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Research Article

Approximations for Linear Tenth-order Boundary Value Problems through Polynomial and Non-polynomial Cubic Spline Techniques

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Abstract: Higher order differential equations have always been a tedious problem to solve for the mathematicians and engineers. Different numerical techniques were carried out to obtain numerical approximations to such problems. This research work presented and illustrated a novel numerical technique to approximate the tenth-order boundary value problems (BVPs). The techniques developed in this research were based on the principle of employing non-polynomial cubic spline method (NPCSM) and polynomial cubic spline method (PCSM) along with the decomposition procedure. The decomposition procedure was used to reduce the tenth-order BVPs into the corresponding system of second-order BVPs. The NPCSM and PCSM schemes were constructed for each second-order ordinary differential equation (ODE), whereas the first-order derivatives were approximated by finite central differences. The performance of the new developed numerical techniques was illustrated by numerical tests that involved comparing numerical solutions with the exact solution on a collection of test problems.

Keywords: Linear tenth-order ordinary differential equation, non-polynomial cubic spline, polynomial cubic spline, boundary value problems, finite central difference, system of algebraic equations

1. INTRODUCTION

A numerical approximation of tenth-order BVPs and the linked characteristic value problem is hardly addressed in the literature. When heating an endless flat film of fluid from below, under the supposition that fluid is subject to the action of rotation and uniform magnetic field across the fluid is implemented in the same direction as gravity, unsteadiness begins. When unsteadiness sets in as usual convection, then it can be modeled by tenth-order BVP, Chandarasekhar [1]. Pervaiz et al [2] discussed the numerical approximations of twelfth-order BVPs by employing non-polynomial cubic spline technique. Omotayo et al [3] proposed non-polynomial spline method for the fourth order BVPs by reducing it to a system of second order BVPs. Usmani [4] presented his work to approximate the fourth order BVPs by utilizing the quartic spline method. Twizell and Boutayeb [5] developed and illustrated the numerical approximations for eighth, tenth, and twelfth-order characteristic value problems emerging in thermal unsteadiness. The approximation of second order BVPs was presented by Alberg and Ito [6]. Siraj-ul-Islam et al [7] presented a non-polynomial spline method to approximate the sixth-order BVPs. Papamichael and Worsey [8] successively proposed the cubic spline algorithm for solving linear fourth-order BVPs. Siddiqui and Twizell [9, 10] developed numerical approximations of tenth and twelfth-order BVPs using tenth and twelfth order splines, respectively.

2. MATERIALS AND METHODS

The primary objective of this paper was to construct a novel cubic spline algorithm using numerical approximations of linear tenth-order BVPs. In this case, non-polynomial and polynomial cubic spline approaches are employed to develop numerical approximations of tenth-order BVPs, which is of the form

$$u^{(x)}(x)+a_1(x)u^{(ix)}(x)+a_2(x)u^{(viii)}(x)+a_3(x)\,u^{(vii)}(x)+a_4(x)u^{(vi)}(x)+a_5(x)u^{(v)}(x)+ a_6(x)u^{(iv)}(x)+a_7(x)u'''(x)+a_8(x)u''(x)+a_9(x)u'(x)+a_{10}(x)\,u(x)=\ f(x)\,,\ x\in[a,b], \tag{1}$$

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with boundary conditions:

$u(a) = \alpha_0, \qquad u(b) = \beta_0,$ (2)

$u''(a) = \alpha_1, \qquad u''(b) = \beta_1,$ (3)

$u^{(iv)}(a) = \alpha_2, \qquad u^{(iv)}(b) = \beta_2,$ (4)

$u^{(vi)}(a) = \alpha_3, \qquad u^{(vi)}(b) = \beta_3,$ (5)

