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POCKLINGTON EQUATION AND THE METHOD OF MOMENTS

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Abstract: In this paper the authors make an analysis of different function combinations for application in the method of moments, especially to dipole antenna, in order to select the most suitable pair of functions which could provide better results in getting the current distribution along the antenna.

***Keywords*:** Pocklington equation; dipole antenna; method of moments; base and weight functions comparative analysis

Introduction

Method of Moments (MM) is widely used in the solution of field equations to obtain current distribution of wire antennas. Both, Hallén equation defined by magnetic and electric potentials and the electric field Pocklington equation, use the MM procedure. Since the beginning, when Harrington [1] established the method, a main task has been the selection of base and weight functions to obtain a computational efficient and reliable solution. Even though the actual articles write mainly about MM applications, we think that there is a pendent issue related with the analysis of the best combination of base and weight functions. This paper presents a comparative analysis of 16 combinations for the four more used functions; as a matter of comparison we use the feed point impedance obtained for the classical analysis and the MM solution of Pocklington's general equation for arbitrary bent wires [2], applied to the well known half wavelength dipole.

The Pocklington general equation

Starting with Maxwell equations, it is possible to obtain the simplified general equation for arbitrary bent wires [3], given by:

$$E^{i}_{\tan} = -\frac{1}{j\omega\varepsilon}\int_{s'}\Big[R^2\left(k^2R^2-1-jkR\right)\mathbf{s}\bullet\mathbf{s'} + \left(3+3jkR-k^2R^2\right)(\mathbf{R}\bullet\mathbf{s})(\mathbf{R}\bullet\mathbf{s'})\Big]\frac{e^{-jkR}}{4\pi R^5}\, I(s')ds' . \qquad (1)$$

Pocklington's procedure applied to wire antennas, supposes the current to be located over a thin filament over the conductor, while the rest of it is part of the free space (Fig. 1), considering constant the transversal current distribution. As Fig. 1 shows, $\mathbf{s'}$ represents a unit vector parallel to the conductor surface and $\mathbf{s}$ a unit vector over the conductor axis, the conductor axis $\mathbf{r}(s)$ and the current filament $\mathbf{r'}(s')$ are given by:

$$\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j} + z(s)\mathbf{k} , \qquad \mathbf{r'}(s') = \mathbf{r}(s') + a\mathbf{n}(s) , \qquad (2)$$

where $\mathbf{n}(s)$ represents a unit normal vector to the wire axis.

As is seen, both the filament curve and axis curve are parallel to each other. Although it is possible to choose an infinite number of filaments, in practice the one which makes the easiest calculation is selected. The Pocklington's general equation, given

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by (1) can be used for any geometry. The MM solution is obtained after definition of unit vectors of equation (2). The dot product $\mathbf{s} \bullet \mathbf{s}'$ and position of any point over the conductor's surface $|\mathbf{r}-\mathbf{r}'|$ is:

$$\begin{aligned} R &= |\mathbf{R}| = |\mathbf{r}-\mathbf{r}'| \\ &= \sqrt{[x(s)-x'(s')]^2+[y(s)-y'(s')]^2+[z(s)-z'(s')]^2}\ . \end{aligned} \quad (3)$$

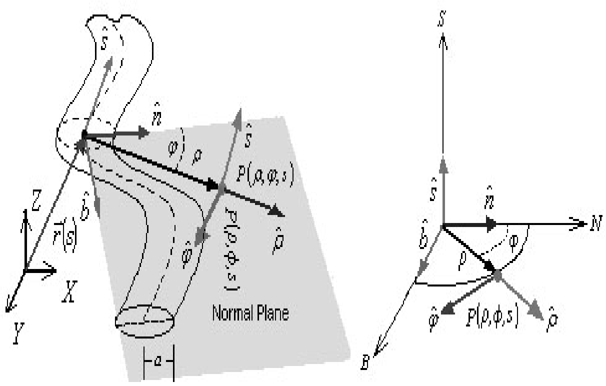

Fig. 1. Geometry for an arbitrary thin wire.

