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FERMIONS IN KERR-NEWMAN-KASUYA SPACE-TIME

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Abstract: The aim of this paper is to build up the U(1)- gauge theory for fermions in the curve space-time such as Kerr-Newman-Kasuya space-time. The Kerr-Newman-Kasuya space-time is not a black hole space-time but it has the common feature with the black hole space-time that it has horizon.

Keywords: Gauge theory, black hole space-time, horizon, magnetic monopole charge, null complex tetrads

Introduction

Carmeli and Carmeli derived the Klein-Gordan, Weyl, and Dirac-type equations on $R\times S^3$ topology by simply going from the momen-tum to the angular momentum representation [1,2,3]. Sen [4] obtained the most general Lagrangians for the Dirac, Weyl, and Majorana fermions. Sen's work offers an excellent description of fermions in the space-time $R\times S^3$. Dariescu *et al.* [4] developed a U(1)-gauge theory for massive fermionic fields minimally coupled to a curved space-time such as Kerr-Newman black hole space-time. Dariescu and Dariescu [5] also developed the tetradic Lorentz-gauge invariant formulation of the SU(2)×U(1) theory in $R\times S^3$ space-time. To develop a U(1)-gauge theory for massive fermionic fields on curved space-time such as Kerr-Newman black hole space-time, they used the Dirac-type equation and the U(1)-gauge invariant Lagrangian. In this paper, the U(1)-gauge theory for massive fermionic fields minimally coupled to a Kerr-Newman-Kasuya space-time is studied. The Kerr-Newman-Kasuya space time is the Kerr-Newman space-time involved with extra magnetic monopole charge. This monopole hypothesis was propounded by Dirac relativity long ago.

The Kerr-Newman-Kasuya space-time

The Kerr-Newman-Kasuya space-time is described by the metric

$$\begin{aligned} ds^2 &= \left(r^2+h^2\cos^2\theta\right)\left(\frac{dr^2}{r^2-2mr+e^2+h^2+l^2}+d\theta^2\right) \\ &+\sin^2\theta\left\{r^2+h^2+\frac{h^2\sin^2\theta\left(2mr-e^2-l^2\right)}{r^2+h^2\cos^2\theta}\right\}d\varphi^2 \\ &-\left(1-\frac{2mr-e^2-l^2}{r^2+h^2\cos^2\theta}\right)dt^2-\frac{2h\sin^2\theta\left(2mr-e^2-l^2\right)}{r^2+h^2\cos^2\theta}dt\,d\varphi \end{aligned} \tag{1}$$

where m, h, e and l are the mass, angular momentum per unit mass, electric charge and magnetic monopole charge parameters respectively. This is a solution to the Einstein-Maxwell equations with electromagnetic vector potential

$$A_\mu dx^\mu = \frac{er\left(dt-h\sin^2\varphi\ d\varphi\right)}{r^2+h^2Cos^2\theta} \tag{2}$$

The space-time given by (1) encompasses all the black hole space-times, which are asymptotically flat. Specially, the metric (1) includes:

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(i) Kerr-Newmann black hole space-time when l=0.
(ii) Kerr black hole space-time for l =e=0.
(iii) Reissner-Nordstrom black hole space-time if l=h=0.
(iv) Schwarzchild black hole space-time when l=e=h=0.

This metric can be transformed to Boyer coordinates under the proper coordinate transformation such as

$$\{x^{\mu}\} = \left\{ p = h\cos\theta \,;\, \sigma = -\frac{\varphi}{h} \,;\, q = r \,;\, \tau = t - h\varphi \right\} \quad (3)$$

with the suitable adjustment of the parameter

$$X(p) = h^2 - p^2 \,;\, Y(q) = q^2 - 2mq + e^2 + h^2 + l^2 \quad (4)$$

Therefore, the metric (1) can be written as

$$ds^2 = \frac{p^2+q^2}{X}dp^2 + \frac{X}{p^2+q^2}(d\tau + q^2 d\sigma)^2 + \frac{p^2+q^2}{Y}dq^2 - \frac{Y}{p^2+q^2}(d\tau - p^2 d\sigma)^2 \quad (5)$$

