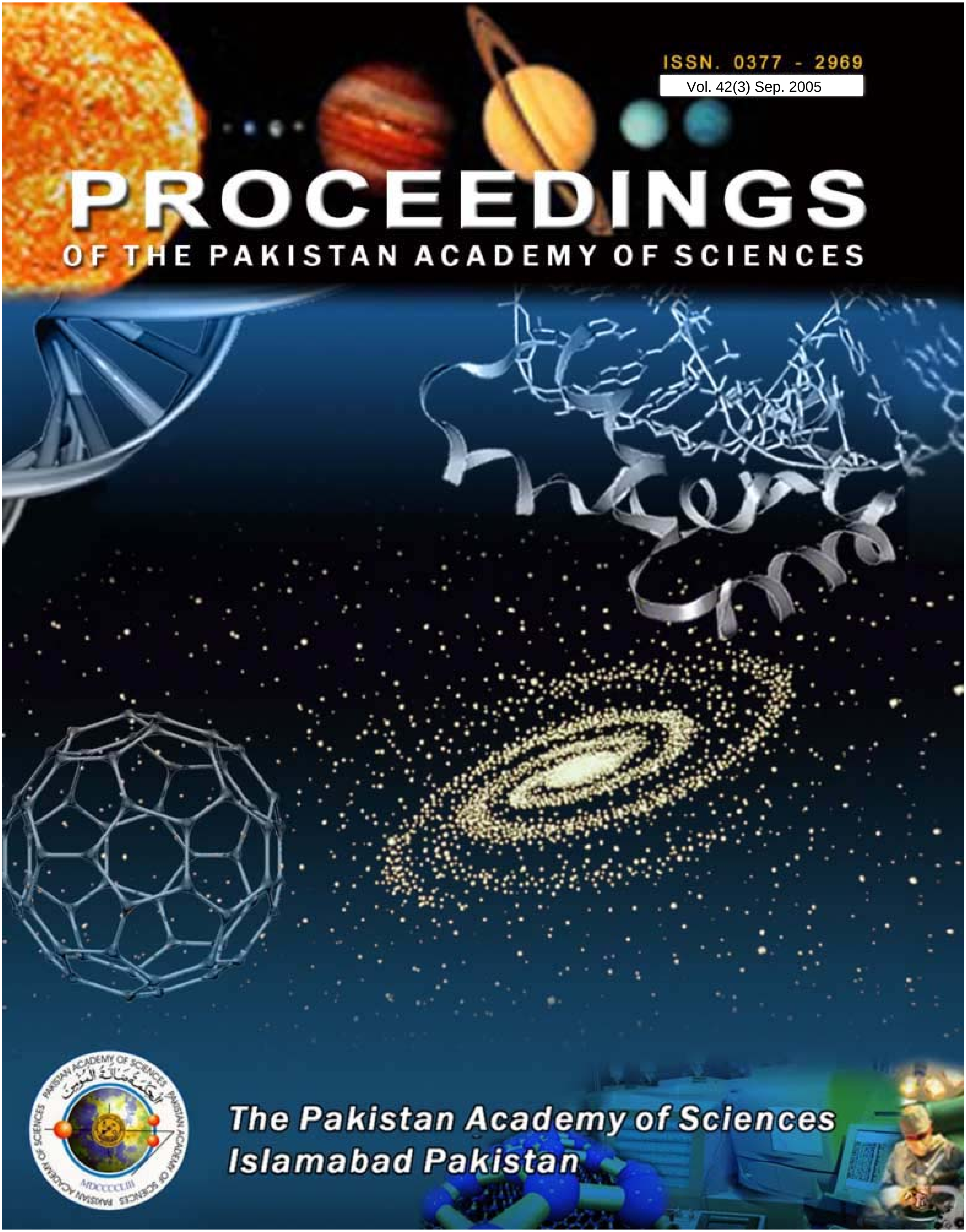

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LEVEL AND TRENDS OF FERTILITY IN BANGLADESH

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Abstract: A parity progression ratio and Bongaarts model, applied to the data of Bangladesh Demographic and Health Survey 1996-97, indicated that fertility of Bangladesh has tremendously declined during the last two decades. For example, a total fertility rate 3.3 in 1992 declined to 2.36 in 1996. The total fertility rate of Bangladesh in two decades 1961-63 and 1979-81 were estimated 6.54 and 4.86 respectively. Moreover, in a short duration of time 1992 to 1996, a female who has a chance of 0.690 for going from parity 3 to 4 in 1992, reduced to 0.677 in 1996. Interestingly, Gompertz curve provided a close estimate of total fertility rate estimated from parity progression ratio. Further, it was observed from Bongaarts' model analysis, it is the rising level of contraceptive use which has a strong effect (about 50 per cent) in reducing total fertility rate in Bangladesh.

Keywords**:** Fertility, total fertility rate, parity progression ratio, Bongaarts model, Gompertz curve

Introduction

The reproduction process in human being typically occurs during teens of life though there is a variation in the age at menarche around the globe [1]. Fertility behavior is changing over time and numerous studies have been done on it all over the world [2.3,4]. Bangladesh is one of the most populous developing countries in the world and it has a remarkable decline in fertility despite a limited affords made for the improvement of bare necessities of human beings, namely, literacy, urbanization, industrialization, status of female, female's age at marriage, infant mortality, level of income etc.[5,6.7]. Due to inadequate vital registration system in Bangladesh, the demographic measures have so far been estimated from data obtained in censuses or surveys. Due to a high rate of illiteracy among female respondents, there is often substantial omission of births due to lack of knowledge of ever born children usually because of omission of dead children or children who have already left the home [7]. Cleland [6] found that older women used to underestimate recent births more frequently than younger women. Conse-quently, under these limitations of data, direct measurements may provide unreliable estimate of fertility rates.

A lot of studies have been conducted to describe the fertility level of Bangladesh by using different indirect techniques proposed by Bongaarts [8], Brass [10], Trussell [28], and many others. The total fertility rate (TFR) is considered to be a refined and reliable measure of fertility in a population and it has recently been studied through parity progression ratios (PPR) which, of course, reflect the tempo of cohort fertility. This approach of measuring period fertility, called period parity progression ratio (PPPR), gives an additional advantage to look at trends in TFR as well as facilitates comparison separately regarding progression from a specific parity to higher order parity [6,11,12,13,14,15,16]. Moreover, fitting of Gompertz curve is already well-known to demographers for estimating mortality as well as fertility parameters [17,18]. Fertility trend can also be estimated by this curve [19]. Bangladesh demographic transition is a faster one [20] and it has created much attention among researchers and policy planners to understand the interior mechanism and to know differentials or determinants of fertility [21]. In fact, the level of fertility in a society is directly influenced by a set of variables called intermediate variables [22]. Bongaarts and Potter [23] revised intermediate variables and proposed a new system of variables termed as 'Proximate determinants of fertility' and his model provides the effect of an individual proximate determinant of fertility as well as an estimate of the level of fertility. Using data from 41 developed and developing countries, Bongaarts and Potter [23] observed that about 96 per cent of total variation of fertility is due to four important proximate determinants namely, marriage, contraception, lactational infecundability and induced abortion. The level of fertility in Bangladesh has been studied using Bongaarts model by Islam *et al.* [14]. Besides these proximate determinants, a number of other demographic and socioeconomic factors have also shown some remarkable effect on declining fertility [6,24,25].

The main objectives of this paper are (i) to study the level of fertility based on cohort parity progression ratio, (ii) to study the trends of fertility based on period parity progression ratio, (iii) to estimate TFR through the estimated value of parity progression ratio by Gompertz curve, (iv) to estimate TFR through Bongaarts' model and the factors acting for declining fertility as determined by this model, and (v) to compare TFR among various methods.

Materials and Methods

The data for this study have been extracted from the 1996-97 Bangladesh Demographic and Heath Survey (BDHS). The sample of this survey has been drawn from the Integrated Multipurpose Master Sample (IMPS) created by Bangladesh Bureau of Statistics (BBS). The primary sampling units in the IMPS were selected with probability proportional to size from the 1991 census frame. Apart from other questionnaires, individual woman's questionnaire was used to collect information from ever-married (eligible) women including such information on fertility as total number of children even born, current age of mother, date of birth of child, survival status and age at death of child. For this study, January 1997 was considered as the reference date and births occurring after this date were eliminated.

Cohort and Period Parity Progression Ratios

The parity progression ratio (PPR) indicates the chance for a woman of parity i^{th} ($i = 0,1,2,....$) to go for higher $(i+1)^{th}$ parity. If this progression is recorded according to cohort year of marriage, year of birth of mother etc., then it is termed as cohort parity progression ratio and if it is recorded for a specific period, it is called Period Parity Progression Ratio (PPPR). Probability of moving from parity i to parity i+1, say, $P'_{i-(i+1)}$ is calculated as $P'_{i-(i+1)} = \frac{F_{i+1}}{F_i}$, where F_i is the number of women passing through parity i. TFR can be calculated either by the cumulated sum of average births per woman by birth order or

$$TFR = P_0 + P_0P_1 + P_0P_1P_2 + ... + P_0P_1P_2....P_{n+} /(1-P_{n+}) \quad (1)$$

where, P_0 denotes progression from marriage to first birth, P_1 denotes progression from first to second birth and so on. The first one is applicable only when the TFR can be replaced by completed family size and the next one is applicable for period parity progression ratio.

Gompertz Curve

Gompertz curve

$$F(i) = FA^{B^{i}} \quad 0 < A,\ B < 1,\ i > 0, \tag{2}$$

has also been used to fit parity distribution of fertility data. The parameters F, A and B of this curve have been estimated by the method of partial sums as follows. Assuming that the cumulative parity specific fertility rates, F_0 F_i, F_{i+1}, F_{i+2}, ..., F_{3n}, are available for parity 0,1,2, ..., 3n, respectively, and S_1, S_2 and S_3 denote the partial sums, then

$$S_1 = \sum_{i=1}^{n} \ln F_{y+i},\ S_2 = \sum_{i=n+1}^{2n} \ln F_{y+i}\ \textit{and}\ S_3 = \sum_{i=2n+1}^{3n} \ln F_{y+i}$$

Then the estimates of the parameters can be obtained according to Pollard and Valkovics [18] as shown below.

$$\hat{B} = \left[\frac{S_3 - S_2}{S_2 - S_1}\right]^{\frac{1}{n}}$$

$$\hat{A} = \exp\left\{\frac{(\hat{B}-1)(S_2 - S_1)}{\left(\hat{B}(\hat{B}^n - 1)^2\right)}\right\}$$

$$\hat{F} = \exp\left\{\frac{(S_1 S_3 - S_2^2)}{[n(S_1 - 2S_2 + S_3)]}\right\}$$

Bongaarts Model

Bongaarts [8] proposed a set of proximate determinants of fertility namely, proportion married among females, prevalence of induced abortion, contraceptive use and effectiveness, duration of post partum infecundability, fecundability (i.e., probability of conception), frequency of intercourse, spontaneous intra uterine mortality rate and prevalence of permanent sterility. Out of these, the first four determinants have been considered to exert a strong effect on fertility. In brief, his model is multiplicative in nature and measures TFR as

$$TFR = Cm \times Ci \times Ca \times Cc \times TF \tag{3}$$

where Cm, Ci, Ca and Cc denote the indices of proportion married, lactational infecundability, abortion, contraception, respectively; and TF indicates the total fecundity rate (defined as the expected number of average live births among those women who are sexually active, fecund, non-contracepting and do not breast feed their children within their reproductive age). TF varies from 12 to 17 [8,26]. However, a low average value of TF is found in a society characterized by poverty, frequent spousal separation, social customs, sexual taboos *etc.* Bangladesh possesses all such charac-teristics even below such average TF value. These indices lie between 0 and 1. A variable has no fertility inhibiting effect, if the index is 1 and zero if the fertility inhibition is complete by the given intermediate variable. Indices of these variables are estimated as

$$Cm = \frac{TFR}{TM} = \frac{\sum m(a) \times g(a)}{\sum g(a)} \tag{4}$$

where, m(a)= proportion of currently married females by age a, g(a) = ASMFR, TM is the total marital fertility rate;

$$Cc = 1 - (S \times e \times u) \tag{5}$$

where, S stands for sterility correction factor with a constant value of 1.08 for developing countries and e stands for use-effectiveness of contraception.

$$Ci = \frac{20}{18.5 + i} \tag{6}$$

where, i represents average duration of post partum amenorrhoea;

and $$Ca = \frac{TFR}{TFR + 0.4\,(1+u)\,TAR} \qquad (7)$$

where, u is contraceptive prevalence rate, TAR is the total abortion rate and the term 0.4(1+u) is an estimate of the births averted by a single abortion. A modification to this index was suggested by Stover [27] and according to him the contraceptive prevalence rate should be multiplied by the effectiveness of contraception to describe the proportion of women protected by contraception more accurately and hence

$$Ca = \frac{TFR}{TFR + 0.4\,(1+u\times e)\,TAR} \qquad (8)$$

where e indicates the average use effectiveness of contraception and is measured by $e = \frac{\sum ei\,ui}{ui}$, where ui indicates proportion of women using ith specific method and ei refers to the estimates of contraceptive use effectiveness for ith method and it is estimated as $ei = 1 - \frac{fi}{fn}$, fi is indicates ith contraceptive method failure rate and fn is the natural fecundability.