The method of moments

Equation (1) has the form [4]:

$$Lf = g. \quad (4)$$

where L represents the integral linear operator, g is a known function (electric field) and f the unknown function (the current), which should be determined and MM represents the unknown f, by a set of functions in the L domain, ($f_1, f_2, f_3, \cdots$), as a linear combination:

$$f = \sum_n \alpha_n f_n\ . \quad (5)$$

The α_n are constants to be determined and the f_n, named base or expansion functions, are
(3) arbitrarily selected. Substituting (4) in (5) and considering the linearity of L, we have:

$$\sum_n \alpha_n L f_n = g\ . \quad (6)$$

Equation (6) has N unknowns and it is necessary to have N independent linear equations, which are obtained taking the internal product of (6) with other set of functions, named weight functions, in the L domain, then:

$$\sum_n \alpha_n \langle w_m, Lf_n \rangle = \langle w_m, g \rangle\ . \quad (7)$$

The internal product is usually an integral of area. Equation (7) can be written in matrix form as:

$$\begin{aligned} [\langle w_m, Lf_n\rangle][\alpha_n] &= [\langle w_m, g\rangle], \\ [l][\alpha_n] &= \mathbf{g}\ . \end{aligned} \quad (8)$$

If the inverse of $[l]$ exists the a_n are obtained by:

$$[\alpha_n] = [l]^{-1}[\langle w_m, Lf_n\rangle]. \quad (9)$$

Comparing (8) and (1), and using the linearity of integral operator, the matrix of (9) can be written as:

$$[Z_{mn}](I_n) = (V_m)\ , \quad (10)$$

where $[Z_{mn}]$ and (V_m) are known as impedance and voltage matrices, respectively, and are defined by [5]:

$$\begin{aligned} Z_{mn} = -\frac{1}{j\omega\varepsilon}\int_{s_m} w_m \int_{s'_n} \alpha_n \big[R^2(k^2R^2-1-jkR)\mathbf{s}\bullet\mathbf{s}' + \\ (3+3jkR-k^2R^2)(\mathbf{R}\bullet\mathbf{s})(\mathbf{R}\bullet\mathbf{s}')\big]\frac{e^{-jkR}}{4\pi R^5}\,ds\,ds'\ , \end{aligned} \quad (11)$$

$$V_m = \int_{s_m} w_m E^i_{\tan}\, ds \quad (12)$$

Using (11) and (12), the current is:

$$(I_n) = [Z_{mn}]^{-1}(V_m) \quad (13)$$

To solve (13) it is necessary to define the base and weight functions; as is known, both functions are selected arbitrarily. The most widely used subdomain functions have been a subject of research. Some discussion of these may be found in [6,7], but the more often used functions are Dirac´s delta, pulse, triangle and piecewise sinusoidal. This paper make 16 combinations of these functions to solve equation (13), trying to find the best one. As a matter of comparison, we use the feed impedance of a half wavelength dipole obtained by the well known analytical method for thin wire antennas.

The four functions

The purpose of this paper is to show the performance of combining the four most used base and weight functions to establish the best combination, considering a time and resources computational efficiency, and reliable solution. The used functions are described in the following paragraphs.

Dirac's delta: This function is the most used as weight function, because it reduces in one the number of integrations:

$$\int_{\Delta s} \delta(s - s_m) ds = \begin{cases} 1 & \text{if} \quad s_m \in \Delta s \\ 0 & \text{elsewhere} \end{cases} \tag{14}$$

Pulse function: This is the most common base function in the literature, and is given by:

$$i_n(s') = \begin{cases} 1 & \text{if} \quad (n-1)\Delta s' \le s' < n\Delta s' \\ 0 & \text{elsewhere} \end{cases} \tag{15}$$

Linear or triangular function: It attempts to use a softer function at the cost of greater complexity, and is defined by:

$$i_n(s') = \begin{cases} \dfrac{s' - s'_{n-1}}{s'_n - s'_{n-1}} & \text{if} \quad s'_{n-1} \le s' \le s'_n \\ \dfrac{s'_{n+1} - s'}{s'_{n+1} - s'_n} & \text{if} \quad s'_n \le s' \le s'_{n+1} \\ 0 & \text{elsewhere} \end{cases} \tag{16}$$

Piecewise sinusoidal: It is a more complicated function with a higher computational complexity, but many authors suppose that it gives a more exact solution. It is represented by:

$$i_n(s') = \begin{cases} \dfrac{sen\left[k\left(s' - s'_{n-1}\right)\right]}{sen\left[k\left(s'_n - s'_{n-1}\right)\right]} & \text{if} \quad s'_{n-1} \le s' \le s'_n \\ \dfrac{sen\left[k\left(s'_{n+1} - s'\right)\right]}{sen\left[k\left(s'_{n+1} - s'_n\right)\right]} & \text{if} \quad s'_n \le s' \le s'_{n+1} \\ 0 & \text{elsewhere} \end{cases} \tag{17}$$

Computational results

The following Figures show the results for the 16 combinations we make, for real and imaginary components; they are presented in eight figures using one as weight function and the other four as base functions. The horizontal coordinate shows the relationship between the length and wire's radius. The horizontal straight line is the reference impedance for a half wavelength dipole, using the analytical method. As is known, the real part in this case is $73\ \Omega$ and $43\ \Omega$ for imaginary part.

