ON CERTAIN CLASS OF ANALYTIC FUNCTIONS

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Received February 2005, accepted September 2005

Communicated by Prof. Dr. Q. K. Ghori

Abstract: $P_k^{\alpha}[A,B]$ and $Q_{\alpha}^{k}[A,B]$ denote classes of functions analytic in the disc $E=\{z:|z|<1\}$ defined by a bounded radius rotation functions. In this paper we have obtained the distortion theorems, coefficients estimate, some radius problems, geometrical properties and studied convolution conditions.

***Keywords**:* Analytic, starlike, convex, positive real part function, bounded radius rotation, convolution

Introduction

Let A denote the class of analytic functions f(z) in $E=\{z:|z|<1\}$, given by

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n\ , \tag{1}$$

and let S, S* and C be classes of functions in A, which are respectively univalent, starlike and convex in the unit disc E.

Janowski [4] introduced the class $P[A,B]$ as follows:

Definition 1

An analytic function in E given by the form $P(z)=1+C_1Z_1+C_2Z^2+\ldots$ belongs to $P[A,B]$ if it satisfies the condition

$$p(z)=\frac{1+Aw(z)}{1+Bw(z)},\ -1\le B<A\le 1,$$

where, w(0)=0 and |w(z)|≤1.P[1,-1]=P (the class of analytic function with positive real part satisfying Rep(z)>0).

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Definition 2

An analytic function in E given by (1) belongs to S*[A,B],-1≤B<A≤1, if and only if, $\frac{zf'(z)}{f(z)} \in P[A,B]$ and S*[1,-1]= S*. Also it is well known that an analytic function given by (1) belongs to C[A,B] , if and only if, $\frac{(zf'(z))'}{f'(z)} \in P[A,B]$ and C[1,-1]=C.

Definition 3

A function f∈A is close to convex denoted by K[A,B,C,D] if ∃ a starlike function g(z)∈S*[C,D] such that $\frac{zf'(z)}{g(z)} \in P[A,B]$ and K[1,-1,1.-1]=K (the well known close to convex class due to Kaplan).

Definition 4

Let $P_k(\alpha), k \geq 2$ and 0<α≤1, be the class of functions p analytic in E and have the representation

$$p(z) = \frac{1}{2}\int_{-\pi}^{\pi} \frac{1+(1-2\alpha)ze^{-it}}{1-ze^{-it}} d\mu(t) \quad ,$$

where $\mu(t)$ is a function with bounded variation on $[-\pi, \pi]$ and satisfies the conditions $\int_{-\pi}^{\pi} d\mu(t) = 2 \quad , \quad \int_{-\pi}^{\pi} |d\mu(t)| \leq k$. We note that $k \geq 2$ and $P_2(\alpha) = P[1-2\alpha, -1] = P(\alpha)$ are the class of analytic function with positive real part greater than α . It can easily be seen [5] that $p \in P_k(\alpha)$, if and only if, there exist two analytic functions $p_1, p_2 \in P(\alpha)$ such that

$$p(z) = \frac{k+2}{4} p_1(z) - \frac{k-2}{4} p_2(z)$$

Let $R_k(\alpha)$ denote a subclass of A of functions of bounded radius rotation of order α . Then f∈ $R_k(\alpha)$, if and only if,

$$\frac{zf'(z)}{f(z)} \in P_k(\alpha) \quad , \quad k \geq 2 \quad , \quad z \in E. \tag{2}$$

I t is clear that $R_2(\alpha)$ =S*(α).

Let f be given by (1) and g given by $g(z) = z + \sum_{n=2}^{\infty} a_n z^n \in$ A . Then the convolution f∗g is defined by (f*g) (z) $= z + \sum_{n=2}^{\infty} a_n b_n z^n$.

Definition 5

Let f∈A. Then f belongs to $P_k^{\alpha}[A, B]$ if it satisfies the condition

$$\frac{f(z)}{g(z)} = \frac{1 + Aw(z)}{1 + Bw(z)},$$

where g ∈ $R_k(\alpha), -1 \le B < A \le 1, w(z)$ is regular, w(0)=0 and w(z)≤1 and 0<α ≤1.

Definition 6

Let $Q_k^{\alpha}[A, B]$ denote the class of functions $F(z) = z^{-1} + c_0 + c_1 z + c_2 z^2 + \ldots$, which are regular in $0 < |z| < 1$ and satisfy the condition

$$\frac{F(z)}{G(z)} = \left[\frac{1 + Aw(z)}{1 + Bw(z)}\right]^{-1},$$

where $-1 \le B < A \le 1, w(z)$ is regular in $0 < |z| < 1$ and $G(z) = z^{-1} + d_0 + d_1 z + d_2 z^2 + \ldots$, is of bounded radius rotation of order α , i.e.

$$-\frac{zG'(z)}{G(z)} \in P_k(\alpha) \quad , \quad 0 < |z| < 1.$$

Distortion theorem for the class $P_k^{\alpha}[A, B]$

Theorem 1

If $f \in P_k^{\alpha}[A, B]$, then for $|z| = r$, $0 < r < 1$

$$\frac{1 - Ar}{1 - Br} \frac{(1 - r)^{(k-2)(1-\alpha)/2}}{(1 + r)^{(k+2)(1-\alpha)/2}} \le |f(z)| \le \frac{1 + Ar}{1 + Br} \frac{(1 + r)^{(k-2)(1-\alpha)/2}}{(1 - r)^{(k+2)(1-\alpha)/2}} \quad \ldots \quad (3)$$

This result is sharp.