$u^{(viii)}(a) = \alpha_4, \qquad u^{(viii)}(b) = \beta_4,$ (6)

where, $\alpha_j, \beta_j,\ \ j = 0, 1, 2, 3, 4$ are arbitrary fixed real constants, $a_j(x),\ j = 1, 2,\ \dots, 10$ and $f(x)$ are continuous functions defined on $[a, b]$. On employing appropriate substitutions, $u''(x) = v(x),\ v''(x) = w(x),\ w''(x) = r(x)$, and $r''(x) = t(x)$ along with the boundary conditions (2-6), equation (1) was expressed in second-order ODEs form as under:

$$t''(x) + a_1(x)t'(x) + a_2(x)\,t(x) + a_3(x)r'(x) + a_4(x)\,r(x) + a_5(x)\,w'(x) + a_6(x)\,w(x) + a_7(x)v'(x) + a_8(x)\,v(x) + a_9(x)u'(x) + a_{10}(x)\,u(x) = f(x), \quad a \le x \le b, \tag{7}$$

$$u''(x) - v(x) = 0, \tag{8}$$

$$v''(x) - w(x) = 0, \tag{9}$$

$$w''(x) - r(x) = 0, \tag{10}$$

$$r''(x) - t(x) = 0. \tag{11}$$

The boundary conditions can be written in reduced form as:

$u(a) = \alpha_0, \qquad u(b) = \beta_0,$ (12)

$v(a) = \alpha_1, \qquad v(b) = \beta_1,$ (13)

$w(a) = \alpha_2, \qquad w(b) = \beta_2,$ (14)

$r(a) = \alpha_3, \qquad r(b) = \beta_3,$ (15)

$t(a) = \alpha_4, \qquad t(b) = \beta_4.$ (16)

Equations (7-11) along with the boundary conditions in equations (12-16) form a complete system of ODEs. This system of ODEs can be solved by simple numerical algorithms.

3. CONSTRUCTION OF NON-POLYNOMIAL CUBIC SPLINE SCHEME

To derive non-polynomial cubic spline approximation S for equations (7-11) with boundary conditions in equations (12-16), the period $[a, b]$ was distributed into n like subintervals:

$\mathrm{x_j = a + jh, j = 0,1, \dots, n,}$ where $\mathrm{a = x_0,\ b = x_n,}$ and $\mathrm{h = \frac{b-a}{n}}$.

Suppose the approximation to the exact solution $\mathrm{u(x_j)}$ was considered to be $\mathrm{u_j}$, which was obtained using non-polynomial cubic spline $\mathrm{S_j(x)}$, passing by the points $\mathrm{(x_j, u_j)}$ and $\mathrm{(x_{j+1}, u_{j+1})}$. Then $\mathrm{S_j(x)}$was required to satisfy the interpolating conditions at $\mathrm{(x_j, x_{j+1})}$, the boundary conditions in equations (12-16), and also the continuity condition of first derivative at the grid point $\mathrm{(x_j, u_j)}$. For every section $\mathrm{[x_j, x_{j+1}]}$, $\mathrm{j = 0, 1, \dots, n-1}$, the non-polynomial spline $\mathrm{S_j(x)}$ can be written in the form

$$S_j(x) = a_j + b_j(x - x_j) + c_j \sin\tau\,(x - x_j) + d_j \cos\tau\,(x - x_j),\ \ j = 0, 1, \dots, n-1, \tag{17}$$

here, a_j , b_j , c_j, and d_j are arbitrary constant values and τ represents the free parameter. The non-polynomial function $S(x)$, chosen from class $C^2[a\,, b]$, interpolates $u(x)$ at the common knots $x_j,\ j =$

$0, 1, \ldots, n$, rely on the parameter τ and then converted to an ordinary cubic spline $S(x)$ over $[a, b]$ when $\tau \to 0$.

Let, $S_j(x_j) = u_j,\ S_j(x_{j+1}) = u_{j+1},\ S_j(x_{j-1}) = u_{j-1},\ S_j''(x_j) = L_j,\ S_j''(x_{j+1}) = L_{j+1},\ S_j''(x_{j-1}) = L_{j-1}$.

