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Research Article

Certain Classes of Meromorphic p-Valent Functions Associated with Mittag-Leffler Function

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Abstract: In this paper, using definition of subordination and Mittag-Leffler function we introduce classes of p-valent meromorphic functions and obtain some subordination results for these classes.

Keywords: Differential Subordination, Mittag-Leffler Function, Meromorphic p-valent Functions. 2010 Mathematics Subject Classification: 30C45.

1. INTRODUCTION

Let Σ_p denote the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=1-p}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, \ldots\}), \tag{1.1}$$

Which are analytic and p-valent in the punctured unit disc $U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = U \setminus \{0\}$. For functions $f(z) \in \Sigma_p$ given by (1.1) and $g(z) \in \Sigma_p$ given by

$$g(z) = z^{-p} + \sum_{k=1-p}^{\infty} b_k z^k \quad (p \in \mathbb{N}), \tag{1.2}$$

The Hadamard product (or convolution) of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z^{-p} + \sum_{k=1-p}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.3}$$

For two analytic functions f , $g \in \Sigma_p$, we say that f is subordinate to g , written $f(z) \prec g(z)$ if there exists a Schwartz function $w(z)$, which is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z))$, $z \in U$. Furthermore, if the function $g(z)$ is univalent in U, then we have the following equivalence (see [1] and [2]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

The Mittag-Leffler function [3] $E_\alpha (\alpha \in \mathbb{C},\ \mathrm{Re}(\alpha) > 0)$, is defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha + 1)} z^n. \tag{1.4}$$

A more general function $E_\alpha(z)$ is $E_{\alpha,\beta}(z)$ was introduced by Wiman (see [4, 5]) and given by

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$$\mathrm{E}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)} \quad (z\in \mathrm{C}). \tag{1.5}$$

Srivastava and Tomovski [6] generalized Mittag-Leffler function in the form

$$\mathrm{E}_{\alpha,\beta}^{\gamma,k}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{nk}z^n}{\Gamma(\alpha n+\beta)n!} \ (\alpha,\beta,\gamma\in \mathrm{C},\ \mathrm{Re}(\alpha)>\max\{0,\mathrm{Re}(k)-1\},\ \mathrm{Re}(k)>0). \tag{1.6}$$

Now, by using (1.6) Mostafa and Aouf [7] defined the function $\mathrm{K}_{\alpha,\beta}^{\gamma,k}(z)$ by:

$$\begin{aligned}\mathrm{K}_{\alpha,\beta,p}^{\gamma,k}(z)&=\Gamma(\beta)z^{-p}\mathrm{E}_{\alpha,\beta}^{\gamma,k}(z)\\ &=z^{-p}+\sum_{n=1-p}^{\infty}\frac{\Gamma[\gamma+(n+p)k]\Gamma(\beta)z^n}{\Gamma[\alpha(n+p)+\beta]\Gamma(\gamma)(n+p)!} \ (z\in \mathrm{C},\ p\in \mathrm{N})\end{aligned} \tag{1.7}$$

and defined the operator

$$\begin{aligned}\mathrm{K}_{\alpha,\beta,p}^{\gamma,k}f(z)&=\mathrm{K}_{\alpha,\beta,p}^{\gamma,k}(z)*f(z),\\ &=z^{-p}+\sum_{n=1-p}^{\infty}\frac{\Gamma[\gamma+(n+p)k]\Gamma(\beta)}{\Gamma[\alpha(n+p)+\beta]\Gamma(\gamma)(n+p)!}a_nz^n.\end{aligned} \tag{1.8}$$