This metric represents the Kerr-Newman-Kasuya space-time in Boyer coordinates, which has been studied in detail by Plebanski [7]. After a suitable choice of the null complex tetrads { ω^a } which consists of two complex conjugate null vectors $m, \overline{m}$ and two real null vectors $k_1, k_2 \,;\, \{\omega^a\} = \{m, \overline{m}, k_1, k_2\}$:

$$\begin{aligned}
\omega^1 = m &= \frac{1}{\sqrt{2}}\left[\sqrt{\frac{p^2+q^2}{X}}dp + i\sqrt{\frac{X}{p^2+q^2}}\left(d\tau + q^2 d\sigma\right)\right] \\
\omega^2 = \overline{m} &= \frac{1}{\sqrt{2}}\left[\sqrt{\frac{p^2+q^2}{X}}dp - i\sqrt{\frac{X}{p^2+q^2}}\left(d\tau + q^2 d\sigma\right)\right] \\
\omega^3 = k_1 &= \frac{1}{\sqrt{2}}\left[\sqrt{\frac{Y}{p^2+q^2}}\left(d\tau - p^2 d\sigma\right) - \sqrt{\frac{p^2+q^2}{Y}}dq\right] \\
\omega^4 = k_2 &= \frac{1}{\sqrt{2}}\left[\sqrt{\frac{Y}{p^2+q^2}}\left(d\tau - p^2 d\sigma\right) + \sqrt{\frac{p^2+q^2}{Y}}dq\right]
\end{aligned} \quad (6)$$

the metric (1) becomes in the simple form

$$ds^2 = 2(\omega^1\omega^2 - \omega^3\omega^4) = g_{ab}\omega^a\omega^b \quad (7)$$

with

$$\left(g_{ab}\right) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad (8)$$

Field equations

For the massive fermionic complex fields ψ the U(1)-gauge invariant Lagrangian is given by

$$L = \overline{\psi}\gamma^{\mu}D_{\mu}\psi + M\overline{\psi}\psi + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad (9)$$

where γ^{μ} is the generalized Dirac gamma metrics, the U(1)-gauge field-strength tensor is defined as

$$F^{\mu\nu} = g^{\mu\alpha}\partial_{\alpha}A^{\nu} - g^{\nu\alpha}\partial_{\alpha}A^{\mu} - \left(g^{\mu\alpha}\partial_{\alpha}g^{\nu\beta} - g^{\nu\alpha}\partial_{\alpha}g^{\mu\beta}\right)g_{\beta\sigma}A^{\sigma} \quad (10)$$

and the gauge-covarint derivative is defined as

$$D_{\mu}\psi = \nabla_{\mu}\psi + igA_{\mu}\psi \,, \text{ and its h.c.} \quad (11)$$

here $\nabla_{\mu}\psi$ be the Levi-Civita covarint derivative and g is the gauge coupling constant. Under these assumptions, the Dirac-type equation is obtained in the covarint expression

$$\gamma^{\mu}\left(\partial_{\mu} + igA_{\mu}\right)\psi - \frac{1}{4}\Gamma_{\alpha\beta\mu}\gamma^{\mu}\gamma^{\alpha}\gamma^{\beta}\psi + M\psi = 0 \quad (12)$$

The Dirac-type equation governs the particle in curved space-time, and the Maxwell equations with sources can be expressed in the standard form

$$\gamma^{\mu}\left(\partial_{\mu} + igA_{\mu}\right)\psi - \frac{1}{4}\Gamma_{\alpha\beta\mu}\gamma^{\mu}\gamma^{\alpha}\gamma^{\beta}\psi + M\psi = 0 \quad (13)$$

which will be generalized for the case of a null tetradic base $\{e_a\}, a = 1,4$. To build up a U(1)-gauge theory of a massive fermionic complex field in the curved space-time described by the metric (7) we use the U(1)-gauge invariant Lagrangian. The general expression for the covarint derivative (11) becomes,

$$D_a\psi = \nabla_a\psi + igA_a\psi\text{, and its h.c.} \tag{14}$$

and the Lagrangian (9) as

$$L = \overline{\psi}\gamma^a D_a\psi + M\overline{\psi}\psi + \frac{1}{4}F_{ab}F^{ab} \tag{15}$$