Results and Discussion

Cohort parity progression ratio is used to see the level of fertility by taking cohorts of women who got married during 1961-63, 1969-71 and 1979-81 (Table 1). Marriage year has been taken instead of current age of women because marriage is a significant event of a woman especially in developing countries. So data on this item seem to be more reliable than age data. For the marriage cohort 1979-81, having completed fifteen years of marriage duration till the survey date (1996-97), there is no harm in comparing the parity progression ratios up to 3-4 parities of this cohort with the marriage cohorts 1961-63 and 1969-71. Table 1 shows that the progression ratios from marriage to first birth drop only about 2% in both the cohorts 1969-71 and 1979-81 compared with 1961-63. Moreover, progression from 1st to 2nd birth for 1979-81 cohort was 0.960 which indicates that 96% of women proceeded to the second birth. However, a considerable reduction (about 11%) was observed in proceeding to 3rd parity which might be due to the introduction of 2-children family policy by the Government of Bangladesh. The probability of progressing from 3rd to 4th parity dropped significantly from 0.934 to 0.855 and from 0.934 to 0.733 from marriage cohorts 1961-63 to 1969-71 and from 1961-63 to 1979-81, respectively (Fig. 1). This may be due to increased age at marriage over time where there is less chance of adolescent sterility as well as usually couples try to set their family size to be completed as early as possible. A declining trend in TFR is visible from this analysis over time. The TFR was 6.54, 5.39 and 4.86 for the marriage cohorts 1961-63, 1969-71 and 1979-81, respectively, giving a decreasing trend of TFR and its amount of reduction is about 18 percent and 10 percent during the decades 1963 to 1971 and 1971 to 1981, respectively. However, cohort based fertility has been found relatively insensitive to elucidate recent trend in fertility [6] and as such annual total fertility rate is required. Based on period parity progression ratio, TFR is computed for each calendar year from 1992 to 1996 considering reproductive age of women in a single year (15-49). Table 2 shows a falling trend in fertility where TFR is dropped slightly from 3.30 in 1992 to 2.36 in 1996. However, a result for the

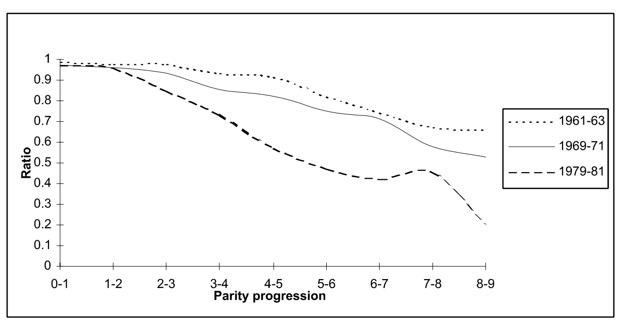

Figure 1. Cohort parity progression ratio.

Table 1. Cohort parity progression ratios (per 1000) and total fertility rate (TFR): BDHS 1996-97 data.

	Cohort parity progression ratios									
Year of marriage	P_{0-1}	P_{1-2}	P_{2-3}	P_{3-4}	P_{4-5}	P_{5-6}	P_{6-7}	P_{7-8}	P_{8-9}	**TFR**
1961-63	990	980	976	934	914	820	745	671	660	6.54
1969-71	974	962	934	855	823	748	713	580	529	5.39
1979-81	970	960	851	733	573	473	423	455	200	4.86

Table 2. Period parity progression ratios (per 1000) and total fertility rate (TFR): BDHS 1996-97 data.

	Period parity progression ratios										
Year	P_{0-1}	P_{1-2}	P_{2-3}	P_{3-4}	P_{4-5}	P_{5-6}	P_{6-7}	P_{7-8}	P_{8-9}	**TFR**	**TFR obtained by Gompertz curve**
1992	841	912	747	690	699	640	655	648	523	3.30	3.47
1993	812	892	710	673	664	715	617	590	663	3.07	3.00
1994	762	838	711	688	651	640	650	657	443	2.68	2.84
1995	897	753	654	593	630	607	480	367	000	2.66	2.74
1996	636	731	677	677	650	700	615	574	561	2.36	2.82
1992-96	941	827	705	671	666	669	626	612	548	3.29	3.16

1996 may be speculative especially for progression from marriage to 1st birth. Reason being that a large number of women who got married at the end of 1996 may be included in the analysis that did not get sufficient time of exposure to give birth. The chance of progressing from 1st to 2nd parity was found highest among all progressions which are consistent with other findings [14]. However, the progression ratios from marriage to 1st birth drop slightly than those from 1st to 2nd birth over the preceding five-year period from 1992 to 1996, except 1995. Thus, it is observed that about 18% reduction in total fertility occurred during the period early 60's to early 70's and about 32% during early 1980's to early 1990's. The last column of Table 2 represents TFR obtained by fitting Gompertz curve based on observed parity distribution. TFR obtained from Gompertz curve also provided close estimate of the fertility as found by the method of period parity progression ratio.

For application of Bongaarts model, necessary measures of fertility are given in section A and the estimated indices are shown in section B of Table 3. An estimate of current age specific marital fertility rate has been computed from the average reported number of births during 36 months preceding the reference date of January 1997 of BDHS 1996-97, and as such it came out to be 3.87. For computing the average use effectiveness (e) of contraception, an estimate of natural fecundability (fn), i.e. monthly probability of becoming pregnant is required while unprotected. Such information was not available from this data and fn = 0.45 has been taken from Bairagi *et al*. [5]. The average use effectiveness of contraception allowing fn = 0.45 was found to be 0.93 (Table 4). The first column of Table 3 (TFR by Bongaarts model) has been taken from the findings of Islam *et al*. [14] to document changes among the indices over the period from 1993 to 1996. An estimate of induced abortion rate was 0.13 (BDHS 1996-97), which seems to be under-reported because of its sensitivity. Fortunately, an estimate of TAR was 0.18 [28] based on the data of

Matlab Demographic surveillance system of ICDDR, B (comparison area), which may be comparable to other parts of Bangladesh during the period 1982 to 1991. So, TAR has been taken as 0.18. Taking values of all these indices as mentioned above, Bongaarts model produced a TFR equal to 4.37 (Table 3).

Table 3. Various fertility measures and TFR by Bongaarts model: BDHS 1993-94 and BDHS 1996-97 data.

A. Reproductive measures	**BDHS 1993-94**	**BDHS 1996-97**	**% Change during 1993-96**
Proportion of contraceptive use (u)	0.45	0.49	
Contraceptive use effectiveness (e)	0.81	0.93	
Mean duration of PPA (I)	12.11	9.17	
Total abortion rate (TAR*)	0.18	0.18	
B. Model indices			
C_m	0.761	0.808	6.2
C_c	0.610	0.508	-16.7
C_i	0.653	0.720	10.3
C_a	0.971	0.967	-0.4
Combined effect of four indices (C_m x C_c x C_i x C_a)	0.294	0.286	-2.7
TF	15.3	15.3	
TFR (estimated)	4.50	4.37	-2.89

*Values taken from Matlab Demographic Surveillance System of ICDDR, B

Table 4. Levels of current use of different contraceptive methods and their use effectiveness: BDHS1996-97 data.

Methods	**Ever in use**	**Current use**	**Annual failure rate (fi)**	**ei= 1-(fi/fn)**
Pill	48.9	20.8	2.9	0.935
IUD	6.9	1.8	0.0	1.0
Injectable	15.7	6.2	1.3	0.971
Condom	15.0	3.9	6.4	0.858
Sterilization	8.8	8.7	0.0	1.0
Periodic abstinence	16.7	5.0	9.9	0.78
Withdrawal	9.5	1.9	4.8	0.893
Any method (CPR)		48.3		0.93

A lower value of an index among the four principal factors of Bongaarts model indicated a higher fertility reducing impact. Among these four principal indices, Cc showed a strong effect (about 50%) in reduction of total natural fertility relative to total marital fertility rate (Table 3). This may be due to a well-designed network established to provide door to door family planning services by the female field workers and this family planning program of Bangladesh is now being considered a model for developing countries [20]. Ci played the role of second important factor for reducing total fecundity rate by 28 percent, followed by Cm which reduced actual fertility level below marital fertility rate by 19.2 percent. The changing pattern of four proximate determinants between 1993 and 1996 is shown in the last column of Table 3. The significant declining change in TFR over this period may be due to the increased use of contraception. However, post partum non susceptibility performed negative role in reduction of fertility over this period. It may be due to shortening of the duration of lactational infecundability or some of the women may be ‘doubly protected’ (women in amenorrhoea or using contraception) but reported only the use of contraception. Fertility inhibiting effects of each proximate determinant under study are presented in Table 5. The findings indicate that a total of 10.82 births in 1996 are being inhibited by 15.7 percent due to the effect of marriage variable, 54.9 percent due to contraception, 26.6 percent due to lactational infecundability and 2.7 percent due to abortion.

In conclusion, a decreasing trend of fertility has been found in Bangladesh according to the methods, cohort parity progression ratio, period parity progression ratio and Gompertz curve. An estimate of total fertility rate (TFR) computed by different methodologies ranges from 2.36 to 6.54. Obviously, it was lowest under period parity progression ratio (1996) and highest under cohort parity progression ratio (marriage year 1961-63). Evidence has been presented that Bangladesh has earned a remarkable decline (more than 42%

over the period of 25 years) in fertility level from 5.4 in 1971 to 2.36 in 1996. A high level of contraceptive use may be one of the responsible factors for such a reduction in fertility. Even in a small span of time a gentle downward trend in progression ratios progressing to third and higher order parities has been obtained for 1992 to 1996. A total fertility rate of 6.54 found in 1963 declined to 4.86 in 1981 (a decline of about 25%) and to 3.30 in 1992, a noticeable decline (about 32%) during 1981 to 1992. An estimate of TFR obtained by Gompertz curve was found consistent with the estimate found by period parity progression ratio and the reported TFR. Where the fertility showed a declining trend, PPR provided a good estimate of TFR. Usually other indirect techniques gives one birth higher than the reported TFR. For example, Bongaarts model provided an estimate of TFR equal to 4.37, almost an estimate consistent with the previous studies conducted in Bangladesh [14,29]. Such type of discrepancy has also been observed in other developing countries such as India, Nepal and Pakistan [30]. This shows a tendency of under reporting of births prevalent in societies having a low level literacy among females. Population scientists are trying to understand the factors/mechanisms that have contributed in reduction of fertility in a very short span in Bangladesh. Use of contraception has been identified as an important factor for such a reduction by the Bongaarts model.

Table 5. Magnitude of the total inhibiting effect accounted by each proximate fertility determinant: BDHS 1993-94 and BDHS 1996-97 data.

Proximate determinants (indices)	Fertility inhibiting effects*			
	Births per woman		Percent	
	1993	1996	1993	1996
Marriage C_m	2.41	1.70	22.3	15.7
Contraception C_c	4.36	5.94	40.4	54.9
Lactational infecundability C_i	3.76	2.88	34.8	26.6
Induced abortion C_a	0.26	0.29	2.4	2.7
Total: 15.3-TFR (estimated)	10.80	10.82	-	-

* Measured by the logarithm of each index as for example: effect of marriage = [TF - TFR (estimated)] × {log Cm/(log Cc+ log Ci + log Ca)}

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BUTYRYLCHOLINESTERASE INHIBITORY LIGNANS FROM *SARCOSTEMMA VIMINALE*

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Abstract. The investigation of the chemical constituents of *Sarcostemma viminale* led to the isolation of two lignans, pinoresinol (**1**) and 9α–hydroxypinoresinol (**2**), which are reported for the first time from this species alongwith β–sitosterol (**3**), β-amyrin (**4**), lupeol (**5**), oleanolic acid (**6**) and ursolic acid (**7**). The structures of the these compounds were determined by 1D and 2D- homonuclear and heteronuclear NMR spectroscopy, chemical evidences, and by comparison with the published data of these compounds. The pinoresinol **1** and 9α–hydroxypinoresinol **2** displayed *in vitro* inhibitory activity against butyrylcholinesterase (BChE) enzyme with IC_{50} value of (15.0 ± 0.1) and (275.1 ± 2.4) μM respectively.

Keywords: Asclepiadaceae, *Sarcostemma viminale*, pinoresinol, 9α-hydroxypinoresinol

Introduction

The genus *Sarcostemma* belongs to the family Asclepiadaceae and consists of about 10 species, distributed in the tropical and sub-tropical regions of the world but represented in Pakistan by one species, *S. viminale* [1]. The milky juice of this plant will instantly allay the intense pain of the human eye caused by the accidental entry of the juice of any Euphorbiaceous plant [2].

Cholinesterases are enzymes that share extensive sequence homology and distinct substrate specificity and inhibitor sensitivity. Cholinesterases are potential target for the symptomatic treatment of Alzheimer's disease and related dementias. It has been found that butyrylcholinesterase (BChE, E.C 3.1.1.8) inhibition is an effective tool for the treatment of AD and related dementias [3]. BChE is found in significantly higher quantities in Alzheimer's plaques than in plaques of normal age-related non-demented brains. BChE is produced in the liver and enriches blood circulation. In addition, it is also present in adipose tissue, intestine, smooth muscle cells, white matter of the brain and many other tissues [4].