For $\alpha,\beta,\gamma\in \mathrm{C},\ \mathrm{Re}(\alpha)>\max\{0,\mathrm{Re}(k)-1\},\mathrm{Re}(k)>0,\eta\ge 0$ and $m\in \mathrm{N}_0=\mathrm{N}\cup\{0\}$, we define the linear operator $\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m}f(z):\ \sum_p\to\sum_p$ as follows :

$$\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,0}f(z)=\mathrm{K}_{\alpha,\beta,p}^{\gamma,k}f(z),$$

$$\begin{aligned}\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,1}f(z)&=(1-\eta)\mathrm{K}_{\alpha,\beta,p}^{\gamma,k}f(z)+\eta z^{-p}\,[z^{p+1}\mathrm{K}_{\alpha,\beta,p}^{\gamma,k}f(z)]'\\ &=z^{-p}+\sum_{n=1-p}^{\infty}\frac{\Gamma[\gamma+k(n+p)]\,\Gamma(\beta)}{\Gamma(\gamma)\Gamma[\alpha(n+p)+\beta](n+p)!}[1+\eta(n+p)]a_nz^n,\end{aligned}$$

$$\begin{aligned}\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,2}f(z)&=\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,1}[\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,1}f(z)]=(1-\eta)\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,1}f(z)+\eta z^{-p}\,[z^{p+1}\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,1}f(z)]'\\ &=z^{-p}+\sum_{n=1-p}^{\infty}\frac{\Gamma[\gamma+k(n+p)]\,\Gamma(\beta)}{\Gamma(\gamma)\Gamma[\alpha(n+p)+\beta](n+p)!}[1+\eta(n+p)]^2a_nz^n\end{aligned}$$

and in general

$$\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m}f(z)=z^{-p}+\sum_{n=1-p}^{\infty}\frac{\Gamma[\gamma+k(n+p)]\,\Gamma(\beta)}{\Gamma(\gamma)\Gamma[\alpha(n+p)+\beta](n+p)!}[1+\eta(n+p)]^m a_nz^n. \tag{1.9}$$

For $f\in\sum_p$, *it is easy to see that* $\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m}f(z)$ *achieve the following relations*

$$(i)\quad z[\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m}f(z)]'=\frac{\gamma}{k}\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma+1,k,m}f(z)-\left(\frac{\gamma+pk}{k}\right)\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m}f(z), \tag{1.10}$$

$$(ii)\quad z\alpha[\mathrm{M}_{\alpha,\beta+1,\eta,p}^{\gamma,k,m}f(z)]'=\beta\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m}f(z)-(p\alpha+\beta)\mathrm{M}_{\alpha,\beta+1,\eta,p}^{\gamma,k,m}f(z), \tag{1.11}$$

$$(iii)\quad z[\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m}f(z)]'=\frac{1}{\eta}\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m+1}f(z)-\left(p+\frac{1}{\eta}\right)\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m}f(z), \tag{1.12}$$

$$(\alpha,\beta,\gamma\in\mathbb{C},\ \mathrm{Re}(\alpha)>\max\{0,\mathrm{Re}(k)-1\},\mathrm{Re}(k)>0,\eta>0,m\in\mathbb{N}_0).$$

Remark 1

- ✓ $\mathrm{M}^{1,1,0}_{0,\beta,\eta,p}f(z)=f(z)\quad (f\in\sum_p);$
- ✓ $\mathrm{M}^{2,1,0}_{0,\beta,\eta,p}f(z)=(p+1)f(z)+zf'(z)\quad (f\in\sum_p);$
- ✓ $\mathrm{M}^{2,1,0}_{0,\beta,\eta,1}f(z)=2f(z)+zf'(z)\quad (f\in\sum_1);$
- ✓ $\mathrm{M}^{1,1,0}_{1,1,\eta,p}(\frac{1}{z^p(1-z)})=z^{-p}e^z;$
- ✓ $\mathrm{M}^{1,1,0}_{1,1,\eta,1}(\frac{1}{z(1-z)})=z^{-1}e^z;$
- ✓ $\mathrm{M}^{1,1,0}_{2,1,\eta,1}(\frac{1}{z(1-z)})=z^{-1}\cosh\left(\sqrt{z}\right);$
- ✓ $\mathrm{M}^{1,1,0}_{2,1,\eta,p}(\frac{1}{z^p(1-z)})=z^{-p}\cosh\left(\sqrt{z}\right);$
- ✓ $\mathrm{M}^{1,1,0}_{2,2,\eta,1}(\frac{1}{z(1-z)})=\frac{\sinh\left(\sqrt{z}\right)}{\sqrt{z^3}};$
- ✓ $\mathrm{M}^{1,1,0}_{2,2,\eta,p}(\frac{1}{z^p(1-z)})=\frac{\sinh\left(\sqrt{z}\right)}{z^p\sqrt{z}}.$