By using the conditions of continuity on first and second derivatives at the grid points (x_j, y_j) and through simple algebraic manipulation, the arbitrary constants in equation (17) can be obtained in the form

$a_j = u_j + \frac{L_j}{\tau^2},\ b_j = \frac{u_{j+1}-u_j}{h} + \frac{L_{j+1}-L_j}{\tau\theta},\ c_j = \frac{L_j \cos\theta - L_{j+1}}{\tau^2 \sin\theta},\ d_j = -\frac{L_j}{\tau^2}$, where, $\theta = \tau h,\ j = 0, 1, \ldots, n-1$.
Further at the grid points (x_j, y_j), we obtained the following consistency relation

$$\alpha L_{j+1} + 2\beta L_j + \alpha L_{j-1} = \frac{1}{h^2}\left(u_{j+1} - 2u_j + u_{j-1}\right),\ j = 1, \ldots, n-1, \tag{18}$$

where, $\alpha = \frac{1}{\theta\sin\theta} - \frac{1}{\theta^2},\ \beta = -\frac{1}{\theta^2} - \frac{\cos\theta}{\theta}$, and $\theta = \tau h$.

Similarly, relations for $v(x), w(x), r(x)$, and $t(x)$, respectively, can be obtained as follows

$$\alpha M_{j+1} + 2\beta M_j + \alpha M_{j-1} = \frac{1}{h^2}\left(v_{j+1} - 2v_j + v_{j-1}\right) \tag{19}$$

$$\alpha N_{j+1} + 2\beta N_j + \alpha N_{j-1} = \frac{1}{h^2}\left(w_{j+1} - 2w_j + w_{j-1}\right), \tag{20}$$

$$\alpha P_{j+1} + 2\beta P_j + \alpha P_{j-1} = \frac{1}{h^2}\left(r_{j+1} - 2r_j + r_{j-1}\right), \tag{21}$$

$$\alpha W_{j+1} + 2\beta W_j + \alpha W_{j-1} = \frac{1}{h^2}\left(t_{j+1} - 2t_j + t_{j-1}\right), \tag{22}$$

where we have substituted

$$u''(x) = L,\quad v''(x) = M,\quad w''(x) = N,\quad r''(x) = P,\quad \text{and } t''(x) = W.$$

To illustrate the applications of the developed scheme, we discretized equations (7-11) at the knots (x_j, u_j), (x_j, v_j), (x_j, w_j), (x_j, r_j), and (x_j, t_j), we had

$$t_j'' + a_1(x_j)t_j' + a_2(x_j)t_j + a_3(x_j)r_j' + a_4(x_j)r_j + a_5(x_j)w_j' + a_6(x_j)w_j + a_7(x_j)v_j' + a_8(x_j)v_j + a_9(x_j)u_j' + a_{10}(x_j)u_j = f(x_j) \tag{23}$$

$$u_j'' = v_j, \tag{24}$$

$$v_j'' = w_j, \tag{25}$$

$$w_j'' = r_j, \tag{26}$$

$$r_j'' = t_j. \tag{27}$$

Take $u_j'' = L_j, v_j'' = M_j, w_j'' = N_j, r_j'' = P_j$, and $t_j'' = W_j$. The equations (23-27) were as under

$$W_j = f(x_j) - a_1(x_j)t_j' - a_2(x_j)t_j - a_3(x_j)r_j' - a_4(x_j)r_j - a_5(x_j)w_j' - a_6(x_j)w_j - a_7(x_j)v_j' - a_8(x_j)v_j - a_9(x_j)u_j' + a_{10}(x_j)u_j, \tag{28}$$

$$L_j = v_j, \tag{29}$$

$$M_j = w_j, \tag{30}$$

$$N_j = r_j, \tag{31}$$

$$P_j = t_j. \tag{32}$$

Approximating the first derivatives of $u, v, w, r,$ and t in (28) by the central finite differences at $j,\ j+1,$ $j-1$, we had

$$u'_j \cong \frac{u_{j+1}-u_{j-1}}{2h}, \qquad u'_{j+1} \cong \frac{3u_{j+1}-4u_j+u_{j-1}}{2h}, \qquad u'_{j-1} \cong \frac{-u_{j+1}+4u_j-3u_{j-1}}{2h},$$