Materials and Methods

General

For column chromatography (CC), silica gel (70-230 mesh) and for flash chromatography (FC), silica gel (230-400 mesh) was used. TLC was performed on pre-coated silica gel G-25-UV_{254} plates. Detection was carried out at 254 nm, and by ceric sulphate reagent. The optical rotations were measured on a Jasco-DIP-360 digital polarimeter. The UV and IR Spectra were recorded on Hitachi-UV-3200 and Jasco-320-A spectrophotometer. ^{1}H-NMR, ^{13}C-NMR, COSY, HMQC and HMBC spectra were run on Bruker spectrometers operating at 500, 400 and 300 MHz. The chemical shifts () are given in ppm and coupling constants in Hz. EI-MS, HR- EI-MS spectra were recorded on a JMS-HX-110 spectrometer, with a data system.

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Plant Collection

Sarcostemma viminale (Family: Asclepiadaceae) was collected from the hills near Hub in 1998 and was identified by Prof. Surryia Khatoon, Department of Botany, University of Karachi.

Extraction and Purification

The shade-dried ground whole plant (6 Kg) was exhaustively extracted with methanol at room temperature. The extract was evaporated to yield the residue (200 g). The whole residue was extracted with *n*-hexane, chloroform, ethyl acetate and *n*-butanol. The ethyl acetate soluble fraction (56 g) was subjected to CC over a silica gel column using hexane with gradient of acetone up to 100 % and followed by methanol. Ten fractions (Fr. 1-10) were collected. The Fr. 1 was loaded on silica gel (flash silica 230-400 mesh) and eluted with chloroform: hexane (1.5: 8.5) to purify **3** (20 mg). The Fr. 2 was loaded on flash silica gel and eluted with ethyl acetate: hexane (1.0: 9.0) to afford **4** (17.6 mg). The Fr. 3 was loaded on flash silica gel and eluted with ethyl acetate: hexane (1.5: 8.5) to afford **5** (29.3 mg). The Fr. 4 was subjected to flash chromatography and eluted with acetone: hexane (2.0: 8.0) to isolate **1** (18.7 mg). The Fr. 5 was subjected to flash chromatography and eluted with acetone: hexane (2.2: 7.8) to isolate **6** (20.7 mg). The Fr. 6 was subjected to flash chromatography and eluted with acetone: hexane (2.5: 7.5) to isolate **7** (30.7 mg). The Fr. 7 was subjected to flash chromatography and eluted with methanol: chloroform (1.5: 8.5) to isolate **2** (15.7 mg).

Pinoresinol (**1**). Crystalline solid, UV λ_{max} nm (log ε) (MeOH): 276 (1.92), 264 (1.54) 212 (4.24), 196 (2.80), 198 (4.91) nm; IR ν_{max} ($CHCl_3$): 2960 (C-H, Ar), 2866 (C-H, Aliphat), 1638, 1455 (C=C, Ar), 1322, 1080 (C-O-C), 830, 738, 672 cm^{-1}; EI-MS *m/z* : 358 $[M]^+$, 206, 194, 166, 151, 124. HR-EI-MS = $C_{20}H_{22}O_{6.}$ ^{1}H-NMR (400 MHz, MeOD), 7.26 (1H, (d, *J* = 2.0, H-2), 7.25 (1H, (d, *J* = 8.0, H-5), 7.14 (1H, (dd, *J* = 8.0, 2.0, H-6), 5.28 (1H, (d, *J* = 6.0, H-7), 3.48, (m, H-8), , 4.22, (dd, *J* = 9.0, 2.0, 9-â), 4.36, (dd, *J* = 9.0, 6.0, 9-α), 7.61, (1H, (d, *J* = 2.0, H-22), 7.26 (1H, (d, *J* = 8.0, H-52), 7.30 (1H, (dd, *J* = 8.0, 2.0, H-62), 5.29 (1H, (d, *J* = 6.0, H-72), 3.47, (m, H-82), 4.24, (dd, *J* = 9.0, 2.0, 92 ù–0+ù–□â), 4.38, (dd, *J* = 9.0, 6.0, 92-α), 3.81 (3H each, s, -OMe).

^{13}C-NMR (100 MHz, MeOD), 134.3 (C-1), 111.0 (C-2), 147.6 (C-3), 148.5 (C-4), 116.0 (C-5), 119.3 (C-6), 84.1 (C-7), 63.1 (C-8), 72.1 (C-9), 135.5 (C-12), 112.0 (C-22), 147.7 (C-32), 148.6 (C-42), 116.5 (C-52), 120.0 (C-62), 87.6 (C-72), 54.4 (C-82), 72.0 (C-92), 56.1, 56.2 (-OMe).

9α–Hydroxypinoresinol (**2**). Gummy solid, $[\alpha]^{23}_{D}$ + 55.0^0 (*c* = 0.95, MeOH); UV $_{max}$ nm (log ε) (MeOH): 278 (1.94), 262 (1.51) 210 (4.27), 198 (2.83), 196 (4.90) nm; IR ν_{max} (KBr): 2964 (C-H, Ar), 2863 (C-H, Aliphat), 1634, 1450 (C=C, Ar), 1322, 1080 (C-O-C), 832, 736, 670 cm^{-1}; EI-MS *m/z* : 374 $[M]^+$, 356, 206, 194, 166, 163, 153, 151, 124. HR-EI-MS = $C_{20}H_{22}O_{7,}$ (Calcd. for $C_{20}H_{22}O_7$, 384. 1234),

^{1}H-NMR (400 MHz, MeOD), 7.27 (1H, (d, *J* = 2.0, H-2), 7.26 (1H, (d, *J* = 8.0, H-5), 7.15 (1H, (dd, *J* = 8.0, 2.0, H-6), 5.27 (1H, (d, *J* = 6.0, H-7), 3.46, (m, H-8), 6.12 (1H, (d, *J* = 1.0, H-9), 7.62, (1H, (d, *J* = 2.0, H-22), 7.28 (1H, (d, *J* = 8.0, H-52), 7.31 (1H, (dd, *J* = 8.0, 2.0, H-62), 5.29 (1H, (d, *J* = 6.0, H-72), 3.47, (m, H-82), 4.24, (dd, *J* = 9.0, 2.0, 92 Đ□0+ù–□â), 4.38, (dd, *J* = 9.0, 6.0, 92-α), 3.83 (3H each, s, -OMe).

^{13}C-NMR (100 MHz, MeOD), 134.5 (C-1), 111.0 (C-2), 147.6 (C-3), 148.7 (C-4), 116.1 (C-5), 119.3 (C-6), 84.1 (C-7), 63.1 (C-8), 102.3 (C-9), 135.3 (C-12), 112.1 (C-22), 147.9 (C-32), 148.8 (C-42), 116.7 (C-52), 120.1 (C-62), 87.8

(C-72), 54.6 (C-82), 72.3 (C-92), 56.1, 56.2 (-OMe).

In Vitro Cholinesterase Inhibition Assay

Horse-serum BChE (E.C 3.1.1.8), butyrylthiocholine chloride, 5,5′-dithiobis [2-nitrobenzoic acid] (DTNB) and galanthamine were purchased from Sigma (St. Louis, MO, USA). All other chemicals were of analytical grade. butyrylcholinesterase inhibiting activitiy was measured by the spectrophotometric method developed by *Ellman et al* [4]. Butyrylthiocholine chloride was used as substrate to assay butyrylcholinesterase. The reaction mixture contained 150 µL of (100 mM) sodium phosphate buffer (pH 8.0), 10 µL of DTNB, 10 µL of test-compound solution and 20 µL of butyrylcholinesterase solution, mixed and incubated for 15 minutes (25 ^{o}C). The reaction was then initiated by the addition of 10-µL butyrylthiocholine. The hydrolysis of butyrylthiocholine was monitored by the formation of yellow 5-thio-2-nitrobenzoate anion as the result of the reaction of DTNB with thiocholine, released by the enzymatic hydrolysis of butyrylthiocholine, respectively, at a wavelength of 412 nm (15 min.). The test compound and the positive control (Galanthamine) were dissolved in EtOH. All reactions were performed in triplicate in 96-well micro-titer plates in *SpectraMax 340* (Molecular Devices, USA).

Determination of IC_{50} value

The percentage (%) inhibition was calculated as follows (E – S)/ E x 100, where E is the activity of the enzyme without the test compound and S is the activity of enzyme with the test compound. The concentrations of the test compounds that inhibited the hydrolysis of substrate (butyrylthiocholine) and oxidation of substrate (linoleic acid) by 50 % (IC_{50}) were determined by monitoring the effect of various concentrations of these compounds in the assays on the inhibition values. The IC_{50} value was then calculated using the EZ-Fit Enzyme Kinetics program (*Perrella Scientific Inc., Amherst*, USA).

Results and Discussion

Pinoresinol (Compound **1**) was isolated from the ethyl acetate soluble fraction (Fig. 1). The molecular ion peak at 358.153 was assigned to the molecular formula $C_{20}H_{22}O_6$ on the basis of high resolution election impact (HR-EI-MS) showing ten degrees of unsaturation. A lignan frame-work of the pinoresinol made up of two benzene moieties, two methoxy, three hydroxyl groups along with furofuran group was suggested by the ^{1}H and ^{13}C NMR spectra as mentioned in Table 1. This compound is now commercially available as ArboNova Product List (catalogue name). It showed inhibitory activity against butyrylcholinesterase (BChE) enzyme with the IC_{50} value of 15.0 ± 0.1 µM as compared to the Galanthamine (8.5 ± 0.01) taken as standard.

Compound **2** (9α-hydroxypinoresinol) was isolated as a gummy solid. The molecular ion peak at 374.123 corresponding to the molecular formula $C_{20}H_{22}O_7$ on the basis of high resolution election impact (HR-EI-MS) showed ten degrees of unsaturation. A lignan framework of the pinoresinol-type made up of two benzene moieties, two methoxy, three hydroxyl groups along with furofuran group was suggested by the ^{1}H and ^{13}C NMR spectra as mentioned in Table 2. One of the hydroxyl groups was assigned to C-9 on the basis of an acetal proton signal at ä 6.12 and an acetal carbon signal at ä 102.3 [5].

The structure of the 9α-hydroxypinoresinol (Fig. 1) was further supported by the mass fragment ions at m/z 163 and m/z 166 obtained as a result of vertical and horizontal cleavages, respectively, of the compound. These fragments permitted the placement of the two aryl groups at different heterocyclic rings [6]. The orientation of the hydroxyl group was

Table 1. ^{1}H and ^{13}C NMR data of compound **1.**

No.	Multiplicity (DEPT)	^{13}C-NMR (δ)	^{1}H-NMR (δ)	$^{1}J_{HH}$ (Hz)
1	C	134.3	-	-
2	CH	111.0	7.26	(d, J = 2.0)
3	C	147.6	-	-
4	C	148.5	-	-
5	CH	116.0	7.25	(d, J = 8.0)
6	CH	119.3	7.14	(dd, J = 8.0, 2.0)
7	CH	84.1	5.28	(d, J = 6.0)
8	CH	63.1	3.48	m
9-β	CH	72.1	4.22	(dd, J = 9.0, 2.0)
9-α			4.36	(dd, J = 9.0, 6.0)
1′	C	135.5	-	-
2′	CH	112.0	7.61	(d, J = 2.0)
3′	C	147.7	-	-
4′	C	148.6	-	-
5′	CH	116.5	7.26	(d, J = 8.0)
6′	CH	120.0	7.30	(dd, J = 8.0, 2.0)
7′	CH	87.6	5.29	(d, J = 6.0)
8′	CH	54.4	3.47	m
9′-β	CH	72.0	4.24	(dd, J = 9.0, 2.0)
9′-α			4.38	(dd, J = 9.0, 6.0)
-	OMe	56.1	3.83 (3H each)	s
	OMe	56.2		

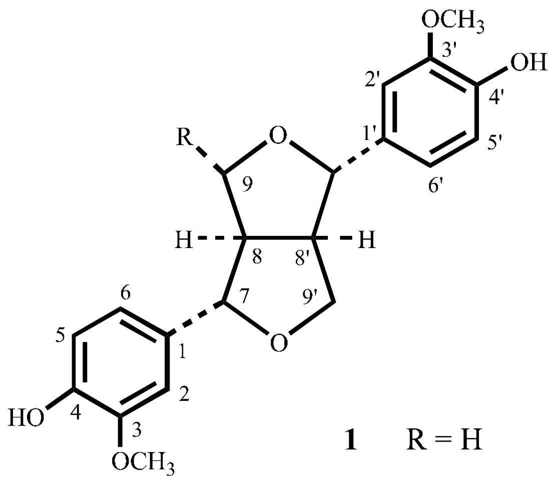
All spectra were recorded at 500 MHz (^{1}H) and 100 MHz (^{13}C); assignment were aided by 2D NMR COSY, HMQC and HMBC experiments, ^{13}C NMR multiplicities were determined by DEPT 135°.

