FERMIONS IN KERR-NEWMAN-KASUYA SPACE-TIME

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Abstract: The aim of this paper is to build up the U(1)- gauge theory for fermions in the curve space-time such as Kerr-Newman-Kasuya space-time. The Kerr-Newman-Kasuya space-time is not a black hole space-time but it has the common feature with the black hole space-time that it has horizon.

Keywords: Gauge theory, black hole space-time, horizon, magnetic monopole charge, null complex tetrads

Introduction

Carmeli and Carmeli derived the Klein-Gordan, Weyl, and Dirac-type equations on \( R \times S^3 \) topology by simply going from the momentum to the angular momentum representation [1,2,3]. Sen [4] obtained the most general Lagrangians for the Dirac, Weyl, and Majorana fermions. Sen’s work offers an excellent description of fermions in the space-time \( R \times S^3 \). Dariescu et al. [4] developed a U(1)-gauge theory for massive fermionic fields minimally coupled to a curved space-time such as Kerr-Newman black hole space-time. Dariescu and Dariescu [5] also developed the tetradic Lorentz-gauge invariant formulation of the SU(2)xU(1) theory in \( R \times S^3 \) space-time. To develop a U(1)-gauge theory for massive fermionic fields on curved space-time such as Kerr-Newman black hole space-time, they used the Dirac-type equation and the U(1)-gauge invariant Lagrangian. In this paper, the U(1)-gauge theory for massive fermionic fields minimally coupled to a Kerr-Newman-Kasuya space-time is studied. The Kerr-Newman-Kasuya space time is the Kerr-Newman space-time involved with extra magnetic monopole charge. This monopole hypothesis was propounded by Dirac relativity long ago.

The Kerr-Newman-Kasuya space-time

The Kerr-Newman-Kasuya space-time is described by the metric

\[
ds^2 = \left( r^2 + h^2 \cos^2 \theta \right) \left( \frac{dr^2}{r^2 - 2mr + e^2 + h^2 + l^2} + d\theta^2 \right) + \sin^2 \theta \left( \frac{r^2 + h^2 + h^2 \sin^2 \theta (2mr - e^2 - l^2)}{r^2 + h^2 \cos^2 \theta} \right) d\phi^2 - \left( 1 - \frac{2mr - e^2 - l^2}{r^2 + h^2 \cos^2 \theta} \right) dt^2 - \frac{2h \sin^2 \theta (2mr - e^2 - l^2)}{r^2 + h^2 \cos^2 \theta} dt \ d\phi
\]

where \( m, h, e \) and \( l \) are the mass, angular momentum per unit mass, electric charge and magnetic monopole charge parameters respectively. This is a solution to the Einstein-Maxwell equations with electromagnetic vector potential

\[
A_\mu dx^\mu = \frac{er}{r^2 + h^2 \cos^2 \theta} \left( dt - h \sin^2 \varphi \ d\varphi \right)
\]

The space-time given by (1) encompasses all the black hole space-times, which are asymptotically flat. Specially, the metric (1) includes:

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(i) Kerr-Newmann black hole space-time when $l=0$.
(ii) Kerr black hole space-time for $l=e=0$.
(iii) Reissner-Nordstrom black hole space-time if $l=h=0$.
(iv) Schwarzchild black hole space-time when $l=e=h=0$.

This metric can be transformed to Boyer coordinates under the proper coordinate transformation such as

$$\{\varphi, \tau, \phi, \sigma, \theta, \mu, \nu\} = \left\{ p = h \cos \theta ; \sigma = -\frac{\varphi}{h} ; q = r ; \tau = t - h \varphi \right\} \quad (3)$$

with the suitable adjustment of the parameter

$$X(p) = h^2 - p^2 ; Y(q) = q^2 - 2mq + e^2 + h^2 + l^2 \quad (4)$$

Therefore, the metric (1) can be written as

$$ds^2 = \frac{p^2 + q^2}{X} dp^2 + \frac{X}{p^2 + q^2} (d\tau + q^2 d\sigma)^2 +$$

$$\frac{p^2 + q^2}{Y} dq^2 - \frac{Y}{p^2 + q^2} (d\tau - p^2 d\sigma)^2 \quad (5)$$

This metric represents the Kerr-Newman-Kasuya space-time in Boyer coordinates, which has been studied in detail by Plebanski [7]. After a suitable choice of the null complex tetrads $\{\omega^a\}$ which consists of two complex conjugate null vectors $m, \overline{m}$ and two real null vectors $k_1, k_2 : \{\omega^a\} = \{m, m, k_1, k_2\} :$

$$\omega^a = m = \frac{1}{\sqrt{2}} \left[ \frac{p^2 + q^2}{X} dp + \frac{X}{p^2 + q^2} (d\tau + q^2 d\sigma) \right]$$

$$\omega^\overline{a} = m = \frac{1}{\sqrt{2}} \left[ \frac{p^2 + q^2}{X} dp - \frac{X}{p^2 + q^2} (d\tau + q^2 d\sigma) \right]$$

$$\omega^a = k_1 = \frac{1}{\sqrt{2}} \left[ \frac{Y}{p^2 + q^2} (d\tau - p^2 d\sigma) - \frac{p^2 + q^2}{Y} dq \right]$$

$$\omega^\overline{a} = k_1 = \frac{1}{\sqrt{2}} \left[ \frac{Y}{p^2 + q^2} (d\tau - p^2 d\sigma) + \frac{p^2 + q^2}{Y} dq \right] \quad (6)$$

the metric (1) becomes in the simple form

$$ds^2 = 2(\omega^1 \omega^2 - \omega^3 \omega^4) = g_{ab} \omega^a \omega^b \quad (7)$$

with

$$\left( g_{ab} \right) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad (8)$$