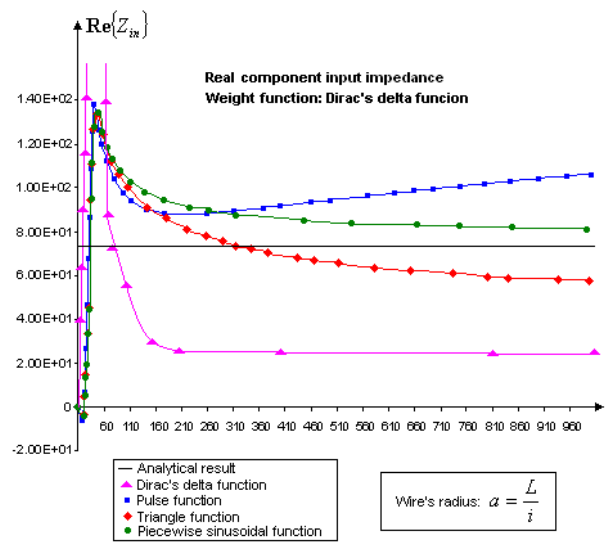

Fig. 2. Real component input impedance for Dirac's delta as weight function.

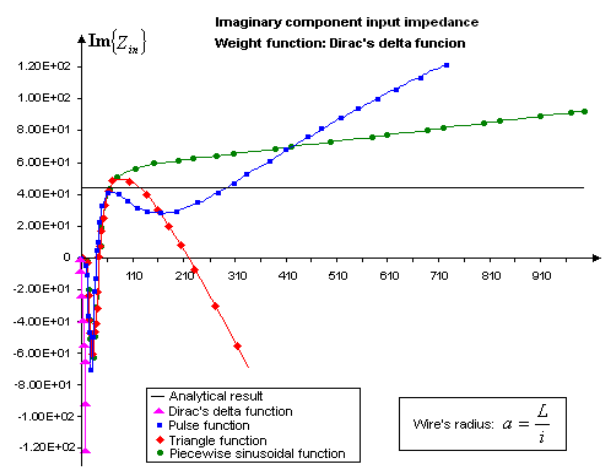

Fig. 3. Imaginary component input impedance for Dirac's delta as weight function.

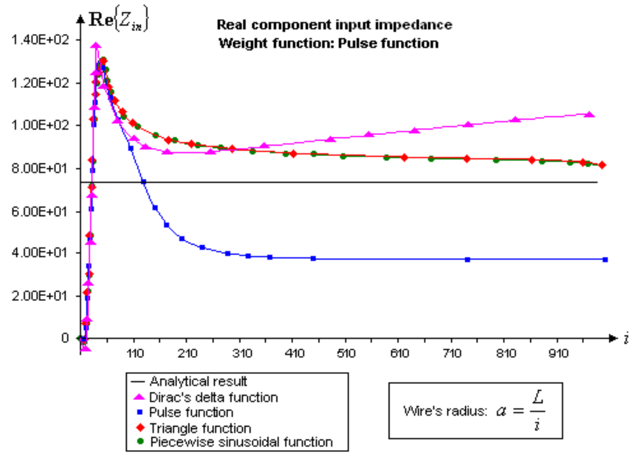

Fig. 4. Real component input impedance for pulse function as weight function.

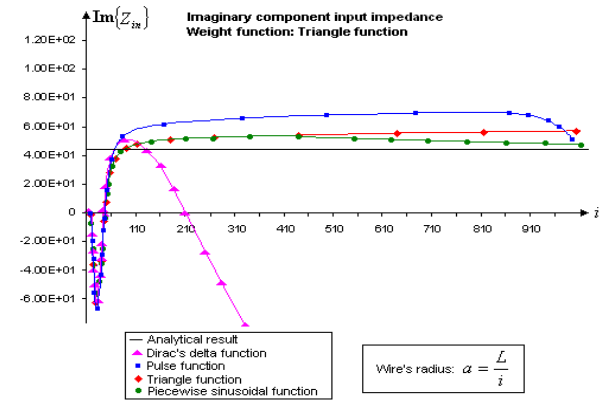

Fig. 7. Imaginary component input impedance for triangular function as weight function.

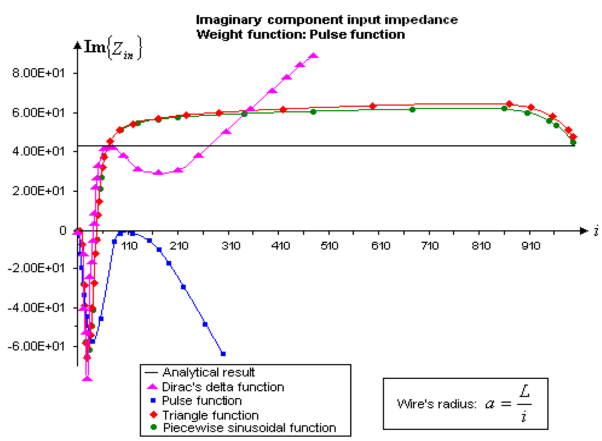

Fig. 5. Imaginary component input impedance for pulse function as weight function.