Table 2. ^{1}H and ^{13}C NMR data of compound **2.**

No.	Multiplicity (DEPT)	^{13}C-NMR (δ)	^{1}H-NMR (δ)	$^{1}J_{HH}$ (Hz)
1	C	134.5	-	-
2	CH	111.0	7.27	(d, J = 2.0)
3	C	147.6	-	-
4	C	148.7	-	-
5	CH	116.1	7.26	(d, J = 8.0)
6	CH	119.3	7.15	(dd, J = 8.0, 2.0)
7	CH	84.1	5.27	(d, J = 6.0)
8	CH	63.1	3.46	m
9	CH	102.3	6.12	(d, J = 1.0)
1′	C	135.3	-	-
2′	CH	112.1	7.62	(d, J = 2.0)
3′	C	147.9	-	-
4′	C	148.8	-	-
5′	CH	116.7	7.28	(d, J = 8.0)
6′	CH	120.1	7.31	(dd, J = 8.0, 2.0)
7′	CH	87.8	5.29	(d, J = 6.0)
8′	CH	54.6	3.47	m
9′-β	CH	72.3	4.24	(dd, J = 9.0, 2.0)
9′-α			4.38	(dd, J = 9.0, 6.0)
-	OMe	56.1	3.83 (3H each)	s
	OMe	56.2		

All spectra were recorded at 500 MHz (^{1}H) and 100 MHz (^{13}C); assignment were aided by 2D NMR COSY, HMQC and HMBC experiments, ^{13}C NMR multiplicities were determined by DEPT 135°.

assigned as á, since a small coupling constant (J = 1.0 Hz) was observed between H-9 and H-8 of **2**. This was also confirmed by NOE (nuclear overhauser effect) experiment by the interaction of H-9 with H-7. The compound **2** showed inhibitory activity against butyrylcholine-sterase (BChE) enzyme with IC_{50} value of 275.1 ± 2.4 µM as compared to the Galanthamine (8.5 ± 0.01) taken as standard.



1 R = H

2 R = OH

Figure 1. Structure of Compounds 1 and 2

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MATRIX ELEMENTS FOR THE MORSE POTENTIAL

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Abstract: We show the usefulness of the hypervirial theorem to obtain the matrix elements $\langle m|e^{-\gamma au}|n\rangle$ for the Morse interaction.

Keywords: Morse potential, hypervirial theorem, matrix elements.
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Introduction

In [1] hypervirial theorem (HT) was applied to determine matrix elements for the one-dimensional harmonic oscillator. Here we shall employ HT to obtain $\langle m|e^{-\gamma au}|n\rangle$ for the Morse field [2]. Since in the formulation of HT, we find the presence of the potential V and the energy levels, we do not need to have explicitly the corresponding wave function. HT allows the following: a) to show that $\langle n|e^{-au}|n\rangle = \langle n|e^{-2au}|n\rangle$, and b) to calculate $\langle m|u|n\rangle$ and $\langle m|e^{-\gamma au}|n\rangle, \gamma = 3,4,5,...,$ if we know $\langle m|e^{-\beta au}|n\rangle, \beta = 1,2$. Besides, we consider the general expressions of [3,4] for $\langle m|e^{-\gamma au}|n\rangle$, with the comment that the total equivalence between the result of [3] and the associated relation of [4] has not been verified in [5].

Hypervirial theorem

The principal aim of our work is calculation of matrix elements for the Morse potential:

$$\langle m|f(r)|n\rangle = \int_0^\infty \psi_m f(r)\psi_n dr\,, \tag{1}$$

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where $\frac{1}{r}\psi_n$ is the radial wave function satisfying the Schrödinger equation (in natural units mass $=1$ and the Planck's constant $=1$):

$$\frac{d^2}{dr^2}\psi_n + 2[E_n - V(r)]\psi_n = 0\,. \tag{2}$$

The analytical procedure employs the explicit formulae of ψ_n and *f(r)*, and it makes directly the integral (1); however, in [1] we see that HT evaluates (1) for the harmonic oscillator without the explicit form of the wave function. The Schrödinger equation has all information on our quantum system, and HT has a part of this information which permits to study (1) without the explicit use of ψ_n.

From (2) it is easy to obtain HT [1]:

$$\begin{gathered}(E_m - E_n)^2\langle m|f|n\rangle + \frac{1}{4}\langle m|f''''|n\rangle + \\ (E_m + E_n)\langle m|f''|n\rangle - 2\langle m|f''V|n\rangle - \langle m|f'V'|n\rangle = 0\end{gathered}\,, \tag{3}$$

where the prime means $\frac{d}{dr}$, with the participation of the energy spectrum associated to potential $V(r)$. The following procedure permits to deduce

(3): The eq. (2) is equivalent to $H|n\rangle = E_n|n\rangle$ (or $\langle n|H = E_n\langle n|$) for its corresponding Hamiltonian H. From this it is evident that the identity $(E_m - E_n)\langle m|f|n\rangle = \langle m|Hf - fH|n\rangle$ motivates the relation $(E_m - E_n)^2 \quad \langle m|f|n\rangle = \langle m|H(Hf - fH) - (Hf - fH)H|n\rangle$. If here we substitute the expression for H and make the integration by parts, then it immediately yields (3). Here we consider the Morse interaction [2,6] which represents an approximation to vibrational motion of diatomic molecule:

$$V(u) = D(e^{-2au} - 2e^{-au}), \qquad u = r - r_0,$$
$$E_n = -\frac{a^2}{8}b^2, \qquad b = k - 2n - 1, \quad k = \frac{2}{a}\sqrt{2D}, \qquad (4)$$

where D is the dissociation energy (potential depth), r_0 is the nuclear separation (characteristic length), and a is a parameter associated with the well width (range of the interaction), being $\frac{a}{2\pi}\sqrt{2D}$ the frequency of small classical vibrations around r_0.

Now we shall take some examples for the particular functions *f(r)* to illustrate how the HT (3) gives information on matrix elements.

Let us choose $f(r) = u(r) = r - r_0$. (4')

Then from (3), (4) and (4') we deduce that:

$$2aD\langle m|e^{-au} - e^{-2au}|n\rangle = (E_m - E_n)^2\langle m|u|n\rangle, \quad (5)$$

with two cases:

a) $m = n$.

Thus (5) implies an identity for diagonal elements

$$\langle n|e^{-au}|n\rangle = \langle n|e^{-2au}|n\rangle, \qquad (6)$$

and we observe that here (6) was obtained without the explicit knowledge of ψ_n and E_n: In Sec 3 we shall employ the formulae of [3, 4] to show (6) with $\langle n|e^{-au}|n\rangle = \frac{1}{k}(k - 2n - 1)$, which was demonstrated by Huffaker-Dwivedi [8] with the factorization method.

b) $m < n$

The relation (5) leads to:

$$\langle m|u|n\rangle = 2aD(E_m - E_n)^{-2}\langle m|e^{-au} - e^{-2au}|n\rangle, \qquad (7)$$

which means that all elements $\langle m|u|n\rangle$ are determined if we know $\langle m|e^{-\beta au}|n\rangle$, $\beta = 1,2$. From the expressions of [3,4] we can obtain the values for $m \le n$:

$$\langle m|e^{-au}|n\rangle = \frac{(-1)^{n+m}}{k}\left[\frac{b_1 b_2 n!\Gamma(k-n)}{m!\Gamma(k-m)}\right]^{1/2}, \quad \begin{matrix} b_1 = k - 2n - 1 \\ b_2 = k - 2m - 1 \end{matrix} \qquad (8)$$
$$\langle m|e^{-2au}|n\rangle = \frac{1}{k}[(n+1)(k-n) - m(k-m-1)]\langle m|e^{-au}|n\rangle$$

where Γ denotes the gamma function, then (7) and (8) imply the result:

$$\langle m|u|n\rangle = \frac{k}{a}[(m-n)(k-n-m-1)]^{-1}, \quad m<n \qquad (9)$$

This expression was deduced analytically by Gallas [7] without the use of HT; Gallas employs the explicit structure of ψ_n and the following non-trivial identity:

$$\frac{m!\Gamma(k-m)}{n!\Gamma(k-n)}\sum_{j=0}^{m}\frac{(n-m+j-1)!\Gamma(k-n-m+j-1)}{j!\Gamma(k-2m+j)} =$$
$$[(n-m)(k-n-m-1)]^{-1} \qquad (10)$$

For our work, the relation (10) is irrelevant. The equation (6) also is a consequence of (8) when $m = n$. In [9] eq. (9) is proved via ladder operators.

Next, by substituting $f(r)=e^{-\gamma a(r-r_0)}, \gamma=1,2,\ldots,$ from (3) and (4) we obtain:

$$\left[(E_m-E_n)^2+\frac{\gamma^4a^4}{4}+\gamma^2a^2(E_m+E_n)\right]\left\langle m\left|e^{-\gamma au}\right|n\right\rangle- \\ 2\gamma Da^2(1+\gamma)\left\langle m\left|e^{-(\gamma+2)au}\right|n\right\rangle+ \\ 2\gamma Da^2(1+2\gamma)\left\langle m\left|e^{-(\gamma+1)au}\right|n\right\rangle=0. \quad (11)$$

Thus if $\gamma=1$, then (11) gives us:

$$\left[(E_m-E_n)^2+\frac{a^4}{4}+a^2(E_m+E_n)\right]\left\langle m\left|e^{-au}\right|n\right\rangle- \\ -4Da^2\left\langle m\left|e^{-3au}\right|n\right\rangle+6Da^2\left\langle m\left|e^{-2au}\right|n\right\rangle=0. \quad (12)$$

Then substituting (4) into (8):

$$\left\langle m\left|e^{-3au}\right|n\right\rangle= \\ k^{-2}\left\{\begin{array}{l}(n+1)(k-n)\left[\frac{1}{2}(n+2)(k-n+1)-m(k-m-1)\right] \\ +\frac{m}{2}(m-1)(k-m-1)(k-m-2)\end{array}\right\}\left\langle m\left|e^{-au}\right|n\right\rangle, \quad (13)$$

Eq. (13) can be deduced from [3,4]. Our following eqs. (14) and (15) also determine the matrix elements $\left\langle m\left|e^{-3au}\right|n\right\rangle$ given above which demonstrates here the usefulness of HT method, alternatively.

Matrix elements $\left\langle e^{-\gamma au}\right\rangle$

Here we exhibit general formulae for $\left\langle m\left|\exp(-\gamma a(r-r_0))\right|n\right\rangle, \gamma=1,2\ldots$ In fact, Vasan-Cross [3] substitute into (1) the Morse's wave function [6] and exponential function $f=e^{-\gamma a(r-r_0)}$. Furthermore, they use known integrals satisfied by the Laguerre polynomials to obtain the following expression for $m\le n$:

$$\left\langle m\left|e^{-\gamma au}\right|n\right\rangle=\frac{(-1)^{m+n}}{k^\gamma}\left[\frac{b_1b_2m!\Gamma(k-m)}{n!\Gamma(k-n)}\right]^{1/2}\bullet \\ \sum_{j=0}^{m}\frac{(-1)^j(n+\gamma-1-j)!\Gamma(k-n-1+\gamma-j)}{j!(m-j)!(\gamma-1-j)!\Gamma(k-m-j)}, \quad (14)$$

which allows to verify (6) and (8). On the other hand, Berrondo *et al.* [4] employ the relationship between Morse potential and the two-dimensional harmonic oscillator to deduce the corresponding relation (its derivation is similar as in (14), but here we first begin in Cartesian coordinates of xy-plane associated to the harmonic oscillator in two dimensions, and then we change to polar coordinates to make the connection with the Morse's vibrational motion):

$$\left\langle m\left|e^{-\gamma au}\right|n\right\rangle=\frac{(-1)^{m+n}}{k^\gamma}\left[\frac{b_1b_2m!n!}{\Gamma(k-m)\Gamma(k-n)}\right]^{1/2}\bullet \\ \sum_{j=0}^{m}\binom{m-n+\gamma-1}{j}\binom{n-m+\gamma-1}{\gamma-1-j}\frac{\Gamma(k-n-1+\gamma-j)}{(m-j)!} \quad (15)$$

which also reproduces (6) and (8), i.e., we have the equality of (14) and (15) for $\gamma=1,2,$. However, it still remains an open problem requiring evidence that both expressions are totally equivalent for any γ. We can find absolutely everything we might want to know about the properties of diatomic molecules in the monu-mental data compilation of Huber-Herzberg [10]. Finally, [11] contains an account of how Morse arrived at the potential that bears his name.

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SEMI-COMPACTNESS IN FUZZIFYING TOPOLOGY

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Abstract: In this paper we introduce and study the concept of semi-compactness in the framework of fuzzifying topology. We use the finite intersection property to give a characterization of the fuzzifying semi-compactness.

Keywords: Fuzzy logic, fuzzifying topology, fuzzifying compactness, s-topology, Mathematics Subject Classification: 54A40, 54A05.

Introduction

In 1991, Ying [7] used the semantic method of continuous valued logic to propose the so-called fuzzifying topology as a preliminary of the research on bifuzzy topology and elementally develop topology in the theory of fuzzy sets from completely different direction. Briefly speaking, a fuzzifying topology on a set X assigns each crisp subset of X to a certain degree of being open, other than being definitely open or not. Furthermore, Ying [10] introduced the concepts of compactness and established a generalization of Tychonoff's theorem in the framework of fuzzifying topology. In [6] Prasad and Yadav introduced the concepts of s-topological spaces and s-compactness. In [2,3] Hanna and Dorsett used the term semi-compactness instead of s-compactness and completely characterized it. In this paper we introduced and study the concept of semi-compactness in the framework of fuzzifying topology.

Preliminaries

Below, we present the fuzzy logical and corresponding set theoretical notations [7,8] since we need them in this paper.