Proof

Since $f \in P_k^{\alpha}[A, B]$, we have

$$\frac{f(z)}{g(z)} = \frac{1+Aw(z)}{1+Bw(z)} \quad , \quad -1 \le B < A \le 1 \quad .,$$

where g∈. $R_k(\alpha)$. By Schwarz's lemma, we have $|w(z)| \le |z|$.

If $p(z) = \frac{1+Aw(z)}{1+Bw(z)}$, $-1 \le B < A \le 1$,then it is well known [4] that p ∈ P[A,B] and satisfies

$$\frac{1-Ar}{1-Br} \le |p(z)| \le \frac{1+Ar}{1+Br} \quad \ldots \quad (4)$$

Further if $g(z)$ is a function of bounded radius rotation of order α , then by [7]

$$\frac{(1-r)^{(k-2)(1-\alpha)/2}}{(1+r)^{(k+2)(1-\alpha)/2}} \le |g(z)| \le \frac{(1+r)^{(k-2)(1-\alpha)/2}}{(1-r)^{(k+2)(1-\alpha)/2}} \quad \ldots \quad (5)$$

equations (4),(5) together imply the inequality (3).

This result is sharp, if we take

$$p(z) = \frac{1+Az}{1+Bz} \text{ and } g(z) = \frac{(1+\theta_1 z)^{(k-2)(1-\alpha)/2}}{(1+\theta_2 z)^{(k+2)(1-\alpha)/2}} \quad , \quad |\theta_1| = |\theta_2| = 1.$$

Remarks

1. On taking $k = 2$, we have a result of Ganesan [2].
2. On taking $k = 2$, $B = -\lambda\beta$ and $A = \beta$ with $w(z)$ replaced by $-w(z)$, we get the result of Goel and Sohi [3].

Coefficient estimates for the class $P_k^\alpha[A,B]$

To find the coefficient estimates for the class $P_k^\alpha[A,B]$, we need the following lemmas:

Lemma 1 [4]

Let $p \in P[A,B]$ and $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$. Then $|c_n| \le A - B$.

Lemma 2

If $p \in P_k(\alpha)$, $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$, then $|c_n| \le k(1-\alpha)$.

Proof

This can be easily seen using Lemma 1 and the relation

$$p(z)=\frac{k+2}{4}p_1(z)-\frac{k-2}{4}p_2(z)$$

. with A= 1- 2α and B=-1.

Using Lemma 2, we can prove Lemma 3

Lemma 3

Let $g\in R_k(\alpha), g(z)=z+b_2z^2+b_3z^3+\dots$. Then

$|b_2|\le k(1-\alpha)$ and $|b_3|\le \frac{k(1-\alpha)}{2}(k-k\alpha+1)$.

Proof

Let $g\in R_k(\alpha)$. Then $zg'(z)=P(z)g(z), P(z)\in P_k(\alpha)$. If $g(z)=z+b_2z^2+\dots$ and $p(z)=1+c_1z+c_2z^2+\dots$, then

$$z+2b_2z^2+3b_3z^3+\dots=(z+b_2z^2+b_3z^3+\dots)(1+c_1z+c_2z^2+\dots)$$

Equating the coefficient of z^2 and z^3 on both sides and using Lemma1 and Lemma 2, we have

$$2b_2=c_1+b_2$$
$$|b_2|=|c_1|\le k(1-\alpha)$$

and $3b_3=b_3+c_1b_2+c_2$

$$|b_3|=\left|\frac{b_2c_1+c_2}{2}\right|\le\frac{k^2(1-\alpha)^2+k(1-\alpha)}{2}=\frac{k(1-\alpha)}{2}(k(1-\alpha)+1).$$

Theorem 2

Let $f\in P_k^{\alpha}[A,B]$, where $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$. Then

$$|a_2|\le(1-\alpha)k+(A-B)$$

and

$$|a_3|\le(A-B)+k(1-\alpha)(A-B)+\frac{k(1-\alpha)}{2}(k-k\alpha+1)$$

These bounds are sharp.

Proof

Since $f \in P_k^{\alpha}[A,B]$, there exists a function $g \in R_k(\alpha)$ such that $f(z) = g(z)p(z) \quad , \quad p \in P[A,B]$.

If $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$

and $\quad p(z) = 1 + c_1 z + c_2 z^2 + \dots ,$

then

$$z + a_2 z^2 + a_3 z^3 + \dots = (z + b_2 z^2 + b_3 z^3 + \dots)(1 + c_1 z + c_2 z^2 + \dots)$$

Equating the coefficient of z^2 and z^3 on both sides and using Lemma 1 and Lemma 3 we have

$$a_2 = b_2 + c_1$$
$$|a_2| = (1-\alpha)k + (A-B)$$

and
$$a_3 = c_2 + b_2 c_1 + b_3$$
$$|a_3| \le (A-B) + k(1-\alpha)(A-B) + \frac{k(1-\alpha)}{2}(k - k\alpha + 1) .$$

This result is sharp as can be seen by the function

$$f(z) = \frac{(1-z)^{(k-2)(1-\alpha)/2}}{(1+z)^{(k+2)(1-\alpha)/2}} \frac{1+Az}{1+Bz} .$$

Remarks

i. If $k = 2$, this result agrees with the result of Ganesan [2] and when $k = 2$, $A = \beta$, $B = -\lambda\beta$, these results correspond to the result of Goel and Sohi [3].

ii. If $B = 0$, we get $\dfrac{f(z)}{g(z)} = 1 + Aw(z)$ and if $k = 2$ in E, the inequality $|a_n| \le A(n-1) + n$, $n \ge 2$ with sharp bounds as discussed in [3] is also obtainable.

