(k-p)]^n[\lambda(k-p)-(p-\alpha)(1-(p+1)\lambda)]}{(p-\alpha)(1-(p+1)\lambda)^{n+1}}\right)\sum_{j=k-p+1}^{\infty}|a_j|}{2-2\sum_{j=-p+1}^{k-p}|a_j|-\left(\frac{[1+\lambda(k-p)]^n[\lambda(k-p)-(p-\alpha)(1-(p+1)\lambda)]}{(p-\alpha)(1-(p+1)\lambda)^{n+1}}\right)\sum_{j=k-p+1}^{\infty}|a_j|}\le 1.$$

This last inequality is equivalent to

$$\sum_{j=-p+1}^{-p+k}|a_j|+\sum_{j=k-p+1}^{\infty}\frac{[1+\lambda(k-p)]^n[\lambda(k-p)-(p-\alpha)(1-(p+1)\lambda)]}{(p-\alpha)(1-(p+1)\lambda)^{n+1}}|a_j|\le 1.$$

Making use of (2.1) we get (4.3). Finally, equality holds in (4.3) for the extremal function f given by (2.13).

Remark 4.1

Different choices of λ ,*n*,*p* and α give the above result for many well known classes.

Corollary (5)

Let $\lambda = 1$, $n = 0$ in Theorem 2.1. $f\in \sum_p^*(\alpha)$ is given by (1.6) and

$$\sum_{k=1}^{\infty}[k-p+(p-\alpha)p]|a_{k-p}|\le p(\alpha-p).$$

Then for $z\in U$

$$\Re e\left\{\frac{f(z)}{f_k(z)}\right\}\ge\frac{(k-p+(p-\alpha)p)+(p-\alpha)p}{(k-p+(p-\alpha)p)}$$

and

$$\Re e\left\{\frac{f_k(z)}{f(z)}\right\} \geq \frac{(k-p+(p-\alpha)p)}{-p(p-\alpha)+(k-p+(p-\alpha)p)}.$$

This result is sharp for

$$f_1(z)=\frac{1}{z^p}+\frac{(p-\alpha)(p)}{(k-p+(p-\alpha)p)}z^{k-p} \ ,k=1,2,3,... \quad (4.7)$$

Corollary (6)

Let $\lambda = 1$ $n = 1$, $f \in \Sigma_k(\alpha)$ and if it satisfies

$$\sum_{k=1}^{\infty}[k-p+1][k-p+(p-\alpha)p]\left|a_{k-p}\right| \leq p^2(\alpha-p).$$

then for z∈ U

$$\Re e\left\{\frac{f(z)}{f_k(z)}\right\} \geq \frac{(k-p+1)(k-p+(p-\alpha)p)+p^2(p-\alpha)}{(k+1-p)(k-p+(p-\alpha)p)}$$

and

$$\Re e\left\{\frac{f_k(z)}{f(z)}\right\} \geq \frac{(k+1-p)(k-p+(p-\alpha)p)}{(p-\alpha)p^2+(k+1-p)(k-p+(p-\alpha)p)}.$$

The result is sharp for the function given by

$$f_k(z)=\frac{1}{z^p}-\frac{(p-\alpha)p^2}{(1+k-p)(k-p+(p-\alpha)p)}z^{k-p}. \quad (4.8)$$

Theorem 4.2

If f of the form (1.6) satisfies the condition (2.1), and

$$\frac{[1+\lambda(m-p)]^n}{(1-(p+1)\lambda)^{n+1}}[\lambda(m-p)-(1-(p+1)\lambda)(p-\alpha)] \geq \begin{cases} m(p-\alpha) & m=1,2,3,... \\ m[1+\lambda(k-p)]^n\left[\dfrac{\lambda(m-p)-(p-\alpha)}{k+1}\right] & m=k+1,k+2,... \end{cases}$$

then

$$\Re e\left\{\frac{f'(z)}{f_k'(z)}\right\} \geq \frac{[1+\lambda(k-p)]^n[\lambda(k-p)-(p-\alpha)(1-(p+1)\lambda)]-(p-\alpha)(k-p+1)(1-(p+1)\lambda)^{n+1}}{[1+\lambda(k-p)]^n[\lambda(k-p)-(p-\alpha)(1-(p+1)\lambda)]}, \quad (4.9)$$

and

$$\Re e\left\{\frac{f_k'(z)}{f'(z)}\right\} \geq \frac{[1+\lambda(k-p)]^n[\lambda(k-p)-(1-(p+1)\lambda)(p-\alpha)]}{(p-\alpha)(1-(p+1)\lambda)^{n+1}(k-p+1)+[1+\lambda(k-p)]^n[\lambda(k-p)-(p-\alpha)(1-(p+1)\lambda)]}, \quad (4.10)$$

where z ∈ U and k = 1,2,....