For any formula φ, the symbol $[\varphi]$ means the truth value of φ, where the set of truth values is the unit interval [0, 1]. We write $\vDash \varphi$ if $[\varphi] = 1$ for any interpretation. By $\vDash^{w} \varphi$ (φ is feebly valid) we mean that for any valuation it always holds that $[\varphi] > 0$, and we mean that $[\varphi] > 0$ implies $[\psi] = 1$. The truth valuation rules for primary fuzzy logical formulae and corresponding set theoretical notations are:

(1) (a) $[\alpha] = \alpha\ (\alpha \in [0,1])$;
 (b) $[\varphi \wedge \psi] = \min([\varphi],[\psi])$;
 (c) $[\varphi \rightarrow \psi] = \min(1, 1 - [\varphi] + [\psi])$.

(2) If $\widetilde{A} \in \Im(X)$, $[x \in \widetilde{A}] := \widetilde{A}(x)$.

(3) If X is the universe of discourse, then $[\forall x \varphi(x)] := \inf_{x \in X} [\varphi(x)]$.

In addition the truth valuation rules for some derived formulae are:

(1) $[\neg\varphi] := [\varphi \rightarrow 0] = 1 - [\varphi]$;
(2) $[\varphi \vee \psi] := [\neg(\neg\varphi \wedge \neg\psi)] = \max([\varphi],[\psi])$;
(3) $[\varphi \leftrightarrow \psi] := [(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)]$;
(4) $[\varphi \dot{\wedge} \psi] := [\neg(\varphi \rightarrow \neg\psi) = \max(0, [\varphi] + [\psi] - 1)$;
(5) $[\exists x \varphi(x)] := [\neg\forall x \neg\varphi(x)] := \sup_{x \in X} [\varphi(x)]$;
(6) If $\widetilde{A}, \widetilde{B} \in \Im(X)$, then
$[\widetilde{A} \subseteq \widetilde{B}] := [\forall x (x \in \widetilde{A} \rightarrow x \in \widetilde{B})]$
$= \inf_{x \in X} \min(1, 1 - \widetilde{A}(x) + \widetilde{B}(x))$,.

where $\Im(X)$ is the family of all fuzzy sets in X.

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Definition 2.1.[7]. Let X be a universe of discourse, $\tau \in \Im(P(X))$ satisfy the following conditions:

(1) $\tau(X)=1, \tau(\phi)=1$;

(2) for any $A, B, \tau(A \cap B) \geq \tau(A) \wedge \tau(B)$;

(3) for any $\{A_\lambda : \lambda \in \Lambda\}, \tau\left(\bigcup_{\lambda \in \Lambda} A_\lambda\right) \geq \bigwedge_{\lambda \in \Lambda} \tau(A_\lambda)$.

Then τ is called a fuzzifying topology and (X,τ) is a fuzzifying topological space.

Definition 2.2.[7]. The family of all fuzzifying closed sets, denoted by $F \in \Im(P(X))$, is defined as
$A \in F := X - A \in \tau$, where $X - A$ is the complement of A.

Definition 2.3.[7]. The fuzzifying neighborhood system of a point $x \in X$ is denoted by $N_x \in \Im(P(X))$ and defined as $N_x(A) = \sup_{x \in B \subseteq A} \tau(B)$.

Definition 2.4.[7, Lemma 5.2]. The closure $\overline{A}$ of A is defined as $\overline{A}(x) = 1 - N_x(X-A)$.
In Theorem 5.3[7], Ying proved that the closure $^{-} : P(X) \to \Im(X)$ is a fuzzifying closure operator (see Definition 5.3 [7]) because its extension

$$^{-} : \Im(X) \to \Im(X), \overline{\mathcal{A}} = \bigcup_{\alpha \in [0,1]} \alpha \overline{\mathcal{A}_\alpha}, \mathcal{A} \in \Im(X),$$

where $\mathcal{A}_\alpha = \{x : \mathcal{A}(x) \geq \alpha\}$ is the α-cut of A and $\alpha\mathcal{A}(x) = \alpha \wedge \mathcal{A}(x)$ satisfies the following Kuratowski closure axioms:

(1) $\models \overline{\phi} = \phi$;

(2) for any $\mathcal{A} \in \Im(X), \models \mathcal{A} \subseteq \overline{\mathcal{A}}$;

(3) for any $\mathcal{A}, \widetilde{B} \in \Im(X), \models \overline{\mathcal{A} \cup \widetilde{B}} \equiv \overline{\mathcal{A}} \cup \overline{\widetilde{B}}$;

(4) for any $\mathcal{A}, \widetilde{B} \in \Im(X), \models \overline{\left(\overline{\mathcal{A}}\right)} \subseteq \mathcal{A}$.

Definition 2.5.[8]. For any $A \subseteq X$, the fuzzy set of interior points of A is called the interior of A, and given as $A^\circ(x) := N_x(A)$.
From Lemma 3.1 [7] and the definitions of $N_x(A)$ and A° we have $\tau(A) = \inf_{x \in A} A^\circ(x)$.

Definition 2.6.[4]. For any
$\mathcal{A} \in \Im(X), \models \left(\mathcal{A}\right)^\circ \equiv X - \overline{\left(X - \mathcal{A}\right)}$.

Lemma 2.1 [4]. If $[\mathcal{A} \subseteq \widetilde{B}] = 1$, then
(1) $\models \overline{\mathcal{A}} \subseteq \overline{\widetilde{B}}$ (2) $\models \left(\mathcal{A}\right)^\circ \subseteq \left(\widetilde{B}\right)^\circ$.

Definition 2.7.[4]. Let (X,τ) be a fuzzifying topological space.

(1) The family of all fuzzifying semi-open sets, denoted by $\tau_S \in \Im(P(X))$, is defined as

$A \in \tau_S := \forall x(x \in A \to x \in A^{\circ-})$, i.e.,
$\tau_S(A) = \inf_{x \in A} A^{\circ-}(x)$

(2) The family of all fuzzifying semi-closed sets, denoted by $F_S \in \Im(P(X))$, is defined as
$A \in F_S := X - A \in \tau_S$.

(3) The fuzzifying semi-neighborhood system of a point $x \in X$ is denoted by
$N_x^S \in \Im(P(X))$ and defined as
$N_x^S(A) = \sup_{x \in A} A^{\circ-}(x)$.

(4) The fuzzifying semi-closure of a set $A \in P(X)$, denoted by $Cl_S \in \Im(X)$, is defined as $Cl_S(A)(x) = 1 - N_x^S(X-A)$.

(5) Let (X,τ) and (Y,σ) be two fuzzifying topological spaces and let $f \in Y^X$. A unary fuzzy predicate $C_S \in \Im(Y^X)$, called fuzzifying semi-continuity, is given as
$C_S(f) := \forall B(B \in \sigma \to f^{-1}(B) \in \tau_S)$.

Definition 2.8. [1]. Let (X,τ) and (Y,σ) be two fuzzifying topological spaces and let $f \in Y^X$. A unary fuzzy predicate $I \in \Im(Y^X)$, called fuzzifying irresolute, is given as
$I(f) := \forall B(B \in \sigma_S \to f^{-1}(B) \in \tau_S)$.

Definition 2.9. [5]. Let Ω be the class of all fuzzifying topological spaces. The unary fuzzy predicate $T_2^S \in \Im(\Omega)$ is defined as

$$T_2^S(X,\tau) := \forall x \forall y((x \in X \wedge y \in X \wedge x \neq y) \to \exists B \exists C(B \in N_x^S \wedge C \in N_y^S \wedge B \cap C = \phi)).$$

Definition 2.10.[10]. Let X be a set. If $\widetilde{A} \in \Im(X)$ such that the support supp $\widetilde{A} = \{x \in X : \widetilde{A}(x) > 0\}$ of A is finite, then $\widetilde{A}$ is said to be finite and we write $F(\widetilde{A})$. A unary fuzzy predicate $FF \in \Im(\Im(X))$, called fuzzy finiteness, is given as

$FF(\widetilde{A}) := (\exists \widetilde{B})(F(\widetilde{B}) \wedge (\widetilde{A} \equiv \widetilde{B}))$
$= 1 - \inf\{\alpha \in [0,1] : F(\widetilde{A}_\alpha)\}$
$= 1 - \inf\{\alpha \in [0,1] : F(\widetilde{A}_{[\alpha]})\}$,
where $\widetilde{A}_\alpha = \{x \in X : \widetilde{A}(x) \geq \alpha\}$ and
$\widetilde{A}_{[\alpha]} = \{x \in X : \widetilde{A}(x) > \alpha\}$.

Definition 2.11.[8]. Let X be a set.

(1) A binary fuzzy predicate $K \in \Im(\Im(P(X)) \times P(X))$, called fuzzifying covering, is given as

$K(\Re, A) := \forall x(x \in A \to \exists B(B \in \Re \wedge x \in B))$.

(2) Let (X,τ) be a fuzzifying topological space. A binary fuzzy predicate $K_\circ \in \Im(\Im(P(X)) \times P(X))$, called fuzzifying open covering, is given as

$K_\circ(\Re, A) := K(\Re, A) \dot{\wedge} (\Re \subseteq \tau)$.

Definition 2.12.[10]. Let Ω be the class of all fuzzifying topological spaces. A unary fuzzy predicate $\Gamma \in \Im(\Omega)$, called fuzzifying compactness, is given as

$(X,\tau) \in \Gamma := (\forall \Re)(K_\circ(\Re, X)$
$\to (\exists \wp)((\wp \leq \Re) \wedge K(\wp, A) \dot{\wedge} FF(\wp)))$.

Definition 2.13.[10]. Let X be a set. A unary fuzzy predicate $\mathrm{fI} \in \Im(\Im(P(X)))$, called fuzzifying finite intersection property, is given as

$\mathrm{fI}(\Re) := (\forall \beta)((\beta \leq \Re) \wedge FF(\beta)$
$\to (\exists x)(\forall B)((B \in \beta) \to (x \in B)))$.

Lemma 2.2.[4]. Let (X,τ) be a fuzzifying topological space. Then

(1) $\models \tau \subseteq \tau_S$; (2) $\models F \subseteq F_S$,

(3) $\models F_S\left(\bigcap_{\lambda \in \Lambda} A_\lambda\right) \geq \bigwedge_{\lambda \in \Lambda} F_S(A_\lambda)$.

Semi-compactness in fuzzifying topology

Definition 3.1. A fuzzifying topological space (X,τ) is said to be s-fuzzifying topological space if $\tau_s(A \cap B) \geq \tau_s(A) \wedge \tau_s(B)$.

Definition 3.2. A binary fuzzy predicate $K_S \in \Im(\Im(P(X)) \times P(X))$, called fuzzifying semi-open covering, is given as
$K_S(\Re, A) := K(\Re, A) \dot{\wedge} (\Re \subseteq \tau_s)$.

Definition 3.3. Let Ω be the class of all fuzzifying topological spaces. A unary fuzzy p-redicate $\Gamma_S \in \Im(\Omega)$, called fuzzifying semi-compactness, is given as

(1) $(X,\tau) \in \Gamma_S := (\forall \Re)(K_S(\Re, X)$
$\to (\exists \wp)((\wp \leq \Re) \wedge K(\wp, A) \dot{\wedge} FF(\wp)))$;

(2) If $A \subseteq X$, then $\Gamma_S(A) := \Gamma_S(A, \tau_{/A})$.

Lemma 3.1. $\models K_\circ(\Re, A) \to K_S(\Re, A)$.

Proof. Since from Lemma 2.2 $\models \tau \subseteq \tau_S$, then we have $[\Re \subseteq \tau] \leq [\Re \subseteq \tau_S]$. So, $[K_\circ(\Re, A)] \leq [K_S(\Re, A)]$.

The following theorem compares compacttness and semi-compactners for fuzzifying topology.

Theorem 3.1. $\models (X,\tau) \in \Gamma_S \to (X,\tau) \in \Gamma$.

Proof. From Lemma 3.1 the proof is immediate.

Theorem 3.2. For any fuzzifying topological space (X,τ) and $A \subseteq X$,

$\Gamma_S(A) \leftrightarrow (\forall \Re)(K_S(\Re, A)$
$\to (\exists \wp)((\wp \leq \Re) \wedge K(\wp, A) \dot{\wedge} FF(\wp)))$,

where K_S is related to τ.