Argument of $\boldsymbol{\dfrac{f(z)}{z}}$ when $\boldsymbol{f \in P_k^{\alpha}[A,B]}$

To discuss the argument of the class $P_k^{\alpha}[A,B]$, we need the following Lemma:

Lemma 4

Let $f \in R_k(\alpha)$. Then

$$\left|\arg \frac{f(z)}{z}\right| \le k(1-\alpha)\sin^{-1} r .$$

Proof

It is well known that if $f \in R_k(\alpha)$, then there exist two functions $s_1, s_2 \in S^*(\alpha)$ such that

$$f(z) = \frac{(s_1(z))^{\frac{k+2}{4}}}{(s_2(z))^{\frac{k-2}{4}}} .$$

Thus
$$\left|\arg \frac{f(z)}{z}\right| = \left|\frac{k+2}{4}\arg \frac{s_2(z)}{z} - \frac{k-2}{4}\arg \frac{s_2(z)}{z}\right|$$
$$\le \frac{k+2}{4}\left|\arg \frac{s_2(z)}{z}\right| + \frac{k-2}{4}\left|\arg \frac{s_2(z)}{z}\right| .$$

It is known [8] that if $s \in S^*(\alpha)$, then

$$\left|\arg \frac{s(z)}{z}\right| \le 2(1-\alpha)\sin^{-1} r .$$

Hence
$$\left|\arg \frac{f(z)}{z}\right| \le k(1-\alpha)\sin^{-1} r .$$

Sharpness is satisfied for $f(z) = \dfrac{(1+\theta_1 z)^{(1-\alpha)\left(\frac{k-2}{2}\right)}}{(1+\theta_2 z)^{(1-\alpha)\left(\frac{k+2}{2}\right)}}$.

Lemma 5 [4]

Let $p \in P[A, B]$. Then

$$\left|\arg \frac{p(z)}{z}\right| \le \sin^{-1} \frac{(A-B)r}{1-ABr^2} .$$

Using Lemma 4 and Lemma 5, we can prove

Theorem 3

Let $f \in P_k^{\alpha}[A,B]$. Then

$$\left|\arg\frac{f(z)}{z}\right| \le k(1-\alpha)\sin^{-1} r + \sin^{-1}\frac{(A-B)r}{1-ABr^2}.$$

Proof

Since $f \in P_k^{\alpha}[A,B]$, therefore

$f(z) = g(z)p(z)$, $p(z) \in P[A,B]$ and $g \in R_k(\alpha)$. By Lemma 4, we have

$$\left|\arg\frac{g(z)}{z}\right| \le k(1-\alpha)\sin^{-1} r \qquad \dots \qquad (6)$$

and by Lemma 5, we have

$$\left|\arg p(z)\right| \le \sin^{-1}\frac{(A-B)r}{1-ABr^2} \qquad \dots \qquad (7)$$

Using (6) and (7), we have the result.

Sharpness follows by taking

$$\frac{f(z)}{g(z)} = \frac{1+A\theta_1 z}{1+B\theta_1 z} \quad , \quad |\theta_1| = 1 \qquad \dots \qquad (8)$$

and

$$g(z) = \frac{(1+\theta_2 z)^{(1-\alpha)(k-2)/2}}{(1+\theta_2 z)^{(1-\alpha)(k+2)/2}} \quad , \quad |\theta_2| = 1.$$

Then

$$\arg\frac{f(z)}{g(z)} = \sin^{-1}\frac{(A-B)r}{1-ABr^2}$$

and

$$\arg\frac{g(z)}{z} = \arg(1+\theta_2 z)^{(1-\alpha)(k-2)/2} + \arg(1+\theta_2 z)^{(1-\alpha)(k+2)/2}$$

Using Lemma 4, we have

$$\arg \frac{g(z)}{z} = (1-\alpha)k \sin^{-1} r \qquad \ldots \qquad (9)$$

Using (8) and (9), we have that

$$\arg \frac{f(z)}{z} = (1-\alpha)k \sin^{-1} r + \sin^{-1} \frac{(A-B)r}{1-ABr^2}.$$

Remark

For $k = 2$ again this result agrees with the result in [2], and when $A = \beta > 0$, $B = -\lambda\beta$ and replacing $w(z)$ by $-w(z)$, we have the result of Goel and Sohi [3].

Some radius problems for $P_k^\alpha[A,B]$

Lemma 6 [1]

Let $p \in P[A,B]$. Then for $z \in E$

$$\operatorname{Re}\left\{\alpha p(z) + \beta \frac{zp'(z)}{p(z)}\right\} > \begin{cases} \dfrac{\alpha - [(A-B)\beta + 2\alpha A]r + \alpha A^2 r^2}{(1-Ar)(1-Br)} & \textit{if} \quad R_1 < R_2, \\ \beta \dfrac{A+B}{A-B} + \dfrac{2}{(A-B)(1-r^2)}\left\{(L_1 K_1)^{1/2} - \beta(1-ABr^2)\right\} & \textit{if} \quad R_2 < R_1. \end{cases}$$

where

$$R_1 = \left(\frac{L_1}{K_1}\right)^{1/2}, R_2 = \frac{1-Ar}{1-Br}, L_1 = (1-A)(1+Ar^2) \text{ and } K_1 = (1-B)(1+Br^2)$$

This result is sharp.