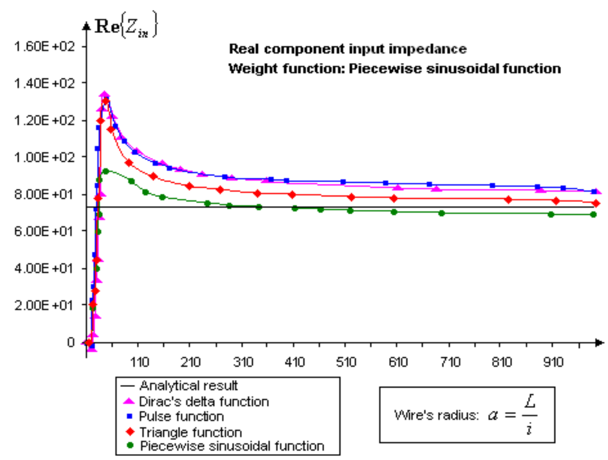

Fig. 8. Real component input impedance for piecewise sinusoidal function as weight function.

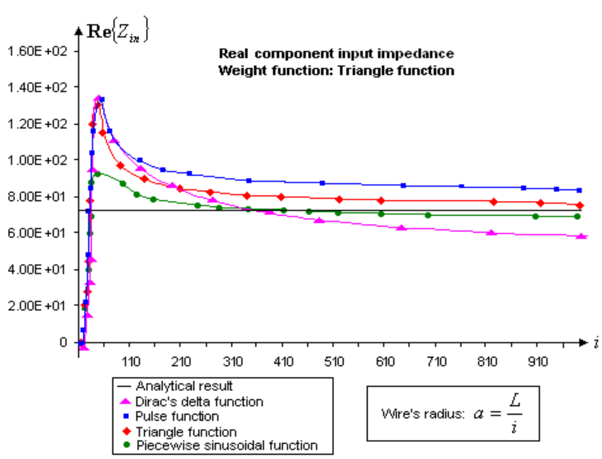

Fig. 6. Real component input impedance for triangular function as weight function.

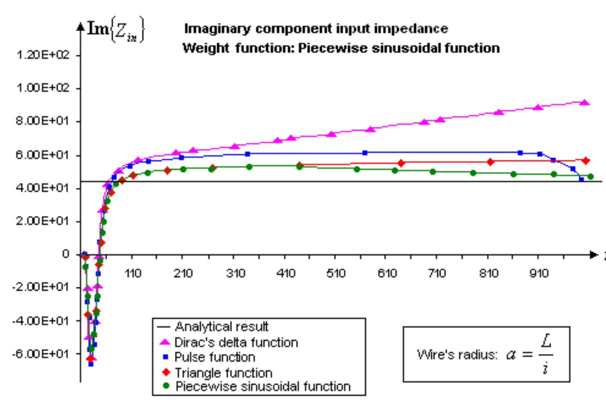

Fig. 9. Imaginary component input impedance for piecewise sinusoidal function as weight function.

As we can see, in computational results the less reliable is de Dirac's delta as weight function, although pulse, triangular and sinusoidal base functions are close to the analytical solution. The curves for the pulse weight functions are closer and softer than the former. It is interesting to see that the triangular and sinusoidal base functions respond almost in the same way, but there is a 15% to 30% deviation.

The triangular and sinusoidal weight functions respond almost in the same way, but Dirac's delta has higher desviation, mainly for the triangle function. It is very easy to conclude that the best solution is the piecewise sinusoidal function for both, the base and weight functions (Galerkin procedure), but the computational time is very high, compared with the pulse or delta procedures, Table 1 shows the computational time difference using a personal computer Pentium 4 running to 1.8 GHz and 256 MB of RAM. We use the pulse as a base and delta as a weight functions and reference time of approximately one minute.

Table 1.
Time comparison

		Base Functions			
Weight Functions		Delta	Pulse	Linear	Sine
	Delta	0.1	1	2	2
	Pulse	1	40	80	80
	Linear	2	80	160	160
	Sine	2	80	160	160

The conclusion is evident. We have presented a comparison analysis for the most popular base and weight functions, considering the reliability and computer time consumption. As can be seen, the reliability runs contrary to time consumption. Under our machine conditions, it takes almost fifteen minutes for the best solution. This is something that requires attention, although we think that it is necessary to search for a more efficient integration procedure than the Simpson's rule we are using. Actually we are working with other integration methods to reduce the inefficient times.

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