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LANDAU LIFSHITZ ENERGY MOMENTUM PSEUDOTENSOR FOR METRICS WITH SPHERICAL SYMMETRY

J. H. Caltenco, J. I. López-Bonilla, R. Peña-Rivero and J. Rivera-Rebolledo

Sección de Estudios de Posgrado e Investigación, Escuela Superior de Ingeniería Mecánicay Eléctrica, Instituto Politécnico Nacional, Edif. Z-4, 3er Piso, Col. Lindavista, C.P. 07738 México, DF

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Abstract: We have calculated the Landau-Lifshitz pseudotensor for various spherically symmetric systems as preparation for a later study for the case of rotation, which may be of interest in astrophysics. The systems we considered are the static spherical star, the Schwarzschild geometry, the collapsing spherical dust ball of uniform density and the general pulsating or collapsing star.

Keywords: Landau-Lifshitz psudotensor, spherically symmetric systems, Schwarzschild geometry

Introduction

The point to point distribution of energy-momentum in the gravitational field is non-unique [1,2] in the theory of general relativity. This is inescapable because it is always possible to change coordinates to make the frame locally Lorentz at any chosen event. Gravitation must, however, make a contribution to the energy of a system since, for example, the mass of a star is less than the sum of the rest masses of its individual particles. In proving conservation laws of momentum and angular momentum for isolated systems, one can construct entities, which describe the energy-momentum content of the gravitational field. These entities are called energy-momentum pseudotensors [3-10]. The distribution of energy-momentum depends [1] on the choice of pseudotensor and on the choice of coordinates. Besides, the total momentum and angular momentum or the total energy radiated into the asymptotically flat space surrounding an isolated source also depends on the choice of pseudotensor or coordinates used in the calculation [11].

E-mail: jlopezb@ipm.mx

Einstein [12] was the first to introduce a pseudotensor, which is not symmetric and does not give a volume integral for the total angular momentum. In this work we have chosen the Landau-Lifshitz pseudotensor [2,13,14], which is symmetric and leads to volume integrals for momentum and angular momentum. Pseudotensors have been used [15,17] in studies of gravitational self-energy.

Landau-Lifshitz energy-momentum pseudotensor

The Landau-Lifshitz (LL) pseudotensor $t_{LL}^{\mu\nu}$ is defined by writing the Einstein equations in the form [2,13,14]:

$$H_{LL\ \ ,\alpha\beta}^{\mu\alpha\nu\beta} = 16\pi(-g)\left(T^{\mu\nu} + t_{LL}^{\mu\nu}\right) = 16\pi\, T_{LL\ eff}^{\mu\nu} \tag{1}$$

where

$$H_{LL}^{\mu\alpha\nu\beta} = q^{\mu\nu}\, q^{\alpha\beta} - q^{\alpha\nu}\, q^{\mu\beta} \quad , \quad q^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu} \tag{2}$$

and

$$\begin{aligned} 16\pi(-g)t_{LL}^{\alpha\beta} = {} & q^{\alpha\beta}{}_{,\lambda}\, q^{\lambda\mu}{}_{,\mu} - q^{\alpha\lambda}{}_{,\lambda}\, q^{\beta\mu}{}_{,\mu} + \frac{1}{2} g^{\alpha\beta} g_{\lambda\mu} \cdot q^{\lambda\nu}{}_{,\rho}\, q^{\rho\mu}{}_{,\nu} - g_{\mu\nu} q^{\mu\rho}{}_{,\lambda} \cdot \\ & \cdot \left(g^{\alpha\lambda} q^{\beta\nu}{}_{,\rho} + g^{\beta\lambda} q^{\alpha\nu}{}_{,\rho}\right) + g_{\lambda\mu} g^{\nu\rho} q^{\alpha\lambda}{}_{,\nu}\, q^{\beta\mu}{}_{,\rho} + \frac{1}{8}\left(2g^{\alpha\lambda} g^{\beta\mu} - g^{\alpha\beta} g^{\lambda\mu}\right) \cdot \\ & \cdot \left(2g_{\nu\rho} g_{\sigma\tau} - g_{\rho\sigma} g_{\nu\tau}\right) q^{\nu\tau}{}_{,\lambda}\, q^{\rho\sigma}{}_{,\mu} \end{aligned} \tag{3}$$

The conserved momentum and angular momentum of an isolated system are given by:

The conserved momentum and angular momentum of an isolated system are given by:

$$P^{\mu} = \int T_{LL\ \ eff}^{\mu 0}\ d^3x\ , \tag{4}$$

$$J^{\mu\nu} = \int \left(x^{\mu} T_{LL\ \ eff}^{\nu 0} - x^{\nu} T_{LL\ \ eff}^{\mu 0} \right) d^3x\ ,$$

with the conservation law $T_{LL\ \ eff}^{\mu\nu}{}_{,\nu} = 0$ In the expressions above, x^{μ} are asymptotically Minkowskian coordinates.

Some geometries with spherical symmetry

a) The metric for a spherically symmetric star is given by:

$$ds^2 = -e^{2\Phi} dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 . \tag{5}$$

We change coordinates t, r, θ, φ to t, x^1, x^2, x^3 where $r^2 = \left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2$. In the new asymptotically Minkowskian coordinates the metric takes the form:

$$ds^2 = -e^{2\Phi} dt^2 + g_{ij}\ dx^i\ dx^j \tag{6}$$

such that

$$g_{ij} = \frac{A}{r^2} x_i x_j + \delta_{ij} \quad , \quad A = \left(1 - \frac{2m(r)}{r} \right)^{-1} - 1 \tag{7}$$

A short calculation yields the following result for the effective energy density:

$$-g\left(T^{00} + t_{LL}^{00}\right) = \frac{1}{8\pi r^2} (rA)_{,\ r} \tag{8}$$

b) We consider next the Schwarzschild geometry with metric:

$$ds^2 = \frac{32M^3}{r} e^{-\frac{r}{2M}} \left(du^2 - dv^2\right) + r^2\ d\,\Omega^2 \tag{9}$$

in Kruskal-Szekeres coordinates [2,18,19]; *r(u,v)* is the Schwarszchild radial coordinate.