The electromagnetic tensor F^{ab} can be expressed in the base of coordinate (p,q,σ,τ)

$$F^{ab} = \omega^a_\mu \omega^b_\nu F^{\mu\nu} \tag{16}$$

Therefore, the essential components $F^{\mu\nu}$ are given below

$$\begin{aligned} F^{12} = {} & \frac{X}{p^2+q^2}A_p{}^2 - \frac{p^2+q^2}{q^4X-p^4Y}A_{,\sigma}{}^1 - \frac{p^2+q^2}{q^2X+p^2Y}A_{,\tau}{}^1 - \\ & \left[\frac{X(q^4X-p^4Y)}{(p^2+q^2)^2}\left(\frac{p^2+q^2}{q^4X-p^4Y}\right)_{,p} + \frac{X(q^2X+p^2Y)}{(p^2+q^2)^2}\left(\frac{p^2+q^2}{q^2X+p^2Y}\right)_{,p}\right]A^2 \end{aligned} \tag{17.1}$$

$$F^{13} = \frac{X}{p^2+q^2}A_{,p}{}^3 - \frac{Y}{p^2+q^2}A_{,q}{}^1 - \frac{X}{Y}\left(\frac{Y}{p^2+q^2}\right)_{,p}A^3 + \frac{Y}{X}\left(\frac{X}{p^2+q^2}\right)_{,q}A^1 \tag{17.2}$$

$$\begin{aligned} F^{14} = {} & \frac{X}{p^2+q^2}A_p{}^4 - \frac{p^2+q^2}{q^2X+p^2Y}A_{,\sigma}{}^1 - \frac{p^2+q^2}{X-Y}A_{,\tau}{}^1 - \\ & \left[\frac{X(q^4X-p^4Y)}{(p^2+q^2)^2}\left(\frac{p^2+q^2}{q^2X-p^2Y}\right)_{,p} + \frac{X(q^2X+p^2Y)}{(p^2+q^2)^2}\left(\frac{p^2+q^2}{X-Y}\right)_{,p}\right]A^2 - \\ & \left[\frac{X(X-Y)}{(p^2+q^2)^2}\left(\frac{p^2+q^2}{X-Y}\right)_{,p} + \frac{X(q^2X+p^2Y)}{(p^2+q^2)^2}\left(\frac{p^2+q^2}{q^2X+p^2Y}\right)_{,p}\right]A^4 \end{aligned} \tag{17.3}$$

$$\begin{aligned} F^{23} = {} & -\frac{Y}{p^2+q^2}A_{,q}{}^2 + \frac{p^2+q^2}{q^4X-p^4Y}A_{,\sigma}{}^3 + \frac{p^2+q^2}{q^2X+p^2Y}A_{,\tau}{}^3 \\ & + \left[\frac{Y(q^4X-p^4Y)}{(p^2+q^2)^2}\left(\frac{p^2+q^2}{q^4X-p^4Y}\right)_{,q} + \frac{Y(q^2X+p^2Y)}{(p^2+q^2)^2}\left(\frac{p^2+q^2}{q^2X+p^2Y}\right)_{,q}\right]A^2 \\ & + \left[\frac{Y(X-Y)}{(p^2+q^2)^2}\left(\frac{p^2+q^2}{q^2X+p^2Y}\right)_{,q} + \frac{Y(q^2X+p^2Y)}{(p^2+q^2)^2}\left(\frac{p^2+q^2}{q^4X-p^4Y}\right)_{,q}\right]A^4 \end{aligned} \tag{17.4}$$

$$F^{24} = \frac{p^2+q^2}{q^4X-p^4Y}A^4_{,\sigma} + \frac{p^2+q^2}{q^2X+p^2Y}A^4_{,\tau} - \frac{p^2+q^2}{q^2X+p^2Y}A^2_{,\sigma} - \frac{p^2+q^2}{X-Y}A^2_{,\tau} \tag{17.5}$$