We also observe that:

- ✓ $\mathrm{M}^{1,1,m}_{0,\beta,\eta,p}f(z)=D^m_{\eta,p}f(z)$ ([8], [9] and [10] with $l=1$)
- ✓ $\mathrm{M}^{1,1,m}_{0,\beta,\eta,1}f(z)=D^m_{\eta}f(z)$ (see [11],);
- ✓ $\mathrm{M}^{1,1,m}_{0,\beta,1,1}f(z)=D^m_{1}f(z)$ (see [12]);
- ✓ $\mathrm{M}^{\gamma,k,0}_{\alpha,\beta,\eta,p}f(z)=\mathrm{K}^{\gamma,k}_{\alpha,\beta,p}f(z)$ (see [7]).

Unless otherwise mentioned, we assume throughout the paper that:

$$-1\le B<A\le 1,\ \alpha,\beta,\gamma\in\mathbb{C},\ \mathrm{Re}(\alpha)>\max\{0,\mathrm{Re}(k)-1\},\mathrm{Re}(k)>0,\ \eta>0,\ m\in\mathbb{N}_0\ \text{ and }\ \lambda>0.$$

2. MATERIALS AND METHODS

Definition 1
For fixed parameters A and B, with $-1\le B<A\le 1$, we say that $f\in\Sigma_p$ is in the class $\Theta_p^{\eta,k,\alpha}(\lambda,\gamma,\beta,m;A,B)$ if it satisfies

$$-\frac{z^{p+1}}{p}\left\{(1-\lambda)(\mathrm{M}^{\gamma,k,m}_{\alpha,\beta,\eta,p}f(z))'+\lambda(\mathrm{M}^{\gamma+1,k,m}_{\alpha,\beta,\eta,p}f(z))'\right\}\prec\frac{1+Az}{1+Bz}. \tag{2.1}$$

Definition 2. *For fixed parameters* A *and* B, *with* $-1\le B<A\le 1$, *we say that* $f\in\Sigma_p$ *is in the class* $\Omega_p^{\eta,k,\alpha}(\lambda,\gamma,\beta,m;A,B)$ *if it satisfies*

$$-\frac{z^{p+1}}{p}\left\{(1-\lambda)(\mathrm{M}^{\gamma,k,m}_{\alpha,\beta,\eta,p}f(z))'+\lambda(\mathrm{M}^{\gamma,k,m}_{\alpha,\beta-1,\eta,p}f(z))'\right\}\prec\frac{1+Az}{1+Bz}. \tag{2.2}$$

Definition 3
For fixed parameters A *and* B , *with* $-1 \le B < A \le 1$, *we say that* $f \in \Sigma_p$ *is in the class* $\Sigma_p^{\eta,k,\alpha}(\lambda,\gamma,\beta,m;A,B)$ *if it satisfies*

$$-\frac{z^{p+1}}{p}\left\{(1-\lambda)(\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m}f(z))' + \lambda(\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m+1}f(z))'\right\} \prec \frac{1+Az}{1+Bz}. \tag{2.3}$$

To prove our main results, we need the following lemmas.

Lemma 1 [13] *For analytic, convex (univalent)* h in U, $h(0)=1$ *and* φ *on the form*

$$\varphi(z) = 1 + c_1 z + c_2 z^2 + \ldots, \tag{2.4}$$

analytic in U, if

$$\varphi(z) + \frac{z\varphi'(z)}{\ell} \prec h(z) \ \ (\mathrm{Re}\{\ell\} \ge 0;\ \ell \ne 0), \tag{2.5}$$

Then

$$\varphi(z) \prec \psi(z) = \ell z^{-\ell}\int_0^z t^{\ell-1}h(t)dt \prec h(z) \tag{2.6}$$

where $\psi(z)$ is the best dominant of (2.6).