Proof. For any $\Re \in \Im(\Im(X))$, we set $\overline{\Re} \in \Im(\Im(A))$ defined as $\overline{\Re}(C) = \Re(B)$ with $C = A \cap B, B \subseteq X$. Then

$K(\overline{\Re}, A) = \inf_{x \in A} \sup_{x \in C} \overline{\Re}(C) = \inf_{x \in A} \sup_{x \in C = A \cap B} \Re(B)$
$= \inf_{x \in A} \sup_{x \in B} \Re(B) = K(\Re, A)$, because $x \in A$ and $x \in B$ if and only if $x \in A \cap B$. Therefore

$$[\Re \subseteq \tau_S/_A] = \inf_{C\subseteq A} \min(1, 1 - \Re(C) + \tau_S/_A(C))$$
$$= \inf_{C\subseteq A} \min(1, 1 - \sup_{C=A\cap B, B\subseteq X} \Re(B) + \sup_{C=A\cap B, B\subseteq X} \tau_S(B))$$
$$\geq \inf_{C\subseteq A, C=A\cap B, B\subseteq X} \min(1, 1 - \Re(B) + \tau_S(B))$$
$$\geq \inf_{B\subseteq X} \min(1, 1 - \Re(B) + \tau_S(B)) = [\Re \subseteq \tau_S].$$

For any $\wp \leq \Re$, we define $\wp' \in \Im(P(X))$ as $\wp'(B) = \begin{cases}\wp(B) & \text{if } B\subseteq A,\\ 0 & \text{otherwise.}\end{cases}$

Then $\wp' \leq \Re, FF(\wp') = FF(\wp)$ and $K(\wp', A) = K(\wp, A)$. Furthermore

$$[\Gamma_S(A) \dot\wedge K_S(\Re, A)] \leq [\Gamma_S(A) \dot\wedge K_S'(\Re, A)] \leq$$
$$[(\exists\wp)((\wp \leq \Re) \wedge K(\wp, A) \dot\wedge FF(\wp))]$$
$$\leq [(\exists\wp')((\wp' \leq \Re) \wedge K(\wp', A) \dot\wedge FF(\wp'))] \leq [(\exists ß)((ß \leq \Re) \wedge K(ß, A) \dot\wedge FF(ß))].$$

Then $\Gamma_S(A) \leq [K_S(\Re,A)] \to [(\exists ß)((ß\leq \Re) \wedge K(ß,A) \dot\wedge FF(ß))]$,

where $K_S'(\Re, A) = [K_S(\Re, A) \dot\wedge (\Re \subseteq \tau_S/_A)]$.

Therefore

$$\Gamma_S(A) \leq \inf_{\Re\in\Im(P(X))} [K_S(\Re, A) \to (\exists ß)((ß \leq \Re) \wedge K(ß, A) \dot\wedge FF(ß))]$$
$$= [(\forall\Re)(K_S(\Re, A) \to (\exists ß)((ß \leq \Re) \wedge K(ß, A) \dot\wedge FF(ß)))].$$

Conversely, for any $\Re \in \Im(P(A))$, if

$$[\Re \subseteq \tau_S/_A] = \inf_{B\subseteq A} \min(1, 1 - \Re(B) + \tau_S/_A(B)) = \lambda,$$

then for any $n \in N$ and $B \subseteq A$,

$$\sup_{B=A\cap C, C\subseteq X} \tau_S(C) = \tau_S/_A(B) > \lambda + \Re(B) - 1 - \tfrac{1}{\pi},$$

and there exists $C_B \subseteq X$ such that $C_B \cap A = B$ and $\tau_S(C_B) > \lambda + \Re(B) - 1 - \frac{1}{\pi}$.

Now, we define $\Re \in \Im(P(X))$ as

$\Re(C) = \max_{B\subseteq A} (0, \lambda + \Re(B) - 1 - \frac{1}{\pi})$. Then

$[\overline{\Re} \subseteq \tau_S] = 1$ and

$$K(\overline{\Re}, A) = \inf_{x\in A} \sup_{x\in C\subseteq X} \overline{\Re}(C) = \inf_{x\in A} \sup_{x\in B} \overline{\Re}(C_B)$$
$$\geq \inf_{x\in A} \sup_{x\in B} (\lambda + \Re(B) - 1 - \tfrac{1}{n})$$
$$= \inf_{x\in A} \sup_{x\in B} \Re(B) + \lambda - 1 - \tfrac{1}{n} = K(\Re, A) + \lambda - 1 - \tfrac{1}{n},$$

$$K_S(\Re, A) = [K(\Re, A) \dot\wedge (\Re \subseteq \tau_S)] = [K(\Re, A)$$
$$\geq \max(0, K(\Re, A) + \lambda - 1 - \tfrac{1}{\pi})$$
$$\geq \max(0, K(\Re, A) + \lambda - 1) - \tfrac{1}{\pi} = K_S(\Re', A) - \tfrac{1}{\pi}.$$

For any $\wp \leq \overline{\Re}$, we set $\wp' \in \Im(P(A))$ as $\wp'(B) = \wp(C_B), B \subseteq A$.

Then $\wp' \leq \Re, FF(\wp') = FF(\wp)$ and $K(\wp', A) = K(\wp, A)$. Therefore

$$[(\forall\Re)(K_S(\Re, A) \to (\exists\wp)((\wp \leq \Re) \wedge K(\wp, A) \dot\wedge FF(\wp))] \dot\wedge [K_S'(\Re, A)] - \tfrac{1}{\pi}$$
$$\leq [(\forall\Re)(K_S(\Re, A) \to (\exists\wp)((\wp \leq \Re) \wedge K(\wp, A) \dot\wedge FF(\wp))] \dot\wedge \left([K_S'(\Re, A)] - \tfrac{1}{\pi}\right)$$
$$\leq [K_S(\Re, A) \to (\exists\wp)((\wp \leq \Re) \wedge K(\wp, A) \dot\wedge FF(\wp))] \dot\wedge [K_S(\Re, A)]$$
$$\leq [(\exists\wp)((\wp \leq \Re) \wedge K(\wp, A) \dot\wedge FF(\wp))]$$
$$\leq [(\exists\wp')((\wp' \leq \Re) \wedge K(\wp', A) \dot\wedge FF(\wp'))]$$
$$\leq [(\exists ß)((ß \leq \Re) \wedge K(ß, A) \dot\wedge FF(ß))],$$

Let $n \to \infty$. We obtain

$$[(\forall\Re)(K_S(\Re, A) \to (\exists\wp)((\wp \leq \Re) \wedge K(\wp, A) \dot\wedge FF(\wp)))] \dot\wedge [K_S'(\Re, A)]$$
$$\leq [(\exists ß)((ß \leq \Re) \wedge K(ß, A) \dot\wedge FF(ß))].$$ Then

$$[(\forall\Re)(K_S(\Re, A) \to (\exists\wp)((\wp \leq \Re) \wedge K(\wp, A) \dot\wedge FF(\wp))]$$
$$\leq [K_S'(\Re, A) \to (\exists ß)((ß \leq \Re) \wedge K(ß, A) \dot\wedge FF(ß))]$$
$$\leq \inf_{\Re\in\Im(P(X))} [K_S'(\Re, A) \to (\exists ß)((ß \leq \Re) \wedge K(ß, A) \dot\wedge FF(ß))] = \Gamma_S(A).$$

Theorem 3.3. Let (X,τ) be a fuzzifying topological space.

$$\pi_1 := (\forall \Re)\Big((\Re \in \Im(P(X))) \wedge (\Re \subseteq F_S) \dot{\wedge} \mathrm{fI}(\Re) \to (\exists x)(\forall A)(A \in \Re \to x \in A)\Big);$$

$$\pi_2 := (\forall \Re)(\exists B)(((\Re \subseteq F_S) \wedge (B \in \tau_S)) \dot{\wedge} (\forall \wp)((\wp \le \Re) \dot{\wedge} FF(\wp) \to \neg(\bigcap \wp \subseteq B)) \to \neg(\bigcap \Re \subseteq B))).$$

Then $\models \Gamma_S(X,\tau) \leftrightarrow \pi_i,\ i = 1,2$.

Proof.

(a) First, we prove $\Gamma_S(X,\tau) = [\pi_1]$. For any $\Re \in \Im(P(X))$, we set $\Re^c(X-A) = \Re(A)$.

Then $[\Re \subseteq \tau_S] = \inf_{A\in P(X)} \min(1, 1-\Re(A)+\tau_S(A))$

$= \inf_{X-A\in P(X)} \min(1, 1-\Re^c(X-A)+F_S(X-A))$

$= [\Re^c \subseteq F_S]$,

$FF(\Re) = 1-\inf\{\alpha \in [0,1] : F(\Re_\alpha)\} =$
$1-\inf\{\alpha \in [0,1] : F(\Re^c_\alpha)\} = FF(\Re^c)$

and

$\text{ß} \le \Re^c \Leftrightarrow \text{ß}(M) \le \Re^c(M) \Leftrightarrow$
$\text{ß}^c(X-M) \le \Re(X-M) \Leftrightarrow \text{ß}^c \le \Re$.

Therefore

$$\Gamma_S(X,\tau) = [(\forall \Re)\Big(K_S(\Re,X) \to (\exists \wp)((\wp \le \Re) \wedge K(\wp,X) \dot{\wedge} FF(\wp))\Big)]$$

$$= [(\forall \Re)\Big((\Re \subseteq \tau_S) \dot{\wedge} K(\Re,X) \to (\exists \wp)((\wp \le \Re) \wedge K(\wp,X) \dot{\wedge} FF(\wp))\Big)]$$

$$= [(\forall \Re)\Big((\Re \subseteq \tau_S) \to (K(\Re,X) \to (\exists \wp)((\wp \le \Re) \wedge K(\wp,X) \dot{\wedge} FF(\wp)))\Big)]$$

$$= [(\forall \Re)\Big((\Re^c \subseteq F_S) \to ((\forall x)(\exists A)(A \in \Re \wedge x \in A) \to (\exists \wp)((\wp \le \Re) \wedge K(\wp,X) \dot{\wedge} FF(\wp))\Big)]$$

$$= [(\forall \Re)\Big((\Re^c \subseteq F_S) \to ((\forall x)(\exists A)(A \in \Re \wedge x \in A) \to (\exists \text{ß}^c)((\text{ß}^c \le \Re) \wedge K(\text{ß}^c,X) \dot{\wedge} FF(\text{ß}^c)))\Big)]$$

$$= [(\forall \Re)\Big((\Re^c \subseteq F_S) \to ((\forall x)(\exists A)(A \in \Re \wedge x \in A) \to (\exists \text{ß})((\text{ß} \le \Re^c) \wedge FF(\text{ß}) \dot{\wedge} K(\text{ß}^c,X)))\Big)]$$

$$= [(\forall \Re)(\Re^c \subseteq F_S \to ((\forall x)(\exists A)(A \in \Re \wedge x \in A) \to (\exists \text{ß})((\text{ß} \le \Re^c) \wedge FF(\text{ß}) \dot{\wedge} (\forall x)(\exists B)(B \in \text{ß}^c \wedge x \in B))]$$

$$= [(\forall \Re)(\Re^c \subseteq F_S \to (\neg((\exists \text{ß})(\text{ß} \le \Re^c \wedge FF(\text{ß}) \dot{\wedge} (\forall x)(\exists B)(B \in \text{ß}^c \wedge x \in B)) \to \neg((\forall x)(\exists A)(A \in \Re \wedge x \in A))))]$$

$$= [(\forall \Re)((\Re^c \subseteq F_S) \to (\mathrm{fI}(\Re^c) \to \neg((\forall x)(\exists A)(A \in \Re \wedge x \in A)))]$$

$$= [(\forall \Re)((\Re^c \subseteq F_S) \dot{\wedge} \mathrm{fI}(\Re^c) \to (\exists x)(\forall A)(A \in \Re^c \to x \in A))] = [\pi_1].$$

(b) Let $X-B \in P(X)$. For any $\Re \in \Im(P(X))$.

$$[(\Re \subseteq F_S) \wedge (B \in \tau_S)] = [(\Re \subseteq F_S) \wedge (X-B \in F_S)]$$

$$= \inf_{A\in P(X)} \min(1, 1-\Re(A)+F_S(A)) \wedge F_S(X-B)$$

$$= \inf_{A\in P(X)} \min(1, 1-\Re(A)+F_S(A))$$

$$\wedge \inf_{A\in P(X)} \min(1, 1-[A \in \{X-B\}] + F_S(A))$$

$$= \inf_{A\in P(X)} \min\big(1, 1-[(\Re \cup \{X-B\})(A)] + F_S(A)\big) = [(\Re \cup \{X-B\}) \subseteq F_S].$$

Therefore, for any $\text{ß} \in \Im(P(X))$, let $\wp = \text{ß} \setminus \{X-B\} \in \Im(P(X))$.