$$\begin{aligned} F^{34} = {} & \frac{Y}{p^2+q^2}A_{,q}{}^4 - \frac{p^2+q^2}{q^2X+p^2Y}A_{,\sigma}{}^3 - \frac{p^2+q^2}{X-Y}A_{,\tau}{}^3 \\ & - \left[\frac{Y(q^4X-p^4Y)}{(p^2+q^2)^2}\left(\frac{p^2+q^2}{q^2X-p^2Y}\right)_{,q} + \frac{Y(q^2X+p^2Y)}{(p^2+q^2)^2}\left(\frac{p^2+q^2}{X-Y}\right)_{,q}\right]A^2 \\ & - \left[\frac{Y(X-Y)}{(p^2+q^2)^2}\left(\frac{p^2+q^2}{X-Y}\right)_{,q} + \frac{Y(q^2X+p^2Y)}{(p^2+q^2)^2}\left(\frac{p^2+q^2}{q^2X+p^2Y}\right)_{,q}\right]A^4 \end{aligned} \tag{17.6}$$

These components of $F^{\mu\nu}$ allow us to put the Maxwell equations (13) in the expression

$$e_a F^{ba} = J^b \tag{18}$$

Finally the Dirac-type equation is derived for the spinorial massive complex field ψ coupled to the Kerr-Newman-Kasuya space-time. Using the U(1)-gauge invariant Lagarangian (15) the Dirac-type equation is obtained in the general form

$$\gamma^a(\partial_a + igA_a)\psi - \frac{1}{4}\Gamma_{bca}\gamma^a\gamma^b\gamma^c\psi + M\psi = 0 \tag{19}$$

Hence, by working out the above Dirac-type equation for the metric (5) can be expressed in the form

$$\begin{aligned} & \gamma^a(\partial_a + igA_a)\psi + M\psi - \frac{1}{4\sqrt{2(p^2+q^2)^3}}\times \\ & \left\{\left[\frac{(p^2+q^2)\left(\frac{\partial X}{\partial p}\right) - 2pX}{\sqrt{X}}(\gamma^1+\gamma^2) + \frac{(p^2+q^2)\left(\frac{\partial Y}{\partial q}\right) - 2qY}{\sqrt{Y}}(\gamma^3-\gamma^4)\right] \right. \\ & \left. -8i\left[p\sqrt{Y}\gamma^1\gamma^2(\gamma^3+\gamma^4) - q\sqrt{X}(\gamma^1-\gamma^2)\gamma^3\gamma^4\right]\right\}\psi = 0 \end{aligned} \tag{20}$$

Using (3) and (4) into the equation (20) we obtain the Dirac-type equation for the metric (1) in the following form

$$\begin{aligned} & \gamma^a(\partial_a + igA_a)\psi + M\psi + \left[\frac{\{(r^2+h^2)h\cos\theta\}(\gamma^1+\gamma^2)}{2h\sin\theta\sqrt{2\{r^2+h^2\cos^2\theta\}^3}}\right]\psi \\ & - \left[\frac{\{h^2\cos^2\theta(r-m) + r(rm-e^2-h^2-l^2)\}(\gamma^3-\gamma^4)}{2\sqrt{2\{r^2+h^2\cos^2\theta\}^3}(r^2-2mr+e^2+h^2+l^2)}\right]\psi + \\ & \left[\frac{2i\{h\cos\theta(\sqrt{r^2-2mr+e^2+h^2+l^2})\}(\gamma^3+\gamma^4)\gamma^1\gamma^2}{\sqrt{2\{r^2+h^2\cos^2\theta\}^3}}\right]\psi - \\ & \left[\frac{2irh\sin\theta(\gamma^1-\gamma^2)\gamma^3\gamma^4}{\sqrt{2\{r^2+h^2\cos^2\theta\}^3}}\right]\psi = 0 \end{aligned} \tag{21}$$

The result obtained in this paper corresponds to the result obtained in the case of the Kerr Newman black hole space-time when l =0. Under this observation, we like to claim that this study encompasses the known result of Dariescu *et al*. [4] in the context of Kerr-Newman black hole. So, it is interesting to note that the

U(1)-gauge theory for fermions not only exists in the Kerr-Newman black hole space-time, but also in the Kerr-Newman Kasuya space-time. The Kerr-Newman-Kasuya space-time is not a black hole space-time but it has the common feature with the black hole space-time that it has horizon.

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