The Gaussian hypergeometric function defined by

$$\begin{aligned} {}_2F_1(a,b;c;z) &= 1 + \frac{ab}{c}.\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}.\frac{z^2}{2!} + \ldots. \\ &= \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \end{aligned} \tag{2.7}$$

where a,b and c $(c \notin 0,-1,-2,\ldots)$, $(d)_k = d(d+1)\ldots(d+k-1)$ and $(d)_0 = 1$. We note that the series defined by (2.7) converges absolutely for $z \in \mathrm{U}$ and ${}_2F_1$ represents an analytic function in U (see [14]).

Lemma 2 [14]. For real or complex numbers a,b and c $(c \notin 0,-1,-2,\ldots)$ the following identities hold:

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\ {}_2F_1(a,b;c;z)\ (\mathrm{Re}\{b\}>0,\ \mathrm{Re}\{c\}>0,\ z \in \mathrm{C}\setminus(1,+\infty)), \tag{2.8}$$

and

$${}_2F_1(a,b;c;z) = (1-z)^{-a}\ {}_2F_1(a,c-b;c;\frac{z}{z-1})\ (z \in \mathrm{C}\setminus(1,+\infty)), \tag{2.9}$$

where ${}_2F_1(a,b;c;z)$ is the Gaussian hypergeometric function.
Let $F(\varsigma)$ be the class of functions $\Phi(z)$ given by

$$\Phi(z) = 1 + b_1 z + b_2 z^2 + \ldots, \tag{2.10}$$

which are analytic in U, and $\mathrm{Re}\{\Phi(z)\} > \varsigma$ $(0 \le \varsigma < 1)$.

Lemma 3 [15].*If* $\Phi \in F(\varsigma)$, then

$$\mathrm{Re}\{\Phi(z)\} \ge 2\varsigma - 1 + \frac{2(1-\varsigma)}{1+|z|}\ (0 \le \varsigma < 1).$$

Lemma 4 [16]. If $\Phi_j \in F(\varsigma_j)$ $(0 \le \varsigma_j < 1;\ j = 1,2)$, then

$$\Phi_1 * \Phi_2 \in F(\varsigma_3) \ \ (\varsigma_3 = 1 - 2(1-\varsigma_1)(1-\varsigma_2)).$$

The result is the best possible.

3. RESULTS AND DISCUSSION

Theorem 1. *Let* $\mathrm{Re}\left\{\frac{\gamma}{\lambda k}\right\} \ge 0$, if $f \in \Theta_p^{\eta,k,\alpha}(\lambda,\gamma,\beta,m;A,B)$, *then*

$$-\frac{z^{p+1}}{p}\left(\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m} f(z)\right)' \prec q_1(z) \prec \frac{1+Az}{1+Bz}, \tag{3.1}$$

such that

$$q_1(z) = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1+Bz)^{-1}\ {}_2F_1(1,1;\frac{\gamma}{\lambda k}+1;\frac{Bz}{1+Bz}) & \text{for} \quad B \ne 0, \\ 1 + \frac{\gamma A}{\lambda k + \gamma} z & \text{for} \quad B = 0, \end{cases} \tag{3.2}$$

is the best dominant of (3.1). and

$$\mathrm{Re}\left(-\frac{z^{p+1}\left(\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m} f(z)\right)'}{p}\right) > \rho, \tag{3.3}$$

where

$$\rho = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1-B)^{-1}\ {}_2F_1(1,1;\frac{\gamma}{\lambda k}+1;\frac{B}{B-1}) & \text{for} \quad B \ne 0, \\ 1 - \frac{\gamma A}{\lambda k + \gamma} & \text{for} \quad B = 0. \end{cases} \tag{3.4}$$

The estimate (3.3) is the best possible.