The coordinate transformation:

$$\bar{u} = 2M\ Ln\left[(u+a)^2 - v^2\right] \quad , \quad \bar{v} = 4M \tanh^{-1} \frac{v}{u+a} \tag{10}$$

which covers the region $(u+a) \rangle |v|$ where $a > 0$, puts the metric in the form

$$ds^2 = \frac{2M}{r} e^{-\frac{r}{2M}} \left[(u+a)^2 - v^2\right]\left(d\,\bar{u}^2 - d\bar{v}^2\right) + r^2\ d\,\Omega^2 \tag{11}$$

The additional coordinate transformation from $\bar{v}, \bar{u}, \theta, \varphi$ to asymptotically Minkowskian coordinates $\bar{v}, x^1, x^2, x^3$ where $(\bar{u})^2 = \left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2$, gives

$$ds^2 = -\frac{2M}{r} e^{-\frac{r}{2M}} \left[(u+a)^2 - v^2\right] d\bar{v}^2 + \left(A\delta_{ij} + \frac{B}{\bar{u}^2} x_i x_j \right) dx^i dx^j \tag{12}$$

with $A = \frac{r^2}{\bar{u}^2}$ and $B = -g_{\bar{v}\bar{v}} - A$. The following expression is obtained for the LL energy density:

$$-g\ t_{LL}^{00} = \frac{1}{16\pi \bar{u}^2} \left[2\left(\bar{u} AB\right)_{,\bar{u}} - \left(\bar{u} \left(A^2\right)_{,\ \bar{u}} \right)_{,\ \bar{u}} \right] \tag{13}$$

The space-like hypersurface $\bar{v} =$ *constant* includes the singularity at $r = 0$ when $\bar{v}$ is larger than a positive number which depends on *a*. The integral of $-g\ t_{LL}^{00}$ over any of these hypersurfaces gives *M*, as it should.

c) We now repeat the calculation for the case of the Schwarszchild metric in comoving coordinates.

This is appropriate to connect to an interior Friedman solution for the case of a collapsing ball of dust. In Novikov coordinates [20], the metric is written as:

$$ds^2 = -d\tau^2 + \left(\frac{R^{*2} + 1}{R^{*2}} \right) \left(\frac{\partial r}{\partial R^*} \right)^2 dR^{*2} + r^2\ d\,\Omega^2 \tag{14}$$

and in terms of the new radial variable $R = 2M\left(R^{*2} + 1\right)$:

$$ds^2 = -d\tau^2 + f\left(\tau, R\right) dR^2 + r^2\left(\tau, R\right) d\,\Omega^2 \tag{15}$$

The new coordinates τ, x^1, x^2, x^3, where

$R^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$ are asymptotically Minkowskian and thus (15) adopts the form:

$$ds^2 = -d\tau^2 + \left(A\delta_{ij} + \frac{B}{R^2} x_i x_j \right) dx^i \, dx^j \qquad (16)$$

being $A = \frac{r^2}{R^2}$ and $B = f - A$. Then the *LL* energy density, is given by a relation similar to (13):

$$-g\, t_{LL}^{00} = \frac{1}{16\pi R^2} \left[2(RAB)_{,R} - \left(R(A^2)_{,R} \right)_{,R} \right]. \quad (17)$$

d) The collapsing uniform density ball of dust has an interior Friedman solution:

$$ds^2 = -d\tau^2 + a^2(\tau)\left[d\chi^2 + Sin^2\chi \, d\,\Omega^2 \right], \quad (18)$$

with $a(\tau) = \frac{1}{2} a_m (1 - Cos\,\eta)$ and

$\tau = \frac{1}{2} a_m (\eta + Sin\,\eta)$. This geometry connects at the surface χ_0 of the ball with the exterior Schwarzschild solution (15) using the radial coordinate $R = a_m Sin\,\chi$, then (18) implies:

$$ds^2 = -d\tau^2 + \left(\frac{a^2}{a_m^2 - R^2} \right) dR^2 + \frac{a^2 R^2}{a_m^2} d\,\Omega^2 \quad (19)$$

We further change the coordinates τ, R, θ, φ to τ, x^1, x^2, x^3, where $R^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$, then the new coordinates connect to the exterior asymptotically Minkowskian coordinates which were used in (16). Thus the *LL* effective energy density $-g\left(T^{00} + t_{LL}^{00}\right)$ is given by (17) with:

$$A = \frac{a^2}{a_m^2} \quad , \qquad B = \frac{a^2}{a_m^2 - R^2} \ . \qquad (20)$$

The interior contribution to the total energy *M* on a τ = constant hypersurface is

$$\frac{MR_0}{16(R_0 - 2M)} (1 + Cos\,\eta)^4 , \qquad (21)$$

where R_0 is the radial (comoving) coordinate of the surface of the ball. This contribution decreases and becomes zero when the dust hits the singularity at $a(\tau) = 0$ for $\tau = \frac{\pi}{2} a_m$. The exterior contribution to the total energy is *M* minus the value above for $\tau < \frac{\pi}{2} a_m$ and *M* for later times.

Finally, the *LL* effective energy density of any metric of the form (15) is given by expression (17) setting $A = \frac{r^2}{R^2}$ and $B = f - A$.

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