$$v'_j \cong \frac{v_{j+1}-v_{j-1}}{2h}, \qquad v'_{j+1} \cong \frac{3v_{j+1}-4v_j+v_{j-1}}{2h}, \qquad v'_{j-1} \cong \frac{-v_{j+1}+4v_j-3v_{j-1}}{2h},$$

$$w'_j \cong \frac{w_{j+1}-w_{j-1}}{2h}, \qquad w'_{j+1} \cong \frac{3w_{j+1}-4w_j+w_{j-1}}{2h}, \qquad w'_{j-1} \cong \frac{-w_{j+1}+4w_j-3w_{j-1}}{2h},$$

$$r'_j \cong \frac{r_{j+1}-r_{j-1}}{2h}, \qquad r'_{j+1} \cong \frac{3r_{j+1}-4r_j+r_{j-1}}{2h}, \qquad r'_{j-1} \cong \frac{-r_{j+1}+4r_j-3r_{j-1}}{2h},$$

$$t'_j \cong \frac{t_{j+1}-t_{j-1}}{2h}, t'_{j+1} \cong \frac{3t_{j+1}-4t_j+t_{j-1}}{2h}, \quad t'_{j-1} \cong \frac{-t_{j+1}+4t_j-3t_{j-1}}{2h}.$$

Substitute the above approximations in equation (28), we had

$$W_j = f(x_j) - a_1(x_j)\frac{t_{j+1}-t_{j-1}}{2h} - a_2(x_j)t_j - a_3(x_j)\frac{r_{j+1}-r_{j-1}}{2h} - a_4(x_j)r_j - a_5(x_j)\frac{w_{j+1}-w_{j-1}}{2h} - a_6(x_j)w_j - a_7(x_j)\frac{v_{j+1}-v_{j-1}}{2h} - a_8(x_j)v_j - a_9(x_j)\frac{u_{j+1}-u_{j-1}}{2h} - a_{10}(x_j)u_j. \tag{33}$$

Similarly,

$$W_{j+1} = f(x_{j+1}) - a_1(x_{j+1})\frac{3t_{j+1}-4t_j+t_{j-1}}{2h} - a_2(x_{j+1})t_j - a_3(x_{j+1})\frac{3r_{j+1}-4r_j+r_{j-1}}{2h} - a_4(x_{j+1})\,r_j - a_5(x_{j+1})\frac{3w_{j+1}-4w_j+w_{j-1}}{2h} - a_6(x_{j+1})w_j - a_7(x_{j+1})\frac{3v_{j+1}-4v_j+v_{j-1}}{2h} - a_8(x_{j+1})v_j - a_9(x_{j+1})\frac{3u_{j+1}-4u_j+u_{j-1}}{2h} - a_{10}(x_{j+1})u_j, \tag{34}$$

$$W_{j-1} = f(x_{j-1}) - a_1(x_{j-1})\frac{-t_{j+1}+4t_j-3t_{j-1}}{2h} - a_2(x_{j-1})t_j - a_3(x_{j-1})\frac{-r_{j+1}+4r_j-3r_{j-1}}{2h} - a_4(x_{j-1})\,r_j - a_5(x_{j-1})\frac{-w_{j+1}+4w_j-3w_{j-1}}{2h} - a_6(x_{j-1})w_j - a_7(x_{j-1})\frac{-v_{j+1}+4v_j-3v_{j-1}}{2h} - a_8(x_{j-1})v_j - a_9(x_{j-1})\frac{-u_{j+1}+4u_j-3u_{j-1}}{2h} - a_{10}(x_{j-1})u_j. \tag{35}$$