Proof. Assume

$$\varphi(z) = -\frac{z^{p+1}\left(\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m} f(z)\right)'}{p} \quad (z \in \mathrm{U}), \tag{3.5}$$

where φ is given by (2.4). Differentiating (3.5) with respect to z and using (1.10), we get

$$-\frac{z^{p+1}}{p}\left\{(1-\lambda)(\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m} f(z))' + \lambda(\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma+1,k,m} f(z))'\right\} = \varphi(z) + \frac{\lambda k}{\gamma} z\varphi'(z) \prec \frac{1+Az}{1+Bz}.$$

From Lemma 2, we have

$$\varphi(z) \prec q_1(z) = \frac{\gamma}{\lambda k} z^{-\frac{\gamma}{\lambda k}} \int_0^z t^{\frac{\gamma}{\lambda k}-1}\left(\frac{1+At}{1+Bt}\right) dt.$$

This proves (3.2) of Theorem 1. In order to prove (3.3), we need to show that

$$\inf_{|z|<1}\{\mathrm{Re}(q_1(z))\} = q_1(-1). \tag{3.6}$$

We have

$$\mathrm{Re}\left\{\frac{1+Az}{1+Bz}\right\} \ge \frac{1-Ar}{1-Br} \quad (|z| \le r < 1).$$

Putting

$$G(z,\zeta) = \frac{1+A\zeta z}{1+B\zeta z} \ \text{ and } \ d\nu(\zeta) = \frac{\gamma}{\lambda\gamma}\zeta^{\frac{\gamma}{\lambda\gamma}-1} d\zeta \ (0 \le \zeta \le 1),$$

$$q_1(z)=\int_0^1 G(z,\zeta)d\nu(\zeta).$$

Then

$$\mathrm{Re}\{q_1(z)\}\geq\int_0^1\frac{1-A\zeta r}{1-B\zeta r}d\nu(\zeta)=q_1(-r)\ \ (|z|\leq r<1).$$

Assuming $r\to 1^-$ in the above inequality, we obtain (3.6). The result in (3.3) is the best possible and q_1 is the best dominant of (3.1). This completes the proof of Theorem 1.

Using (1.11) and (1.12) respectively instead of (1.10) we can prove the theorems 2-3, respectively.

Theorem 2. *Let* $\mathrm{Re}(\frac{\beta-1}{\lambda\alpha})\geq 0$, if $f\in\Omega_p^{\eta,k,\alpha}(\lambda,\gamma,\beta,m;A,B)$, *then*

$$\varphi(z)\prec q_2(z)\prec\frac{1+Az}{1+Bz},\tag{3.7}$$

where φ given by (3.5),

$$q_2(z)=\begin{cases}\frac{A}{B}+\left(1-\frac{A}{B}\right)(1+Bz)^{-1}\ {}_2F_1(1,1;\frac{\beta-1}{\lambda\alpha}+1;\frac{Bz}{1+Bz}) & \text{for}\quad B\neq 0,\\ 1+\frac{\beta-1}{\lambda\alpha+\beta-1}Az & \text{for}\quad B=0,\end{cases}\tag{3.8}$$

is the best dominant of (3.7). And

$$\mathrm{Re}\{\varphi(z)\}>\sigma,\tag{3.9}$$

where

$$\sigma=\begin{cases}\frac{A}{B}+\left(1-\frac{A}{B}\right)(1-B)^{-1}\ {}_2F_1(1,1;\frac{\beta-1}{\lambda\alpha}+1;\frac{B}{B-1}) & \text{for}\quad B\neq 0,\\ 1-\frac{\beta-1}{\lambda\alpha+\beta-1}A & \text{for}\quad B=0.\end{cases}\tag{3.10}$$

The estimate (3.9) is the best possible.

Theorem 3. *Let* $\mathrm{Re}(\frac{1}{\lambda\eta})>0$, if $f\in\Sigma_p^{\eta,k,\alpha}(\lambda,\gamma,\beta,m;A,B)$, *then*

$$\varphi(z)\prec q_3(z)\prec\frac{1+Az}{1+Bz},\tag{3.11}$$

where φ given by (3.5),

$$q_3(z)=\begin{cases}\frac{A}{B}+\left(1-\frac{A}{B}\right)(1+Bz)^{-1}\ {}_2F_1(1,1;\frac{1}{\lambda\eta}+1;\frac{Bz}{1+Bz}) & \text{for}\quad B\neq 0,\\ 1+\frac{1}{\lambda\eta+1}Az & \text{for}\quad B=0,\end{cases}\tag{3.12}$$