Proof

We write

$$\frac{1+w(z)}{1-w(z)}=\frac{[1+\lambda(k-p)]^n[\lambda(k-p)-(1-(p+1)\lambda)(p-\alpha)]}{(k-p+1)(p-\alpha)(1-(p+1)\lambda)^{n+1}}$$

$$\times\left[\frac{f'(z)}{f_k'(z)}-\frac{[1+\lambda(k-p)]^n[\lambda(k-p)-(1-(p+1)\lambda)(p-\alpha)]-(k-p+1)(1-(p+1)\lambda)^{n+1}(p-\alpha)}{[1+\lambda(k-p)]^n[\lambda(k-p)-(1-(p+1)\lambda)(p-\alpha)]}\right],$$

where

$$w(z)=\frac{\left(\dfrac{[1+\lambda(k-p)]^n[\lambda(k-p)-(1-(p+1)\lambda)(p-\alpha)]}{(k-p+1)(p-\alpha)(1-(p+1)\lambda)^{n+1}}\right)\sum_{j=1-p}^{\infty} ja_j z^{j+p}}{2+2\sum_{j=1-p}^{k-p} ja_j z^{j+p}+\left(\dfrac{[1+\lambda(k-p)]^n[\lambda(k-p)-(1-(p+1)\lambda)(p-\alpha)]}{(k-p+1)(p-\alpha)(1-(p+1)\lambda)^{n+1}}\right)\sum_{j=k-p+1}^{\infty} ja_j z^{j+p}}$$

Now $|w(z)|\leq 1$ if

$$\sum_{j=1-p}^{k-p} j|a_j| + \left(\frac{[1+\lambda(k-p)]^n[\lambda(k-p)-(p-\alpha)(1-(p+1)\lambda)]}{(k-p+1)(p-\alpha)(1-(p+1)\lambda)^{n+1}}\right)\sum_{j=k-p+1}^{\infty} j|a_j| \leq 1.$$

Now it is sufficient to show that

$$\sum_{j=1}^{k}(j-p)|a_{j-p}| + \left(\frac{[1+\lambda(k-p)]^n[\lambda(k-p)-(1-(p+1)\lambda)(p-\alpha)]}{(k-p+1)(p-\alpha)(1-(p+1)\lambda)^{n+1}}\right)\sum_{j=k+1}^{\infty}(j-p)|a_{j-p}|$$

$$\leq \sum_{j=1}^{\infty}\frac{[1+\lambda(j-p-1)]^n[\lambda(j-p-1)-(1-(p+1)\lambda)(p-\alpha)]}{(p-\alpha)(1-(p+1)\lambda)^{n+1}}|a_{j-p}|,$$

that is

$$\sum_{j=1}^{k}\frac{[1+\lambda(j-p-1)]^n[\lambda(j-p-1)-(p-\alpha)(1-(p+1)\lambda)]-(p-\alpha)(1-(p+1)\lambda)^{n+1}(j-p)}{(p-\alpha)(1-(p+1)\lambda)^{n+1}}|a_{j-p}|$$

$$+\sum_{j=k+1}^{\infty}\frac{(j-p+1)[1+\lambda(j-p-1)]^n[\lambda(j-p-1)-(1-(p+1)\lambda)(p-\alpha)]}{(k-p+1)(p-\alpha)(1-(p+1)\lambda)^{n+1}}|a_{j-p}|$$

$$-\sum_{j=k+1}^{\infty}\frac{[1+\lambda(k-p)]^n[\lambda(k-p)-(p-\alpha)(1-(p+1)\lambda)](j-p+1)}{(k-p+1)(p-\alpha)(1-(p+1)\lambda)^{n+1}}|a_{j-p}|\geq 0.$$

To prove (4.10), let us define w by

$$\frac{1+w(z)}{1-w(z)}=\frac{(k-p+1)(p-\alpha)(1-(p+1)\lambda)^{n+1}+[1+\lambda(k-p)]^n[\lambda(k-p)-(1-(p+1)\lambda)(p-\alpha)]}{(p-\alpha)(1-(p+1)\lambda)^{n+1}}$$

$$\times\left[\frac{f_k'(z)}{f'(z)}-\frac{[1+\lambda(k-p)]^n[\lambda(k-p)-(1-(p+1)\lambda)(p-\alpha)]}{(k-p+1)(p-\alpha)(1-(p+1)\lambda)^{n+1}+[1+\lambda(k-p)]^n[\lambda(k-p)-(1-(p+1)\lambda)(p-\alpha)]}\right],$$