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Lemma 7 [7]

Let g $\in R_k(\alpha)$. Then

$$\operatorname{Re} \frac{zg'(z)}{g(z)} \geq \frac{1 - k(1-\alpha)r + (1-2\alpha)r^2}{1-r^2}.$$

Further, since, g $\in R_k(\alpha)$ implies $\frac{zg'(z)}{g(z)} = f(z) \in P_k(\alpha)$, we have for all f$\in P_k(\alpha)$

$$\text{Re } f(z) \geq \frac{1-k(1-\alpha)r+(1-2\alpha)r^2}{1-r^2}.$$

Theorem 4

Let $f \in P_k^{\alpha}[A,B]$. Then

$$\text{Re}\frac{zf'(z)}{f(z)} \geq \begin{cases} M_1(r) & for \quad R_1 \leq R_2 \\ M_2(r) & for \quad R_2 \leq R_1 \end{cases},$$

where

$$M_1(r) = \frac{1-k(1-\alpha)r+(1-2\alpha)r^2}{1-r^2} - \frac{(A-B)r}{(1-Ar)(1-Br)},$$

$$M_2(r) = \frac{1-k(1-\alpha)r+(1-2\alpha)r^2}{1-r^2} + \frac{A+B}{A-B} + \frac{2}{(1-r^2)(A-B)}\left[(L_1K_1)^{\frac{1}{2}1} - (1-ABr^2)\right]$$

and $R_1, R_2, L_1 and K_1$ are defined in Lemma 6 .

Proof

Since $f \in P_k^{\alpha}[A,B]$, there exists a function $g \in R_k(\alpha)$ such that

$$\frac{f(z)}{g(z)} = P(z) \in P[A,B]$$

Using logarithmic differentiation, we obtain

$$\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \frac{zp'(z)}{p(z)}$$

and

$$\text{Re}\frac{zf'(z)}{f(z)} \geq \min \text{Re}\frac{zg'(z)}{g(z)} + \min \text{Re}\frac{zp'(z)}{p(z)}.$$

Using Lemma 6 with $\alpha = 0$, $\beta = 1$ and Lemma 7, we have the result.

Sharpness of the bounds follow if we choose $g_i(z)(i=1,2)$, of bounded radius rotation of order α such that

Case 1: If $R_1 \le R_2$, we take $P_1(z) = \dfrac{1+Az}{1+Bz}$, and $\dfrac{zg_1'(z)}{g_1(z)} = \dfrac{1+(1-\alpha)z+(1-2\alpha)z^2}{1-r^2}$.

Then $\dfrac{zP_1'(z)}{P_1(z)} = \dfrac{(A-B)z}{(1+Az)(1+Bz)}$ .Thus at z= -r, $\operatorname{Re}\dfrac{zP_1'(z)}{P_1(z)} = \dfrac{-(A-B)r}{(1-Ar)(1-Br)}$.

Case 2: If $R_2 \le R_1$, we take $p_2(z) = \dfrac{f_2(z)}{g_2(z)} = \dfrac{1+Aw_1(z)}{1+Bw_1(z)}$ and

$\dfrac{zg'(z)}{g(z)} = \dfrac{1+k(1-\alpha)w_1(z)+(1-2\alpha)w_1^2}{1-w^2(z)}$ with $w_1(z) = \dfrac{z(z-c_1z)}{(1-c_1z)}$, where c_1 defined by the condition

$\operatorname{Re}\left[\dfrac{1+Aw_1(z)}{1+Bw_1(z)}\right] = R_1$ at z=-r.

Now $\dfrac{zp_2'(z)}{p_2(z)} = \dfrac{(A-B)zw_1'(z)}{(1+Aw_1(z))(1+Bw_1(z))}$.

In fact from the inequalities $R_2 \le R_1 \le c+p$, where $c = \dfrac{1-ABr^2}{1-B^2r^2}$, $p = \dfrac{(A-B)r}{1-B^2r^2}$ and we have $\dfrac{1-Ar}{1-Br} \le \dfrac{1+AT}{1+BT} \le \dfrac{1+Ar}{1+Br}, T = w_1(-r)$.

Hence $|T| \le r$ and $T^2 \le r^2$ which yields

$$\frac{r^2(r+c_1)^2}{(1+rc_1)^2} \le r^2 \quad . \text{ Thus } |c_1| \le 1$$

Further $|zw_1'(z) - w_1(z)| = \dfrac{|z|^2 - |w(z)|^2}{1-|z|^2}$, for $w_1(z) = \dfrac{z(z-c_1)}{(1-c_1z)}, |c_1| \le 1$.

$$w_1(-r) = T = \frac{1-R_1}{BR_1 - A} = \frac{r(r-c_1)}{(1+c^2{}_1)}.$$

Hence $c_1 = \dfrac{r^2-T}{r(T-1)}$ *and* $\dfrac{r^2-T^2}{(1-r^2)} = \dfrac{r^2(1-q^2)}{(1+qr\ )^2}$ and $[zw_1'(z) - w_1(z)]_{z=1r} = \dfrac{r^2-T^2}{(1-r^2)}$.

Now $\operatorname{Re}\left[\frac{zp_2'(z)}{p_2(z)}\right]=\frac{(A-B)}{(1-AT)(1-BT)}\left\{T-\frac{r^2-T^2}{1-r^2}\right\}$

Using $T=\frac{1-R_1}{BR_1-A}$ with $R_1=\sqrt{\frac{(1-A)(1+Ar^2)}{(1-B)(1+Br^2)}}$ (see [1]),

and simplifying, we have $\operatorname{Re}\left[\frac{zp_2'(z)}{p_2(z)}\right]_{z=-r}=\frac{A+B}{A-B}+\frac{2}{(1-r^2)(A-B)}\left\{(L_1K_1)^{\frac{1}{2}}-(1-ABr^2)\right\}$,

where $L_1=(1-A)(1+Ar^2), K_1=(1-B)(1+Br^2)$, (see[1]).