is the best dominant of (3.11). And

$$\mathrm{Re}\{\varphi(z)\}>\varepsilon,\tag{3.13}$$

where

$$\varepsilon=\begin{cases}\frac{A}{B}+\left(1-\frac{A}{B}\right)(1-B)^{-1}\ {}_2F_1(1,1;\frac{1}{\lambda\eta}+1;\frac{B}{B-1}) & \text{for}\quad B\neq 0,\\ 1-\frac{1}{\lambda\eta+1}A & \text{for}\quad B=0.\end{cases}\tag{3.14}$$

The estimate (3.13) is the best possible.

Taking $\varphi(z)=z^p\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m}f(z)$ we can prove Theorems 4-6 below by using the same manner in the proof of Theorem 1.

Theorem 4. *Let* $-1 \le B < A \le 1$. If $f \in \sum_p$ *satisfies*

$$z^p \left\{(1-\lambda)\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m} f(z) + \lambda \mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma+1,k,m} f(z)\right\} \prec \frac{1+Az}{1+Bz}, \tag{3.15}$$

then

$$z^p \mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m} f(z) \prec q_1(z) \prec \frac{1+Az}{1+Bz} \tag{3.16}$$

is the best dominant of (3.15). And

$$\mathrm{Re}\left(z^p \mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m} f(z)\right) > \rho, \tag{3.17}$$

where q_1 *and* ρ are given by (3.2) and (3.4), respectively.

Theorem 5. *If* $f \in \sum_p$ satisfi*es*

$$z^p \left\{(1-\lambda)\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m} f(z) + \lambda \mathrm{M}_{\alpha,\beta-1,\eta,p}^{\gamma,k,m} f(z)\right\} \prec \frac{1+Az}{1+Bz}, \tag{3.18}$$

then

$$z^p \mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m} f(z) \prec q_2(z) \prec \frac{1+Az}{1+Bz} \tag{3.19}$$

is the best dominant of (3.18). And

$$\mathrm{Re}\left(z^p \mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m} f(z)\right) > \sigma, \tag{3.20}$$

where q_2 *and* σ are given by (3.8) and (3.10), respectively.

Theorem 6. *If* $f \in \sum_p$ *satisfies*

$$z^p \left\{(1-\lambda)\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m} f(z) + \lambda \mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m+1} f(z)\right\} \prec \frac{1+Az}{1+Bz}, \tag{3.21}$$

then

$$z^p \mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m} f(z) \prec q_3(z) \prec \frac{1+Az}{1+Bz} \tag{3.22}$$

is the best dominant of (3.21). And

$$\mathrm{Re}\left(z^p \mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m} f(z)\right) > \varepsilon, \tag{3.23}$$

where q_3 *and* ε are given by (3.12) and (3.14), respectively.

Theorem 7. *If* $Re\left\{\frac{\gamma}{\lambda k}\right\} \ge 0$ and $f_j(z) \in \sum_p$ *satisfies*

$$z^p \left\{(1-\lambda)\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m} f_j(z) + \lambda \mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma+1,k,m} f_j(z)\right\} \prec \frac{1+A_j z}{1+B_j z} \quad (j=1,2), \tag{3.24}$$

then

$$z^p \left\{(1-\lambda)\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m} \Psi(z) + \lambda \mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma+1,k,m} \Psi(z)\right\} \prec \frac{1+\left(1-\frac{2\varsigma}{p}\right)z}{1-z}, \tag{3.25}$$

such that

$$\Psi(z) = \mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m}(f_1 * f_2)(z) \tag{3.26}$$

and

$$\varsigma = 1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1-B_1)(1-B_2)}\left[1 - \frac{1}{2}\ {}_2F_1(1,1;\frac{\gamma}{\lambda k}+1;\frac{1}{2})\right]. \tag{3.27}$$

The result is the best possible when $B_1 = B_2 = -1$.