$\wp(A) = \begin{cases} \text{ß}(A), & A \ne X-B \\ 0, & A = X-B. \end{cases}$ Then $\wp \le \text{ß}$,

$\wp \cup \{X-B\} \ge \text{ß}$, $[FF(\wp)] = [FF(\text{ß})]$,
$[\wp \le \Re] = [\text{ß} \le (\Re \cup \{X-B\})]$ and

$$[(\forall \wp)\Big((\wp \le \Re) \dot{\wedge} FF(\wp) \to (\exists x)(\forall A)(A \in (\wp \cup \{X-B\}) \to (x \in A))\Big)]$$

$$= \inf_{\wp \le \Re} \min\Big(1, 1-[FF(\wp)] + \sup_{x\in X} \inf_{A\in P(X)} ((\wp \cup \{X-B\})(A) \to A(x))\Big)$$

$$\le \inf_{\text{ß} \le (\Re \cup \{X-B\})} \min\Big(1, 1-[FF(\text{ß})] + \sup_{x\in X} \inf_{A\in P(X)} \big(\text{ß}(A) \to A(x)\big)\Big)$$

$$= \mathrm{fI}(\Re \cup \{X-B\}).$$

Furthermore

$$\pi_1 \dot{\wedge} \Big[((\Re \subseteq F_S) \wedge (B \in \tau_S)) \dot{\wedge} (\forall \wp)$$

$$\Big((\wp \leq \Re) \dot{\wedge} FF(\wp) \to \neg(\bigcap \wp \subseteq B) \Big) \Big]$$

$$= \pi_1 \dot{\wedge} [(\Re \cup \{X-B\} \subseteq F_S) \dot{\wedge} (\forall \wp)(\wp \leq \Re \dot{\wedge} FF(\wp) \to$$

$$(\exists x)(\forall A)\big(A \in (\wp \cup \{X-B\}\big) \to x \in A))]$$

$$= \pi_1 \dot{\wedge} \Big[(\Re \cup \{X-B\} \subseteq F_S) \dot{\wedge} \mathrm{fI}(\Re \cup \{X-B\}) \Big]$$

$$\leq [(\exists x)(\forall A)(A \in (\Re \cup \{X-B\}) \to x \in A)]$$

$$= [\neg(\bigcap \Re \subseteq B)].$$

Therefore

$$\pi_1 \leq \inf_{\Re \in \Im(P(X))} \sup_{B \subset X} ((\Re \subseteq F_S \wedge B \in \tau_S) \dot{\wedge} (\forall \wp)$$

$$(\wp \leq \Re \dot{\wedge} FF(\wp) \to \neg(\bigcap \wp \subseteq B)) \to \neg(\bigcap \Re \subseteq B)) = \pi_2.$$

Conversely,

$$\pi_2 \dot{\wedge} [(\Re \subseteq F_S) \dot{\wedge} \mathrm{fI}(\Re)]$$

$$= \pi_2 \dot{\wedge} [((\Re \backslash \{B\}) \cup \{B\}) \subseteq F_S] \dot{\wedge} \big[\mathrm{fI}((\Re \backslash \{B\}) \cup \{B\})\big]$$

$$= \pi_2 \dot{\wedge} [\Re' \subseteq F_S \wedge (X-B \in \tau_S) \dot{\wedge} (\forall \wp)$$

$$\Big(\wp \leq \Re' \dot{\wedge} FF(\wp) \to (\exists x)$$

$$(A \in (\wp \cup \{B\}) \to x \in A) \Big)]$$

$$= \pi_2 \dot{\wedge} [\Re' \subseteq F_S \wedge (X-B \in \tau_S) \dot{\wedge} (\forall \wp)$$

$$\Big(\wp \leq \Re' \dot{\wedge} FF(\wp) \to \neg(\bigcap \wp \subseteq X-B) \Big)]$$

$$\leq [\neg(\bigcap \Re' \subseteq X-B)]$$

$$= [(\exists x)(\forall A)((A \in (\Re' \cup \{B\}) \to (x \in A))]$$

$$= [(\exists x)(\forall A)(A \in \Re \to (x \in A))].$$

Therefore

$$\pi_2 \leq \inf_{\Re \in \Im(P(X))} [(\Re \subseteq F_S) \dot{\wedge} \mathrm{fI}(\Re)$$

$$\to (\exists x)(\forall A)(A \in \Re \to (x \in A))] = \pi_1.$$

Some properties of fuzzifying semi-compactness

Theorem 4.1. For any fuzzifying topological space (X, τ) and $A \subseteq X$,
$\models \Gamma_S(X,\tau) \dot{\wedge} A \in F_S \to \Gamma_S(A)$.

Proof. For any $\Re \in \Im(P(A))$, we define $\overline{\Re} \in \Im(P(X))$ as $\overline{\Re}(B) = \begin{cases} \Re(B) & \text{if } B \subseteq A, \\ 0 & \text{otherwise.} \end{cases}$ Then

$$FF(\overline{\Re}) = 1 - \inf\{\alpha \in [0,1] : F(\overline{\Re}_\alpha)\}$$

$$= 1 - \inf\{\alpha \in [0,1] : F(\Re_\alpha)\} = FF(\Re)$$ and

$$\sup_{x \in X} \inf_{x \notin B \subseteq X} (1 - \overline{\Re}(B)) = \sup_{x \in X}$$

$$\left(\left(\inf_{x \notin B \subseteq A} (1 - \overline{\Re}(B)) \right) \wedge \left(\inf_{x \notin B \subsetneq A} (1 - \overline{\Re}(B)) \right) \right)$$

$$= \sup_{x \in X} \left(\inf_{x \notin B \subseteq A} (1 - \overline{\Re}(B)) \right)$$

$$\wedge \sup_{x \in X} \left(\inf_{x \notin B \subsetneq A} (1 - \overline{\Re}(B)) \right) = \sup_{x \in X} \left(\inf_{x \notin B \subseteq A} (1 - \overline{\Re}(B)) \right)$$

$$= \sup_{x \in A} \left(\inf_{x \notin B \subseteq A} (1 - \overline{\Re}(B)) \right) \vee \sup_{x \notin A} \left(\inf_{x \notin B \subseteq A} (1 - \overline{\Re}(B)) \right)$$

If $x \notin A$, then for any $x' \in A$ we have

$$\inf_{x \notin B \subseteq A} (1 - \Re(B)) = \inf_{B \subseteq A} (1 - \Re(B))$$

$$\leq \inf_{x' \notin B \subseteq A} (1 - \Re(B)).$$

Therefore,

$$\sup_{x \in X} \inf_{x \notin B \subseteq A} (1 - \overline{\Re}(B)) = \sup_{x \in A} \inf_{x \notin B \subseteq A} (1 - \Re(B)),$$

$$[\mathrm{fI}(\overline{\Re})] = [(\forall \overline{\text{ß}})((\overline{\text{ß}} \leq \overline{\Re}) \wedge FF(\overline{\text{ß}}) \to$$

$$(\exists x)(\forall B)((B \in \overline{\Re}) \to (x \in B)))]$$

$$= \inf_{\overline{\text{ß}} \leq \overline{\Re}} \min\left(1, 1 - FF(\overline{\text{ß}}) + \sup_{x \in X} \inf_{x \notin B \subset X} (1 - \overline{\Re}(B)) \right)$$

$$= \inf_{\text{ß} \leq \Re} \min\left(1, 1 - FF(\text{ß}) + \sup_{x \in A} \inf_{x \notin B \subseteq A} (1 - \text{ß}(B)) \right)$$

$$= [\mathrm{fI}(\Re)].$$

We want to prove that $F_S(A) \dot{\wedge} [\Re \subseteq F_S/_A] \leq [\overline{\Re} \subseteq F_S]$. In fact, from Lemma 2.2 we have

$$F_S(A) \dot{\wedge} [\Re \subseteq F_S/_A]$$

$$= \max\left(0, F_S(A) + \inf_{B \subseteq A} \min(1, 1 - \Re(B) + F_S/_A(B)) - 1 \right)$$

$$\leq \inf_{B \subseteq A} (1 - \Re(B)) + (F_S(A) + F_S/_A(B) - 1)$$

$$\leq \inf_{B \subseteq A} (1 - \Re(B)) + (F_S(A) \wedge F_S/_A(B))$$

$$= \inf_{B \subseteq A} (1 - \Re(B)) + \left(F_S(A) \wedge \sup_{B' \cap A = B, B' \subseteq X} F_S(B') \right)$$

$$= \inf_{B\subseteq A}(1-\Re(B)) + \sup_{B'\cap A=B, B'\subseteq X}(F_S(A)\wedge F_S(B'))$$

$$\le \inf_{B\subseteq A}(1-\Re(B)) + \sup_{B'\cap A=B, B'\subseteq X}(F_S(A\cap B'))$$

$$\le \inf_{B\subseteq A}(1-\Re(B)) + F_S(B) \le \inf_{B\subseteq A}(1-\Re(B)+F_S(B))$$

$$= \inf_{B\subseteq A}\min(1, 1-\Re(B)+F_S(B))$$

$$\le \inf_{B\subseteq A}\min(1, 1-\overline{\Re}(B)+F_S(B)) = [\overline{\Re}\subseteq F_S].$$

Furthermore, from Theorem 3.3 we have

$$\Gamma_S(X,\tau)\underset{\bullet}{\wedge}F_S(A)\underset{\bullet}{\wedge}[\Re\subseteq F_S/_A]\underset{\bullet}{\wedge}\text{fI}(\Re)\le\Gamma_S(X,\tau)\underset{\bullet}{\wedge}[\overline{\Re}\subseteq F_S]\underset{\bullet}{\wedge}\text{fI}(\overline{\Re})$$

$$\le \sup_{x\in X}\inf_{x\notin B\subseteq A}(1-\overline{\Re}(B)) = \sup_{x\in A}\inf_{x\notin B\subseteq A}(1-\Re(B)).$$

Then $\Gamma_S(X,\tau)\underset{\bullet}{\wedge}F_S(A)\le[\Re\subseteq F_S/_A]\underset{\bullet}{\wedge}\text{fI}(\Re)\to\sup_{x\in A}\inf_{x\notin B\subseteq A}(1-\Re(B))$

$$\le \inf_{\Re\in\Im(P(A))}\left([\Re\subseteq F_S/_A]\underset{\bullet}{\wedge}\text{fI}(\Re)\to\sup_{x\in A}\inf_{x\notin B\subseteq A}(1-\Re(B))\right)$$

$$=\Gamma_S(A)$$

Theorem 4.2. Let (X,τ) and (Y,σ) be any two fuzzifying topological spaces and $f\in Y^X$ is surjection. Then $\models \Gamma_S(X,\tau)\underset{\bullet}{\wedge}C_S(f)\to\Gamma(f(X))$.

Proof. For any $ß\in\Im(P(Y))$, we define $\Re\in\Im(P(X))$ as: $\Re(A)=f^{-1}(ß)(A)=ß(f(A))$.

Then $K(\Re,X)=\inf_{x\in X}\sup_{x\in A}\Re(A)=\inf_{x\in X}\sup_{x\in A}ß(f(A))=\inf_{x\in X}\sup_{f(x)\in B}ß(B)=\inf_{y\in f(X)}\sup_{y\in B}ß(B)=K(ß,f(X))$,

$$[ß\subseteq\sigma]\underset{\bullet}{\wedge}[C_S(f)] = \inf_{B\subseteq Y}\min(1,1-ß(B)+\sigma(B))\underset{\bullet}{\wedge}\inf_{B\subseteq Y}\min(1,1-\sigma(B)+\tau_S(f^{-1}(B)))$$

$$=\max\left(0,\inf_{B\subseteq Y}\min(1,1-ß(B)+\sigma(B))+\inf_{B\subseteq Y}\min(1,1-\sigma(B)+\tau_S(f^{-1}(B)))-1\right)$$

$$\le\inf_{B\subseteq Y}\max\left(0,\min(1,1-ß(B)+\sigma(B))+\min(1,1-\sigma(B)+\tau_S(f^{-1}(B)))-1\right)$$

$$\le\inf_{B\subseteq Y}\min\left(1,1-ß(B)+\tau_S(f^{-1}(B))\right)$$

$$=\inf_{A\subseteq X}\inf_{f^{-1}(B)=A}\min\left(1,1-ß(B)+\tau_S(f^{-1}(B))\right)$$

$$=\inf_{A\subseteq X}\inf_{f^{-1}(B)=A}\min\left(1,1-ß(B)+\tau_S(A)\right)$$

$$=\inf_{A\subseteq X}\min\left(1,1-\sup_{f^{-1}(B)=A}ß(B)+\tau_S(A)\right)$$

$$=\inf_{A\subseteq X}\min(1,1-\Re(A)+\tau_S(A))=[\Re\subseteq\tau_S].$$

For any $\wp\le\Re$, we set $\overline{\wp}\in\Im(P(Y))$ defined as $\overline{\wp}(f(A))=f(\wp)(f(A))=\wp(A)$, $A\subseteq X$. Then

$\overline{\wp}(f(A))=f(\wp)(f(A))\le f(\Re)(f(A))=f(f^{-1}(ß))(f(A))\le ß(f(A))$,

$FF(\wp)=1-\inf\{\alpha\in[0,1]:F(\wp_{[\alpha]})\}=1-\inf\{\alpha\in[0,1]:F(f(\wp)_{[\alpha]})\}$

$=FF(f(\wp))\le FF(\overline{\wp})$

and

$$K(\overline{\wp},f(X))=\inf_{y\in f(X)}\sup_{y\in B}\overline{\wp}(B)=\inf_{y\in f(X)}\sup_{y\in B=f(A)}\wp(A)$$

$$\ge\inf_{y\in f(X)}\sup_{f^{-1}(y)\in A}\wp(A)=\inf_{x\in X}\sup_{x\in A}\wp(A)$$

$$=K(\wp,X).$$

Furthermore

$$[\Gamma_S(X,\tau)]\underset{\bullet}{\wedge}[C_S(f)]\underset{\bullet}{\wedge}[K'_\circ(ß,f(X))]$$

$$=[\Gamma_S(X,\tau)]\underset{\bullet}{\wedge}[C_S(f)]\underset{\bullet}{\wedge}[K(ß,f(X))]\underset{\bullet}{\wedge}[ß\subseteq\sigma]$$

$$\le[\Gamma_S(X,\tau)]\underset{\bullet}{\wedge}[\Re\subseteq\tau_S]\underset{\bullet}{\wedge}[K(\Re,X)]$$

$$=[\Gamma_S(X,\tau)]\underset{\bullet}{\wedge}[K_S(\Re,X)]$$

$$\le[(\exists\wp)((\wp\le\Re)\wedge K(\wp,X)\underset{\bullet}{\wedge}FF(\wp))]$$

$$\le[(\exists\wp)((\wp\le\Re)\wedge K(\overline{\wp},f(X))\underset{\bullet}{\wedge}FF(\overline{\wp}))]$$

$$\le[(\exists\wp')((\wp'\le\Re)\wedge K(\wp',f(X))\underset{\bullet}{\wedge}FF(\wp'))],$$

where $K'_\circ$ is related to σ.