Substituting equations (33-35) in equation (22) and on simplifying, we obtained

$$\left(\frac{1}{h^2} - \frac{3\alpha a_1(x_{j-1})}{2h} - \frac{\beta a_1(x_j)}{h} + \frac{\alpha a_1(x_{j+1})}{2h} + \alpha a_2(x_{j-1})\right)t_{j-1} + \left(-\frac{2}{h^2} + \frac{2\alpha a_1(x_{j-1})}{h} - \frac{2\alpha a_1(x_{j+1})}{h} + 2\beta a_2(x_j)\right)t_j + \left(\frac{1}{h^2} - \frac{\alpha a_1(x_{j-1})}{2h} + \frac{\beta a_1(x_j)}{h} + \frac{3\alpha a_1(x_{j+1})}{2h} + \alpha a_2(x_{j+1})\right)t_{j+1} + \left(-\frac{3\alpha a_3(x_{j-1})}{2h} - \frac{\beta a_3(x_j)}{h} + \frac{\alpha a_3(x_{j+1})}{2h} + \alpha a_4(x_{j-1})\right)r_{j-1} + \left(\frac{2\alpha a_3(x_{j-1})}{h} - \frac{2\alpha a_3(x_{j+1})}{h} + 2\beta a_4(x_j)\right)r_j + \left(-\frac{\alpha a_3(x_{j-1})}{2h} + \frac{\beta a_3(x_j)}{h} + \frac{3\alpha a_3(x_{j+1})}{2h} + \alpha a_4(x_{j+1})\right)r_{j+1} + \left(-\frac{3\alpha a_5(x_{j-1})}{2h} - \frac{\beta a_5(x_j)}{h} + \frac{\alpha a_5(x_{j+1})}{2h} + \alpha a_6(x_{j-1})\right)w_{j-1} + \left(\frac{2\alpha a_5(x_{j-1})}{h} - \frac{2\alpha a_5(x_{j+1})}{h} + 2\beta a_6(x_j)\right)w_j + \left(-\frac{\alpha a_5(x_{j-1})}{2h} + \frac{\beta a_5(x_j)}{h} + \frac{3\alpha a_5(x_{j+1})}{2h} + \alpha a_6(x_{j+1})\right)w_{j+1} + \left(-\frac{3\alpha a_7(x_{j-1})}{2h} - \frac{\beta a_7(x_j)}{h} + \frac{\alpha a_7(x_{j+1})}{2h} + \alpha a_8(x_{j-1})\right)v_{j-1} + \left(\frac{2\alpha a_7(x_{j-1})}{h} - \frac{2\alpha a_7(x_{j+1})}{h} + 2\beta a_8(x_j)\right)v_j + \left(-\frac{\alpha a_7(x_{j-1})}{2h} + \frac{\beta a_7(x_j)}{h} + \frac{3\alpha a_7(x_{j+1})}{2h} + \alpha a_8(x_{j+1})\right)v_{j+1} + \left(-\frac{3\alpha a_9(x_{j-1})}{2h} - \frac{\beta a_9(x_j)}{h} + \frac{\alpha a_9(x_{j+1})}{2h} + \alpha a_{10}(x_{j-1})\right)u_{j-1} +$$

$$\left(\frac{2\alpha a_9(x_{j-1})}{h}-\frac{2\alpha a_9(x_{j+1})}{h}+2\beta a_{10}(x_j)\right)u_j+$$
$$\left(-\frac{\alpha a_9(x_{j-1})}{2h}+\frac{\beta a_9(x_j)}{h}+\frac{3\alpha a_9(x_{j+1})}{2h}+\alpha a_{10}(x_{j+1})\right)u_{j+1}=-\alpha f(x_{j-1})-2\beta f(x_j)-\alpha f(x_{j-1}) \quad (36)$$

Now equations (29-32) could also be written as

$$L_j=v_j,\quad L_{j+1}=v_{j+1},\quad L_{j-1}=v_{j-1},$$
$$M_j=w_j,\quad M_{j+1}=w_{j+1},\quad M_{j-1}=w_{j-1},$$
$$N_j=r_j,\quad N_{j+1}=r_{j+1},\quad N_{j-1}=r_{j-1},$$
$$P_j=t_j,\quad P_{j+1}=t_{j+1},\quad P_{j-1}=t_{j-1}.$$