Thus the equality in our theorem holds at z=-r for

$f_1(z)=\frac{1+Az}{1+Bz}g_1(z), if R_1\le R_2$

and for $f_2(z)=\frac{1+Aw_1(z)}{1+Bw_1(z)}g_2(z), if R_2\le R_1$,where $g_1(z), g_2(z)\in R_k(\alpha)$.

Theorem 5

If $f\in P_k^\alpha[A,B]$, then f is starlike in

$$|z|<\begin{cases} r_1 & for \quad R_1\le R_2 \\ r_2 & for \quad R_2\le R_1 \end{cases},$$

where R_1 and R_2 are defined as in Lemma 6 and r_1, r_2 are respectively the positive roots of the following two equations

$$(1-k(1-\alpha)r+(1-2\alpha)r^2)(1-Ar)(1-Br)-(A-B)r(1-r^2)=0$$

$$(1-k(1-\alpha)r+(1-2\alpha)r^2)(A-B)+(1-r^2)(A+B)+2\left[(L_1K_1)^{1/2}-(1-ABr^2)\right]=0 \quad ,$$

where K_1 and L_1 are defined in Lemma 6. This result is sharp.

Proof

It follows from Theorem 4 that if $f\in P_k^\alpha[A,B]$, then $\operatorname{Re}\frac{zf'(z)}{f(z)}\ge M_1(r), if R_1\le R_2$ and

$\operatorname{Re}\frac{zf'(z)}{f(z)} \geq M_2(r), if R_2 \leq R_1$. Then

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \geq \frac{(1-k(1-\alpha)r+(1-2\alpha)r^2)(1-Ar)(1-Br)-(A-B)r(1-r^2)}{(1-r^2)(1-Ar)(1-Br)} > 0, for all |z| < r_1$$

If $R_1 \leq R_2$ and

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \geq \frac{(1-k(1-\alpha)r+(1-2\alpha)r^2)(A-B)+(1-r^2)(A+B)+2\left[(L_1K_1)^{\frac{1}{2}}-(1-ABr^2)\right]}{(1-r^2)(A-B)} > 0. \quad \text{for}$$

all, if $R_2 \leq R_1$.

For special cases see [2] and [3].

Lemma 8

Let $g_1(z)$ and $g_2(z) \in R_k(\alpha)$. Then $G(z) = (g_1(z))^{\rho}(g_2(z))^{\gamma} z^{1-(\rho+\gamma)}$ belongs to $R_k(\alpha_1)$ where $\alpha_1 = 1-(1-\alpha)(\rho+\gamma)$.

Proof

A logarithmic differentiation yields

$$\frac{zG'(z)}{G(z)} = \rho\frac{zg_1'(z)}{g_1(z)} + \gamma\frac{zg_2'(z)}{zg_2(z)} + (1-(\rho+\gamma))$$

$$= \rho K_1(z) + \gamma K_2(z) + (1-(\rho+\gamma))$$

where $K_1 and K_2 \in P_k(\alpha)$. From the definition of $P_k(\alpha)$, there exists $h_i, i = 1,2,3,4 \in P(\alpha)$ such that

$$\frac{zG'(z)}{G(z)} = \rho\left[\frac{k+2}{4}h_1(z) - \frac{k-2}{4}h_2(z)\right] + \gamma\left[\frac{k+2}{4}h_3(z) - \frac{k-2}{4}h_4(z)\right] + (1-(\rho+\gamma))$$

It is well known that if $h \in P(\alpha)$, then h(z) can be written as

$h(z) = (1-\alpha)p(z) + \alpha, where \operatorname{Re} p(z) > 0$

and

$$\frac{zG'(z)}{G(z)} = \rho\left[\frac{k+2}{4}\left[(1-\alpha)p_1(z)+\alpha\right]\right] - \rho\,\frac{k-2}{4}\left[(1-\alpha)p_2+\alpha\right] + \gamma\,\frac{k+2}{4}\left[(1-\alpha)p_3+\alpha\right] - \gamma\,\frac{k-2}{4}\left[(1-\alpha)p_4+\alpha\right] + \left(1-(\rho+\gamma)\right).. \tag{10}$$

Since the class P is a convex set, then

$$\frac{\rho p_1(z)+\gamma p_3(z)}{\rho+\gamma} = H_1(z) \text{ and } \frac{\rho p_2(z)+\gamma p_4(z)}{\rho+\gamma} = H_2(z),$$

where $\operatorname{Re} H_I(z) > 0, i = 1,2.$ Hence(10) can be written as

$$\begin{aligned}\frac{zG'(z)}{G(z)} &= \frac{k+2}{4}\left[(1-\alpha)(\rho+\gamma)H_1(z) + \left[1-(1-\alpha)(\rho+\gamma)\right]\right] \\ &- \frac{k-2}{4}\left[(1-\alpha)(\rho+\gamma)H_2(z) + \left[1-(1-\alpha)(\rho+\gamma)\right]\right] \\ &= \frac{k+2}{4}T_1(z) + \frac{k-2}{4}T_2(z), T_1, T_2 \in P(\alpha_1) \, and \\ &\alpha_1 = 1-(1-\alpha)(\rho+\gamma)\end{aligned}$$

This shows that $G \in R_k(\alpha_1)$.