### Field equations

For the massive fermionic complex fields $\psi$, the U(1)-gauge invariant Lagrangian is given by

$$L = \overline{\psi} \gamma^\mu D_\mu \psi + M \overline{\psi} \psi + \frac{1}{4} F_{\mu \nu} F^{\mu \nu} \quad (9)$$

where $\gamma^\mu$ is the generalized Dirac gamma metrics, the U(1)-gauge field-strength tensor is defined as

$$F^{\mu \nu} = \delta{\mu}{\nu} A^a - \delta{\mu}{\nu} A^a - \left( \delta{\mu}{\nu} A^a - \delta{\mu}{\nu} A^a \right) g_{\mu \nu} A^a \quad (10)$$

and the gauge-covariant derivative is defined as

$$D_\mu \psi = \nabla_\mu \psi + ig A_\mu \psi \quad \text{and its h.c.} \quad (11)$$

here $\nabla_\mu \psi$ be the Levi-Civita covarint derivative and $g$ is the gauge coupling constant. Under these assumptions, the Dirac-type equation is obtained in the covarint expression

$$\gamma^\mu (\partial_\mu + ig A_\mu) \psi - \frac{1}{4} \Gamma_{\mu \nu \rho} \gamma^\mu \gamma^\nu \gamma^\rho \psi + M \psi = 0 \quad (12)$$

The Dirac-type equation governs the particle in curved space-time, and the Maxwell equations with sources can be expressed in the standard form

$$\gamma^\mu (\partial_\mu + ig A_\mu) \psi - \frac{1}{4} \Gamma_{\mu \nu \rho} \gamma^\mu \gamma^\nu \gamma^\rho \psi + M \psi = 0 \quad (13)$$

which will be generalized for the case of a null tetradic base $\{e_a\}, a=1,4$. To build up a U(1)-gauge theory of a massive fermionic complex...
field in the curved space-time described by the metric (7) we use the U(1)-gauge invariant Lagrangian. The general expression for the covariant derivative (11) becomes,

\[ D_a \psi = \nabla_a \psi + igA_a \psi, \]  
(14)

and the Lagrangian (9) as

\[ L = \mathcal{V} \psi^a D_a \psi + M \mathcal{V} \psi + \frac{1}{4} F_{ab} F^{ab}. \]  
(15)

The electromagnetic tensor \( F^{ab} \) can be expressed in the base of coordinate \((p, q, \sigma, \tau)\)

\[ F^{ab} = \omega^a \omega^b F^{\mu\nu}. \]  
(16)

Therefore, the essential components \( F^{\mu\nu} \) are given below

\[ F^{11} = \frac{X}{p^2 + q^2} A^3 \bigg[ \frac{-p^1 + q^2}{q X - p Y} A^1 - \frac{p^2 + q^2}{q X + p Y} A^2 \bigg] - \frac{X}{q^2} \bigg[ \frac{q X}{p^2 + q^2} X - Y \bigg] A^3, \]  
(17.1)

\[ F^{11} = \frac{X}{p^2 + q^2} A^3 \bigg[ \frac{-p^1 + q^2}{q X - p Y} A^1 - \frac{p^2 + q^2}{q X + p Y} A^2 \bigg] - \frac{X}{q^2} \bigg[ \frac{q X}{p^2 + q^2} X - Y \bigg] A^3, \]  
(17.2)

\[ F^{11} = \frac{X}{p^2 + q^2} A^3 \bigg[ \frac{-p^1 + q^2}{q X - p Y} A^1 - \frac{p^2 + q^2}{q X + p Y} A^2 \bigg] - \frac{X}{q^2} \bigg[ \frac{q X}{p^2 + q^2} X - Y \bigg] A^3, \]  
(17.3)

These components of \( F^{\mu\nu} \) allow us to put the Maxwell equations (13) in the expression

\[ e_a F^{ba} = J^b. \]  
(18)

Finally, the Dirac-type equation is derived for the spinorial massive complex field \( \psi \) coupled to the Kerr-Newman-Kasuya space-time. Using the U(1)-gauge invariant Lagrangian (15) the Dirac-type equation is obtained in the general form

\[ \gamma^a (\partial_a + igA_a) \psi + M \psi = 0. \]  
(19)

Hence, by working out the above Dirac-type equation for the metric (5) can be expressed in the form

\[ \gamma^a (\partial_a + igA_a) \psi + M \psi = \gamma^a \gamma^b \gamma^c \gamma^d \psi + M \psi = 0. \]  
(20)

Using (3) and (4) into the equation (20) we obtain the Dirac-type equation for the metric (1) in the following form

\[ \gamma^a (\partial_a + igA_a) \psi + M \psi = \left\{ \left[ (m^2 + h^2) \cos \theta \right] \gamma^a \gamma^a \right\} \psi \] 
+ \left\{ \left[ m \cos \theta \left( r - m \right) + r \sqrt{r^2 - 2mr + e^2} \right] \gamma^a \gamma^a \right\} \psi 
+ \left\{ \left[ \frac{1}{2} \sqrt{r^2 - 2mr + e^2} \right] \left[ \gamma^a \gamma^a \right] \right\} \psi = 0. \]  
(21)

The result obtained in this paper corresponds to the result obtained in the case of the Kerr Newman black hole space-time when \( l = 0 \). Under this observation, we like to claim that this study encompasses the known result of Dariescu et al. [4] in the context of Kerr-Newman black hole. So, it is interesting to note that the
U(1)-gauge theory for fermions not only exists in the Kerr-Newman black hole space-time, but also in the Kerr-Newman Kasuya space-time. The Kerr-Newman-Kasuya space-time is not a black hole space-time but it has the common feature with the black hole space-time that it has horizon.

References