Proof. Let $f_j(z) \in \sum_p$ satisfy (3.24),

$$\Phi_j(z) = z^p\left\{(1-\lambda)\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m} f_j(z) + \lambda \mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma+1,k,m} f_j(z)\right\} \quad (j = 1,2), \tag{3.28}$$

such that $\Phi_j(z) \in F(\varsigma_j)$ $\left(\varsigma_j = \frac{1-A_j}{1-B_j}\right)$ $(j = 1,2)$. By using (1.10) in (3.28), we obtain

$$\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m} f_j(z) = \frac{\gamma}{\lambda k} z^{-p-\frac{\gamma}{\lambda k}} \int_0^z t^{\frac{\gamma}{\lambda k}-1} \Phi_j(t)dt \quad (j = 1,2), \tag{3.29}$$

from (3.26) and (3.29), we have

$$\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m}\Psi(z) = \left(\frac{\gamma}{\lambda k} z^{-p-\frac{\gamma}{\lambda k}} \int_0^z t^{\frac{\gamma}{\lambda k}-1}\Phi_1(t)dt\right) * \left(\frac{\gamma}{\lambda k} z^{-p-\frac{\gamma}{\lambda k}} \int_0^z t^{\frac{\gamma}{\lambda k}-1}\Phi_2(t)dt\right) \quad (j = 1,2)$$

$$= \frac{\gamma}{\lambda k} z^{-p-\frac{\gamma}{\lambda k}} \int_0^z t^{\frac{\gamma}{\lambda k}-1}\Phi_0(t)dt, \tag{3.30}$$

such that

$$\Phi_0(z) = z^p\left\{(1-\lambda)\mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma,k,m}\Psi(z) + \lambda \mathrm{M}_{\alpha,\beta,\eta,p}^{\gamma+1,k,m}\Psi(z)\right\}$$

$$= \frac{\gamma}{\lambda k} z^{-\frac{\gamma}{\lambda k}} \int_0^z t^{\frac{\gamma}{\lambda k}-1}(\Phi_1 * \Phi_2)(t)dt. \tag{3.31}$$

Since $\Phi_1(z) \in F(\varsigma_1)$ and $\Phi_2(z) \in F(\varsigma_2)$, it follows from Lemma 4 that:

$$(\Phi_1 * \Phi_2)(z) \in F(\varsigma_3) \ (\varsigma_3 = 1 - 2(1-\varsigma_1)(1-\varsigma_2)). \tag{3.32}$$

According to Lemma 3, we get

$$\mathrm{Re}\{(\Phi_1 * \Phi_2)(z)\} \ge 2\varsigma_3 - 1 + \frac{2(1-\varsigma_3)}{1+|z|}. \tag{3.33}$$

Now by using (3.33) in (3.31) and Lemma 2, we have;

$$\mathrm{Re}\{\Phi_0(z)\} = \frac{\gamma}{\lambda k}\int_0^1 u^{\frac{\gamma}{\lambda k}-1}(\Phi_1 * \Phi_2)(uz)du$$

$$\ge \frac{\gamma}{\lambda k}\int_0^1 u^{\frac{\gamma}{\lambda k}-1}\left(2\varsigma_3 - 1 + \frac{2(1-\varsigma_3)}{1+u|z|}\right)du$$

$$> \frac{\gamma}{\lambda k}\int_0^1 u^{\frac{\gamma}{\lambda k}-1}\left(2\varsigma_3 - 1 + \frac{2(1-\varsigma_3)}{1+u}\right)du$$

$$= 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left[1 - \frac{\gamma}{\lambda k}\int_0^1 u^{\frac{\gamma}{\lambda k}-1}(1+u)^{-1}du\right]$$

$$= 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left[1 - \frac{1}{2}\,{}_2F_1(1,1;\frac{\gamma}{\lambda k}+1;\frac{1}{2})\right] = \varsigma$$

which completes the proof of (3.25).
To prove the result is best possible, we assume $B_1 = B_2 = -1$, $f_j(z) \in \sum_p$, which satisfy (3.24),