Theorem 6

Let $f_1, f_2 \in P_k^{\alpha}[A, B]$. Then

$$F(z) = (f_1(z))^{\rho} (f_2(z))^{\gamma} z^{1-(\rho+\gamma)}$$

belongs to $P_k^{\alpha_1}[A, B]$, where $\alpha_1 = 1-(1-\alpha)(\rho+\gamma)$.

Proof

Let G (z) be given by $G(z) = (g_1(z))^{\rho} (g_2(z))^{\gamma} z^{1-(\rho+\gamma)}$. Then

$$\begin{aligned}\frac{F(z)}{G(z)} &= \left(\frac{f_1(z)}{g_1(z)}\right)^{\rho} \left(\frac{f_2(z)}{g_2(z)}\right)^{\gamma} \\ &= (h_1(z))^{\rho} (h_2(z))^{\gamma} \qquad , \quad (\rho+\gamma) \le 1,\end{aligned}$$

where $h_1, h_2 \in P[A, B]$.

Hence $F \in P_k^{\alpha_1}[A,B]$, $\alpha_1 = 1-(1-\alpha)(\rho+\gamma)$.

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Some geometrical properties

In this part we shall investigate the behavior of $\arg f(z)$ at a point $w(\theta) = F(re^{i\theta})$ to the image Γ_r of the circle $Cr = \{z : |z| = r\}$, $0 \le r < 1$ and where θ is any number of the interval $(0, 2\pi)$ under the mapping by means of function f from the class $p_k^{\alpha}[A,B]$. We have

Theorem 7

If $F \in P_k^{\alpha}[A,B]$ and $0 \le r < 1$, then for $\theta_2 < \theta_1, \theta_1, \theta_2 \in [0, 2\pi]$

$$\arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1}) = \int_{\theta_1}^{\theta_2} \mathrm{Re}\left[\frac{re^{i\theta} f'(re^{i\theta})}{(re^{i\theta})}\right]$$

$$\ge -\pi + \{1-(1-\alpha)k + (1-2\alpha)\}(\theta_2 - \theta_1) + 2arcC\,\mathrm{os}\frac{A-B}{1-AB}$$

where -1≤B<A≤1 and 0<α≤1 .

Proof

If $f \in P_k^{\alpha}[A,B]$, then $\dfrac{f(z)}{g(z)} = p(z)$, where $p \in P[A,B]$.

Thus

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} = \mathrm{Re}\,\frac{zg'(z)}{g(z)} + \mathrm{Re}\,\frac{zp'(z)}{p(z)} \qquad \ldots \qquad (11)$$

Let $z = re^{i\theta}$, $0 < r < 1, \theta \in [0, 2\pi]$. Integrating (11) with respect to θ in the interval $1[\theta_1, \theta_2], \theta_1 < \theta_2$, we have

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\frac{re^{i\theta} f'(re^{i\theta})}{f(re^{i\theta})} d\theta = \arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1})$$

$$= \int_{\theta_1}^{\theta_2} \mathrm{Re}\frac{re^{i\theta} g'(re^{i\theta})}{g(re^{i\theta})} d\theta + \int_{\theta_1}^{\theta_2} \mathrm{Re}\frac{re^{i\theta} p'(re^{i\theta})}{p(re^{i\theta})} d\theta$$

Since $f \in R_k(\alpha)$, it follows that

$$\min_{g \in R_k(\alpha)} \int_{\theta_1}^{\theta_2} \mathrm{Re}\frac{re^{i\theta} g'(re^{i\theta})}{g(re^{i\theta})} d\theta \ge \frac{1-k(1-\alpha)r + (1-2\alpha)r^2}{1-r^2}(\theta_2 - \theta_1), \quad \text{See [7].}$$

Now in the second integral, we observe that

$$\frac{\partial}{\partial\theta}\arg p(re^{i\theta}) = \frac{\partial}{\partial\theta}\mathrm{Re}\left\{-i\ln p(re^{i\theta})\right\} = \mathrm{Re}\frac{re^{i\theta}p'(re^{i\theta})}{p(re^{i\theta})}.$$

Consequently

$$\int_{\theta_1}^{\theta_2}\mathrm{Re}\left[\frac{re^{i\theta}p'(re^{i\theta})}{p(re^{i\theta})}\right]d\theta = \arg p(re^{i\theta_2}) - \arg p(re^{i\theta_1})$$

and

$$\max_{p\in P[A,B]}\left|\int_{\theta_1}^{\theta_2}\mathrm{Re}\frac{re^{i\theta}p'(re^{i\theta})}{p(re^{i\theta})}d\theta\right| \le \max_{p\in P[A,B]}\left|\arg p(re^{i\theta_2}) - \arg p(re^{i\theta_1})\right|$$

Using Lemma 5, we have

$$\max_{p\in P[A,B]}\arg p(re^{i\theta}) = \sin^{-1}\frac{(A-B)r}{1-ABr^2}$$

$$\begin{aligned}\max_{p\in P[A,B]}\left|\int_{\theta_1}^{\theta_2}\mathrm{Re}\frac{re^{i\theta}p'(re^{i\theta})}{p(re^{i\theta})}d\theta\right| &\le \max_{p\in P[A,B]}\left|\arg p(re^{i\theta})\right| - \min_{p\in P[A,B]}\left|\arg p(re^{i\theta})\right| \\ &\le 2\sin^{-1}\frac{(A-B)r}{1-ABr} \\ &= \pi - 2\cos^{-1}\frac{(A-B)r}{1-ABr}\end{aligned}$$

Hence

$$\arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1}) \ge -\pi + 2Cos^{-1}\frac{(A-B)r}{1-ABr^2} + \frac{1-k(1-\alpha)r+(1-2\alpha)r^2}{1-r^2}(\theta_2-\theta_1).$$

The value of the right side is depending on the value of r and it takes its smallest value at $r=1$. Thereby we obtain the required result.