Therefore from Theorem 3.1 we obtain

$$[\Gamma_S(X,\tau)]\underset{\bullet}{\wedge}[C_S(f)]\le K'_\circ(ß,f(X))$$

$$\to(\exists\wp')((\wp'\le\Re)\wedge K(\wp',f(X))\underset{\bullet}{\wedge}FF(\wp'))$$

$$\leq \inf_{ß\in\Im(P(X))}\left(K'_{\circ}(ß,f(X)) \rightarrow (\exists \wp')((\wp' \leq \Re) \wedge K(\wp',f(X)) \underset{\bullet}{\wedge} FF(\wp'))\right) = [\Gamma(f(X))].$$

Theorem 4.3. Let (X,τ) and (Y,σ) be any two fuzzifying topological space and $f\in Y^X$ is surjection.

$\vDash \Gamma_S(X,\tau) \underset{\bullet}{\wedge} I(f) \rightarrow \Gamma_S(f(X))$

Proof. From the proof of Theorem 4.2 we have for any $ß\in \Im(P(Y))$, we define $\Re \in \Im(P(X))$ as $\Re(A)=f^{-1}(ß)(A)=ß(f(A))$. Then $K(\Re,X)=K(ß,f(X))$ and $[ß\subseteq\sigma_S]\underset{\bullet}{\wedge}[I(f)]\leq[\Re\subseteq\tau_S]$. For any $\wp\leq\Re$, we set $\overline{\wp}\in\Im(P(Y))$ defined as $\overline{\wp}(f(A))=(\wp)(f(A))=\wp(A), A\subseteq X$ and we have

$FF(\wp)\leq FF(\overline{\wp})$, $K(\overline{\wp},f(X))\geq K(\wp,X)$.

Therefore

$$\Gamma_S(X,\tau)]\underset{\bullet}{\wedge}[I(f)]\underset{\bullet}{\wedge}[K'_S(ß,f(X))]=$$
$$[\Gamma_S(X,\tau)]\underset{\bullet}{\wedge}[I(f)]\underset{\bullet}{\wedge}[K(ß,f(X))]\underset{\bullet}{\wedge}[ß\subseteq\sigma_S]$$
$$\leq[\Gamma_S(X,\tau)]\underset{\bullet}{\wedge}[\Re\subseteq\tau_S]\underset{\bullet}{\wedge}[K(\Re,X)]$$
$$=[\Gamma_S(X,\tau)]\underset{\bullet}{\wedge}[K_S(\Re,X)]$$
$$\leq[(\exists\wp)((\wp\leq\Re)\wedge K(\wp,X)\underset{\bullet}{\wedge}FF(\wp))]$$
$$\leq[(\exists\wp)((\wp\leq\Re)\wedge K(\overline{\wp},f(X))\underset{\bullet}{\wedge}FF(\overline{\wp}))]$$
$$\leq[(\exists\wp')((\wp'\leq ß)\wedge K(\wp',f(X))\underset{\bullet}{\wedge}FF(\wp'))],$$

where K'_S is related to σ. Therefore, from Theorem 3.2 we obtain

$$\Gamma_S(X,\tau)]\underset{\bullet}{\wedge}[I(f)]\leq K'_S(ß,f(X))\rightarrow(\exists\wp')((\wp'\leq ß)\wedge K(\wp',f(X))\underset{\bullet}{\wedge}FF(\wp'))$$

$$\leq \inf_{ß\in\Im(P(X))}\left(K'_S(ß,f(X)) \rightarrow (\exists \wp')((\wp' \leq ß) \wedge K(\wp',f(X)) \underset{\bullet}{\wedge} FF(\wp'))\right) = [\Gamma_S(f(X))].$$

Theorem 4.4. Let (X,τ) be any fuzzifying $s-$topological space and $A,B\subseteq X$. Then

(1) $T_2^S(X,\tau)\underset{\bullet}{\wedge}(\Gamma_S(A)\wedge\Gamma_S(B))\wedge A\cap B=\phi$ $\vDash^{ws}$ $T_2^S(X,\tau)\rightarrow(\exists U)(\exists V)(U\in\tau_S\wedge V\in\tau_S \wedge A\subseteq U\wedge B\subseteq V\wedge U\cap V=\phi)$;

(2) $T_2^S(X,\tau)\underset{\bullet}{\wedge}\Gamma_S(A)$ $\vDash^{ws}$ $T_2^S(X,\tau)\rightarrow A\in F_S$

Proof. (1) Assume $A\cap B=\phi$ and $T_2^S(X,\tau)=t$. Let $x\in A$. Then for any $y\in B$ and $\lambda<t$, we have

$$\sup\{\tau_S(P)\wedge\tau_S(Q):\ x\in P, y\in Q, P\cap Q=\phi\}$$
$$=\sup\{\tau_S(P)\wedge\tau_S(Q):\ x\in P\subseteq U, y\in Q\subseteq V, U\cap V=\phi\}$$
$$=\sup_{U\cap V=\phi}\left\{\sup_{x\in P\subseteq U}\tau_S(P)\wedge\sup_{y\in Q\subseteq V}\tau_S(Q)\right\}$$
$$=\sup_{U\cap V=\phi}\left\{N_x^S(U)\wedge N_y^S(V)\right\}$$
$$\geq\inf_{x\neq y}\sup_{U\cap V=\phi}\{N_x^S(U)\wedge N_y^S(V)\}=T_2^S(X,\tau)=t>\lambda,$$

i.e., there exist P_y, Q_y such that $x\in P_y, y\in Q_y$, $P_y\cap Q_y=\phi$ and $\tau_S(P_y)>\lambda, \tau_S(Q_y)>\lambda$.

Set $ß(Q_y)=\tau_S(Q_y)$ for $y\in B$.

Since $[ß\subseteq\tau_S]=1$, we have $[K_S(ß,B)]=[K(ß,B)]=\inf_{y\in B}\sup_{y\in C}ß(C)\geq\inf_{y\in B}ß(Q_y)=\inf_{y\in B}\tau_S(Q_y)\geq\lambda.$

On the other hand, since $T_2^S(X,\tau)\underset{\bullet}{\wedge}(\Gamma_S(A)\wedge\Gamma_S(B))>0$, then $1-t<\Gamma_S(A)\wedge\Gamma_S(B)\leq\Gamma_S(A)$.

Therefore, for any $\lambda\in(1-\Gamma_S(A),t)$, it holds that

$$1-\lambda<\Gamma_S(A)\leq 1-[K_S(ß,B)]+\sup_{\wp\leq ß}\left\{K(\wp,B)]\underset{\bullet}{\wedge}FF(\wp)\right\}$$
$$\leq 1-\lambda+\sup_{\wp\leq ß}\left\{K(\wp,B)]\underset{\bullet}{\wedge}FF(\wp)\right\},$$

i.e., $\sup_{\wp\leq ß}\left\{K(\wp,B)]\underset{\bullet}{\wedge}FF(\wp)\right\}>0$ and there exists $\wp\leq ß$ such that

$K(\wp,B)+FF(\wp)-1>0$, i.e., $1-FF(\wp)<K(\wp,B)$.

Then, $\inf\{\theta\ :\ F(\wp_\theta)\}<K(\wp,B)$.

Now, there exists θ_1 such that $\theta_1<K(\wp,B)$ and $F(\wp_{\theta_1})$.

Since $\wp\leq ß$, we may write $\wp_{\theta_1}=\{Q_{y_1},...,Q_{y_n}\}$.

We put
$U_x = \{P_{y_1} \cap \ldots \cap P_{y_n}\}$, $V_x = \{Q_{y_1} \cap \ldots \cap Q_{y_n}\}$ and
have $V_x \supseteq B, U_x \cap V_x = \phi$,
$\tau_S(U_x) \geq \tau_S(P_{y_1}) \wedge \ldots \wedge \tau_S(P_{y_n}) > \lambda$ because (X,τ) is fuzzifying s-topological space. Also,
$\tau_S(V_x) \geq \tau_S(Q_{y_1}) \wedge \ldots \wedge \tau_S(Q_{y_n}) > \lambda$.

In fact, $\inf_{y\in B} \sup_{y\in D} \wp(D) = K(\wp, B) > \theta_1$ and for any $y \in B$, there exists D such that $y \in D$ and $\wp(D) > \theta_1, D \in \wp_{\theta_1}$. Similarly, we can find $x_1, \ldots x_m \in A$ with $U_\circ = U_{x_1} \cup \ldots \cup U_{x_m} \supseteq A$ if $\lambda \in (1-[\Gamma_S(A) \wedge \Gamma_S(B)], t)$. By putting $V_\circ = V_{x_1} \cap \ldots \cap V_{x_m}$ we obtain
$V_\circ \supseteq B, U_\circ \cap V_\circ = \phi$ and
$(\exists U)(\exists V)(U \in \tau_S \wedge V \in \tau_S \wedge A \subseteq U \wedge B \subseteq V \wedge U \cap V = \phi) \geq \tau_S(U_\circ) \wedge \tau_S(V_\circ)$
$\geq \min_{i=1,\ldots,n} \tau_S(U_{x_i}) \wedge \min_{i=1,\ldots,n} \tau_S(V_{x_i}) > \lambda$. Finally, we let $\lambda \to t$ and complete the proof.

(2) Assume $\models^w T_2^S(X,\tau) \mathbin{\dot\wedge} \Gamma_S(A)$. for any $x \in X - A$, we obtain from (1)
$\sup_{x\in U \subseteq X-A} \tau_S(U) \geq \sup\{\tau_S(U) \wedge \tau_S(V) : x \in U, A \subseteq V, U \cap V = \phi\} \geq [T_2^S(X,\tau)]$.
From [4, Theorem 7.1] we obtain,
$\tau_S(A) = \inf_{x\in X-A} \sup_{x\in U \subseteq X-A} \tau_S(U) \geq [T_2^S(X,\tau)]$.

Definition 4.1. Let (X,τ) and (Y,σ) be two fuzzifying topological spaces. A unary fuzzy predicate $Q_S \in \Im(Y^X)$, called fuzzifying semi-closedness, is given as
$Q_S(f) := \forall B(B \in F_S^X \to f^{-1}(B) \in F_S^Y)$, where F_S^X and F_S^Y are the fuzzy families of τ, σ-semi-closed in X and Y respectively.

Theorem 4.5. Let (X,τ), a fuzzifying topologyical space, (Y,σ) be an s-fuzzifying topological space and $f \in Y^X$.
Then $\models \Gamma_S(X,\tau) \mathbin{\dot\wedge} T_2^S(X,\tau) \mathbin{\dot\wedge} I(f) \to Q_S(f)$.

Proof. For any $A \subseteq X$, we have the following:
(i) From Theorem 4.1, we have
$[\Gamma_S(X,\tau) \mathbin{\dot\wedge} ?_S^X(A)] \leq \Gamma_S(A)$;

(ii) $I(f_{\backslash A}) = \inf_{U\in P(Y)} \min(1, 1-\sigma_S(U) + \tau_S/_A((f/_A)^{-1}(U)))$
$= \inf_{U\in P(Y)} \min(1, 1-\sigma_S(U) + \tau_S/_A(A \cap f^{-1}(U)))$
$= \inf_{U\in P(Y)} \min\left(1, 1-\sigma_S(U) + \sup_{A\cap f^{-1}(U)=B\cap A} \tau_S(B)\right)$
$\geq \inf_{U\in P(Y)} \min(1, 1-\sigma_S(U) + \tau_S(f^{-1}(U))) = I(f)$.
(iii) From Theorem 4.3, we have
$[\Gamma_S(A) \mathbin{\dot\wedge} I(f_{\backslash A})] \leq \Gamma_S(f(A))$.

(iv) From Theorem 4.4 (2) we have
$T_2^S(Y,\sigma) \mathbin{\dot\wedge} \Gamma_S(f(A)) \models^{ws} T_2^S(Y,\sigma) \to f(A) \in F_S^Y$,
which implies
$FT_2^S(Y,\sigma) \mathbin{\dot\wedge} \Gamma_S(f(A)) \to f(A) \in F_S^Y$.
By combining (i)-(iv) we have
$[\Gamma_S(X,\tau) \mathbin{\dot\wedge} T_2^S(X,\tau) \mathbin{\dot\wedge} I(f)] \leq [(F_S^X(A)$
$\to \Gamma_S(A)) \mathbin{\dot\wedge} I(f_{\backslash A}) \mathbin{\dot\wedge} T_2^S(Y,\sigma)]$
$\leq [(F_S^X(A) \to (\Gamma_S(A)) \mathbin{\dot\wedge} I(f_{\backslash A}))) \mathbin{\dot\wedge} T_2^S(Y,\sigma)]$
$\leq [F_S^X(A) \to \Gamma_S(f(A)) \mathbin{\dot\wedge} T_2^S(Y,\sigma)]$
$\leq [F_S^X(A) \to F_S^Y(f(A))]$.

Therefore
$[\Gamma_S(X,\tau) \mathbin{\dot\wedge} T_2^S(X,\tau) \mathbin{\dot\wedge} I(f)] \leq [F_S^X(A) \to F_S^Y(f(A))]$
$\leq \inf_{A\subseteq X}([F_S^X(A) \to F_S^Y(f(A))]) = Q_S(f)$.

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