$$\mathrm{M}^{\gamma,k,m}_{\alpha,\beta,\eta,p} f_j(z) = \frac{\gamma}{\lambda k} z^{-p-\frac{\gamma}{\lambda k}} \int_0^z t^{\frac{\gamma}{\lambda k}-1} \frac{1+A_j t}{1-t}\,dt \;,\left(\Phi_j(t) = \frac{1+A_j t}{1-t}\right)(j=1,2)$$

and $(\Phi_1 * \Phi_2)(z) = 1 + \frac{(1+A_1)(1+A_2)z}{1-z}$. Thus by using (3.31) and Lemma 2, we have

$$\Phi_0(z) = \frac{\gamma}{\lambda k}\int_0^1 u^{\frac{\gamma}{\lambda k}-1}\left(1-(1+A_1)(1+A_2) + \frac{(1+A_1)(1+A_2)z}{1-uz}\right)du$$

$$= 1-(1+A_1)(1+A_2) + (1+A_1)(1+A_2)(1-z)^{-1}\;{}_2F_1(1,1;\frac{\gamma}{\lambda k}+1;\frac{z}{z-1})$$

$$= 1-(1+A_1)(1+A_2) + (1+A_1)(1+A_2)\;{}_2F_1(1,1;\frac{\gamma}{\lambda k}+1;\frac{1}{2}) \quad (\text{as } z \to -1).$$

This completes the proof of Theorem 7.

Theorem 8. *If* $\mathrm{Re}\left\{\frac{\beta-1}{\lambda\alpha}\right\} \geq 0$ *and* $f_j(z) \in \sum_p$ *satisfies*

$$z^p\left\{(1-\lambda)\mathrm{M}^{\gamma,k,m}_{\alpha,\beta,\eta,p} f_j(z) + \lambda \mathrm{M}^{\gamma,k,m}_{\alpha,\beta-1,\eta,p} f_j(z)\right\} \prec \frac{1+A_j z}{1+B_j z} \quad (j=1,2),$$

Then,

$$z^p\left\{(1-\lambda)\mathrm{M}^{\gamma,k,m}_{\alpha,\beta,\eta,p}\Psi(z) + \lambda \mathrm{M}^{\gamma,k,m}_{\alpha,\beta-1,\eta,p}\Psi(z)\right\} \prec \frac{1+\left(1-\frac{2\varpi}{p}\right)z}{1-z},$$

such that $\Psi(z)$ given by (3.26) *and*

$$\varpi = 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left[1 - \frac{1}{2}\,{}_2F_1(1,1;\frac{\beta-1}{\lambda\alpha}+1;\frac{1}{2})\right].$$

The result is the best possible when $B_1 = B_2 = -1$.

Proof. The proof is similar to that of Theorem 7, so we omit it.

Theorem 9. *If* $\mathrm{Re}\left\{\frac{1}{\lambda\eta}\right\} > 0$ *and* $f_j(z) \in \sum_p$ *satisfies*

$$z^p\left\{(1-\lambda)\mathrm{M}^{\gamma,k,m}_{\alpha,\beta,\eta,p} f_j(z) + \lambda \mathrm{M}^{\gamma,k,m+1}_{\alpha,\beta,\eta,p} f_j(z)\right\} \prec \frac{1+A_j z}{1+B_j z} \quad (j=1,2),$$

then

$$z^p\left\{(1-\lambda)\mathrm{M}^{\gamma,k,m}_{\alpha,\beta,\eta,p}\Psi(z) + \lambda \mathrm{M}^{\gamma,k,m+1}_{\alpha,\beta,\eta,p}\Psi(z)\right\} \prec \frac{1+\left(1-\frac{2\omega}{p}\right)z}{1-z},$$

such that $\Psi(z)$ given by (3.26) *and*

$$\omega = 1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1-B_1)(1-B_2)}\left[1 - \frac{1}{2}\ {}_2F_1(1,1;\frac{1}{\lambda\eta}+1;\frac{1}{2})\right].$$

The result is the best possible when $B_1 = B_2 = -1$.

Proof. The proof is similar to that of Theorem 7, so we omit it.

4. CONCLUSIONS

In conclusion, we given new operator and defined three classes by using this operator. We calculated differential subordination result and introduced special cases. We can apply the new operator in different topics in the future.

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