A convolution conditions for $p_k^{\alpha}[A,B]$

In 1973, Rushweyh and Sheil-Small [9] proved the polya-Schoenberg conjecture, namely, if f is convex or starlike or close to convex and ϕ is convex then $f*\phi$ belongs to the same class. In the following we shall prove the analogue of this conjecture for the class $p_k^{\alpha}[A,B]$ and give some of its applications. We need the following lemma with simple modification.

Lemma 9 [6]

Let f $\in R_k(\alpha)$. Then G $= f * \phi \in R_k(\alpha)$ where ϕ is convex in E

Theorem 8

Let $F \in P_k^{\alpha}[A,B]$ and ϕ is convex. Then $F * \phi \in P_k^{\alpha}[A,B]$.

Proof:

Let F$\in P_k^{\alpha}[A,B]$. Then F (z)=P(z) g(z) , where g belongs to $R_k(\alpha)$ and P(z)$\in$P[A,B]. It follows from the Lemma 9 that g$*\varphi$ $\in R_k(\alpha)$ .Then $\dfrac{F*\phi}{g*\phi} \in P[A,B]$.

Remark

As an application of Theorem 8, we have the following

(1) The family $P_k^{\alpha}[A,B]$ is invariant under the following operators.

$$F_1(f) = \int_0^z \frac{f(\xi)}{\xi} d\xi = (f * \phi_1)(z)$$

$$F_2(f) = \frac{2}{z}\int_0^z f(\xi) d\xi = (f * \phi_2)(z)$$

$$F_3(f) = \int_0^z \frac{f(\zeta) - f(x\zeta)}{\zeta - x\zeta} d\zeta \quad , \quad |x| \le 1 \quad , \quad x \ne 1$$

$$= (f * \phi_3)(z)$$

$$F_4(f) = \frac{1+c}{c}\int_0^z \xi^{c-1} f(\xi) d\xi \quad , \quad \mathrm{Re}\, c > 0$$

where $F(f_i(z)) = (f * \phi_i)(z)$ and $\phi_i\,(i = 1,2,3,4)$ are convex univalent functions which satisfy

$$\phi_1(z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n = -\log(1-z),$$

$$\phi_2(z) = \sum_{n=1}^{\infty} \frac{2}{n+1} z^n = \frac{-2[z + \log(1-z)]}{z},$$

$$\phi_3(z) = \sum_{n=1}^{\infty} \frac{1-x^n}{n(1-x)} z^n = \frac{1}{1-x}\log\frac{1-xz}{1-z} \quad , |x| \le 1, x \ne 1,$$

$$\phi_4(z) = \sum_{n=1}^{\infty} \frac{1+c}{n+c} z^n \quad , \mathrm{Re}\, c > 0.$$

Now let $D_\lambda F(z)=(1-\lambda)F(z)+\lambda zF'(z)=(\psi_\lambda * F)(z)$..(12)

where λ>0 and let $\psi_\lambda(z)=\dfrac{z[1-(1-\lambda z)]}{1-z^2}$. Then $\psi_\lambda(z)$ is convex if

$$|z|=r_\lambda=\frac{1}{2\lambda+\sqrt{4\lambda^2-2\lambda+1}} \quad \ldots \quad (13)$$

Thus, we have

(2) Let $F(z)\in P_k^\alpha[A,B]$. Then $D_\lambda F(z)=\psi_\lambda * F$ belongs to the same class for $|z|<r_\lambda$, where r_λ is given by (13).

Now let $\mu(F)=zF'(z)$. This differential operator can be written as $\mu(F)=\phi * F$,

where

$$\phi(z)=\sum_{n=1}^{\infty} nz^n=\frac{z}{1-z^2} \quad \ldots \quad (14)$$

It can be easily verified that the radius of convexity of ϕ is given by $r_c(\phi)=2-\sqrt{3}$. This fact together with Theorem 8 yields

(3) If $f\in P_k^\alpha[A,B]$ then $\phi * f\in P_k^\alpha[A,B]$ where ϕ is given by (14) if $|z|=r_c<2-\sqrt{3}$.

Radius of starlikeness for the class $Q_k^\alpha[A,B]$

Now we generalize the result of Goel and Sohi [3] and Ganesen [2] for the class $Q_k^\alpha[A,B]$. The following lemma can be easily derived.

Lemma 9

Let $s_i, i=1,2$ be given by $s_1(z)=z^{-1}+c_\circ+c_1z+c_2z^2+\ldots$ and $s_2(z)=z^{-1}+d_\circ+d_1z+d_2z^2+\ldots.$, and let $s_i, i=1,2$ satisfy $-\operatorname{Re}\dfrac{zs_i'(z)}{s_i(z)}>\alpha$. If $G(z)=z^{-1}+b_\circ+b_1z+b_2z^2+\ldots$ such that

$$G(z)=\frac{(s_1(z))^{\frac{k+2}{4}}}{(s_2(z))^{\frac{k-2}{4}}} \quad \ldots \quad (15)$$

then

$$-\frac{zG'(z)}{G(z)} \in P_k(\alpha) \quad .$$

Proof

Differentiating (15) logarithmically yields

$$\frac{zG'(z)}{G(z)} = \frac{k+2}{4}\frac{zs_1'(z)}{s_1(z)} - \frac{k-2}{4}\frac{zs_2'(z)}{s_2(z)} \quad .$$

This implies that

$$-\frac{zG'(z)}{G(z)} = \frac{k+2}{4}\left(-\frac{zs_1'(z)}{s_1(z)}\right) - \frac{k-2}{4}\left(-\frac{zs_2'(z)}{s_2(z)}\right)$$

or $$-\frac{zG'(z)}{G(z)} = \frac{k+2}{4}p_1(z) - \frac{k-2}{4}p_2(z) \ ,$$

where $\operatorname{Re} p_i(z) > \alpha$, $i = 1, 2$ and $-\dfrac{zG'(z)}{G(z)} \in P_k(\alpha)$.