where

$$w(z)=\frac{\left(1+\frac{[1+\lambda(k-p)]^n[\lambda(k-p)-(1-(p+1)\lambda)(p-\alpha)]}{(k-p+1)(p-\alpha)(1-(p+1)\lambda)^{n+1}}\right)\sum_{j=k+1}^{\infty}(j-p)a_{j-p}z^j}{2+2\sum_{j=1}^{k}(j-p)a_{j=p}z^j+\left(1+\frac{[1+\lambda(k-p)]^n[\lambda(k-p)-(1-(p+1)\lambda)(p-\alpha)]}{(k-p+1)(p-\alpha)(1-(p+1)\lambda)^{n+1}}\right)\sum_{j=k+1}^{\infty}(j-p)a_{j-p}z^j}$$

Now $|w(z)|\leq 1$, if

$$\sum_{j=1}^{k}(j-p)|a_{j-p}|$$

$$+\left(1+\frac{[1+\lambda(k-p)]^n[\lambda(k-p)-(1-(p+1)\lambda)(p-\alpha)]}{(k-p+1)(p-\alpha)(1-(p+1)\lambda)^{n+1}}\right)\sum_{j=k+1}^{\infty}(j-p)|a_{j-p}|\leq 1.$$

It is sufficient to show that the left side is bounded by

$$\sum_{j=1}^{\infty}\frac{[1+\lambda(j-p-1)]^n[\lambda(k-p-1)-(p-\alpha)(1-(p+1)\lambda)]}{(p-\alpha)(1-(p+1)\lambda)^{n+1}}|a_{j-p}|,$$

which is equivalent to

$$\sum_{j=1}^{k}\left(\frac{[1+\lambda(j-p-1)]^n[\lambda(k-p-1)-(1-(p+1)\lambda)(p-\alpha)]}{(p-\alpha)(1-(p+1)\lambda)^{n+1}}-(j-p)\right)|a_{j-p}|$$

$$+\sum_{j=k+1}^{\infty}\left(\frac{[1+\lambda(k-p-1)]^n[\lambda(k-p-1)-(1-(p+1)\lambda)(p-\alpha)]}{(p-\alpha)(1-(p+1)\lambda)^{n+1}}\right)|a_{j-p}|$$

$$-\sum_{j=k+1}^{\infty}\left(1+\frac{[1+\lambda(k-p)]^n[\lambda(k-p)-(1-(p+1)\lambda)(p-\alpha)]}{(k-p+1)(p-\alpha)(1-(p+1)\lambda)^{n+1}}\right)(j-p)|a_{j-p}|\geq 0.$$

Remark 4.2

Different choices of $\lambda,n,p,$ give the above result for many well known classes of meromorphic functions. For example

Corollary (7)

Let f be given by (1.6) and if it satisfies the condition

$$\sum_{j=1}^{\infty}[j-p-1+(p-\alpha)p]|a_{j-p}|\leq(\alpha-p)p$$

then $f \in \sum_p^*(\alpha)$ and for $z \in U$

$$\Re e\left\{\frac{f'(z)}{f_n'(z)}\right\} \geq \frac{[n-p+(p-\alpha)p]+(p-\alpha)(n-p+1)p}{(n-p+(p-\alpha)p)}, \; n=1,2,3,4,...$$

and

$$\Re e\left\{\frac{f_n'(z)}{f'(z)}\right\} \geq \frac{(n-p+(p-\alpha)p)}{(\alpha-p)p(n-p+1)+(n-p+(p-\alpha)p)}, \quad n=1,2,3,...$$

In both cases the extremal function is given by (4.7).

Corollary (8)

Let f be of the form (1.6) and satisfy the condition

$$\sum_{k=1}^{\infty}(k-p)(k-p-1+(p-\alpha)p)\left|a_{k-j}\right| \leq p^2(p-\alpha), \; k=1,2,3,...$$

Then $f \in \sum_k(\alpha)$ and for $z \in U$

$$\Re e\left\{\frac{f'(z)}{f_k'(z)}\right\} \geq \frac{(k-p+(p-\alpha)p)-p^2(p-\alpha)}{(k-p+(p-\alpha)p)}, \; k=1,2,3,...$$

and

$$\Re e\left\{\frac{f_k'(z)}{f'(z)}\right\} \geq \frac{(k-p+(p-\alpha)p)}{(p-\alpha)p^2-(k-p+(p-\alpha)p)}, \; k=1,2,3,...$$

In both cases the extremal function is given by (4.8).

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