Theorem 9

If $F \in Q_k^\alpha[A, B]$, then for $|z| = r < 1$

$$-\operatorname{Re}\frac{zF'(z)}{F(z)} \geq \begin{cases} \{M_1(r), \ forR_1 \leq R_2 \\ M_2(r), \ forR_2 \leq R_1 \end{cases} \quad ,$$

where

$$M_1(r) = \frac{1 - k(1-\alpha)r + (1-2\alpha)r^2}{1-r^2} - \frac{(A-B)r}{(1-Ar)(1-Br)},$$

$$M_2(r) = \frac{1 - k(1-\alpha)r + (1-2\alpha)r^2}{1-r^2} + \frac{A+B}{A-B} + \frac{2}{(1-r^2)(A-B)}\left[(L_1K_1)^{\frac{1}{2}1} - (1 - ABr^2)\right]$$

and $R_1, R_2, L_1 andK_1$ are defined in Lemma 6 .

Proof

Since $F \in Q_k^\alpha[A, B]$, therefore

$$p(z)=\left[\frac{F(z)}{G(z)}\right]^{-1}=\frac{1+Aw(z)}{1+Bw(z)}\text{, where } -1\le B<A\le 1 \qquad \ldots \qquad (16)$$

$w(z)$ is analytic in E and satisfies $w(0)=0$, $|w(z)|<1$,

Differentiating (16) logarithmically, we have

$$\frac{zp'(z)}{p(z)}=-\frac{zF'(z)}{F(z)}+\frac{zG'(z)}{G(z)}$$

or $$-\frac{zF'(z)}{F(z)}=-\frac{zG'(z)}{G(z)}+\frac{zp'(z)}{p(z)} \; .$$

Using Lemma 6, we have

$$-\operatorname{Re}\frac{zF'(z)}{F(z)}\ge-\operatorname{Re}\frac{zG'(z)}{G(z)}-\frac{(A-B)r}{(1-Ar)(1-Br)} \quad if \quad R_1\le R_2$$

$$\ge-\operatorname{Re}\frac{zG'(z)}{G(z)}+\frac{(A+B)}{(A-B)}+\frac{2\left[(L_1K_1)^{1/2}-(1-ABr^2)\right]}{(A-B)(1-r^2)}$$

and since G is of bounded radius rotation of order α , using Lemma 7 we have

$$\operatorname{Re}-\frac{zG'(z)}{G(z)}\ge\frac{1-(1-\alpha)kr+(1-2\alpha)r^2}{1-r^2},|z|<r \qquad \ldots \qquad (17)$$

Using (17), we have the required result. The bounds are sharp. This can be seen by choosing $G_1(z)$ of bounded radius variation of order α such that

$$-\frac{zG'(z)}{G(z)}\ge\frac{1-(1-\alpha)kz+(1-2\alpha)z^2}{1-z^2} \quad if \quad R_1\ge R_2,$$

$$-\frac{zG'(z)}{G(z)}\ge\frac{1-(1-\alpha)kw_1(z)+(1-2\alpha)w_1^2(z)}{1-w_1^2(z)} \quad if \quad R_2\ge R_1$$

and take $F_1(z)$ such that it satisfies

$$p_1(z)=\left[\frac{F_1(z)}{G_1(z)}\right]^{-1}=\frac{1+Az}{1+Bz} \quad if \quad R_1\le R_2$$

$$= \frac{1+Aw_1(z)}{1+Bw_1(z)} \quad if \quad R_2 \le R_1,$$

where $w_1(z) = \frac{z(1-c_1 z)}{1-c_1 z}$ with $|c_1| \le 1$. Proceeding in the same way as in proving the sharpness of Theorem 4, we can prove that this result is sharp.

Theorem 10

If $F \in Q_k^{\alpha}[A, B]$, then F is starlike for $|z| = r_i$<1,i=1,2

i. $0 < |z| < r_1 \quad for \quad R_1 \le R_2$
ii. $0 < |z| < r_2 \quad for \quad R_2 \le R_1$

where r_1 and r_2 are the smallest positive roots of the following equations respectively

$$\left[1-k(1-\alpha)r+(1-2\alpha)r^2\right](1-Ar)(1-Br)-(A-B)r(1-r^2)=0$$

$$\left[1-k(1-\alpha)r+(1-2\alpha)r^2\right](A-B)+(1-r^2)(A+B)+2\left[(L_1K_1)^{1/2}-(1-ABr^2)\right]=0$$

Proof

Using Theorem 9, we have

$\text{Re}-\frac{zF'(z)}{F(z)} \ge M(r)_1, whenR_1 \le R_2$ and $\text{Re}-\frac{zF'(z)}{F(Z)} \ge M_2(r) whenR_2 \ge R_1$. Hence

$\text{Re}\frac{zF'(z)}{F(z)} > 0$ For $|z| < r_i, i = 1,2$, and this gives a sufficient condition for any function F to be starlike. Proceeding in the same way as in Theorem 5, we obtain the required result.

Acknowledgements

The author is grateful to the referee for his valuable comments and suggestions.

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