Substituting the above relations in equations (18-21), we had

$$\frac{1}{h^2}\left(u_{j+1}-2u_j+u_{j-1}\right)=\alpha v_{j+1}+2\beta v_j+\alpha v_{j-1}, \quad (37)$$
$$\frac{1}{h^2}\left(v_{j+1}-2v_j+v_{j-1}\right)=\alpha w_{j+1}+2\beta w_j+\alpha w_{j-1}, \quad (38)$$
$$\frac{1}{h^2}\left(w_{j+1}-2w_j+w_{j-1}\right)=\alpha r_{j+1}+2\beta r_j+\alpha r_{j-1}, \quad (39)$$
$$\frac{1}{h^2}\left(r_{j+1}-2r_j+r_{j-1}\right)=\alpha t_{j+1}+2\beta t_j+\alpha t_{j-1}. \quad (40)$$

The equations (36-40) associated with the boundary conditions in equations (12-16) form a complete system of the $5(n+1)$ linear equations in the $5(n+1)$ unknowns. This system can be solved by simple numerical algorithms.

4. CONSTRUCTION OF POLYNOMIAL CUBIC SPLINE SCHEME

To derive polynomial cubic spline approximation for equations (7-11) with boundary conditions in equations (12-16), the interval $[a,b]$ was again divided into n equal subintervals using equally spaced knots: $\mathrm{x_j}=\mathrm{a}+\mathrm{jh},\ \mathrm{j}=0,1,\dots,\mathrm{n}$, where $\mathrm{a}=\mathrm{x_0},\ \mathrm{b}=\mathrm{x_n}$, and $\mathrm{h}=\frac{\mathrm{b-a}}{\mathrm{n}}$. For each segment $[x_j,x_{j+1}],\ j=0,1,\dots,n-1$, we have the polynomial cubic spline

$$S_j(x)=a_j+b_j(x-x_j)+c_j(x-x_j)^2+d_j(x-x_j)^3,\ j=0,1,\dots,n-1. \quad (41)$$

In a similar way to that for the non-polynomial cubic spline scheme, the polynomial cubic spline scheme was constructed as follows

$$\left(\frac{6}{h^2}-\frac{3a_1(x_{j-1})}{2h}-\frac{2a_1(x_j)}{h}+\frac{a_1(x_{j+1})}{2h}+a_2(x_{j-1})\right)t_{j-1}+\left(-\frac{12}{h^2}+\frac{2a_1(x_{j-1})}{h}-\frac{2a_1(x_{j+1})}{h}+4a_2(x_j)\right)t_j+$$
$$\left(\frac{6}{h^2}-\frac{a_1(x_{j-1})}{2h}+\frac{2a_1(x_j)}{h}+\frac{3a_1(x_{j+1})}{2h}+a_2(x_{j+1})\right)t_{j+1}+$$
$$\left(-\frac{3a_3(x_{j-1})}{2h}-\frac{2a_3(x_j)}{h}+\frac{a_3(x_{j+1})}{2h}+a_4(x_{j-1})\right)r_{j-1}+\left(\frac{2a_3(x_{j-1})}{h}-\frac{2a_3(x_{j+1})}{h}+4a_4(x_j)\right)r_j+$$
$$\left(-\frac{a_3(x_{j-1})}{2h}+\frac{2a_3(x_j)}{h}+\frac{3a_3(x_{j+1})}{2h}+a_4(x_{j+1})\right)r_{j+1}+$$
$$\left(-\frac{3a_5(x_{j-1})}{2h}-\frac{2a_5(x_j)}{h}+\frac{a_5(x_{j+1})}{2h}+a_6(x_{j-1})\right)w_{j-1}+\left(\frac{2a_5(x_{j-1})}{h}-\frac{2a_5(x_{j+1})}{h}+4a_6(x_j)\right)w_j+$$

$$\left(-\frac{a_5(x_{j-1})}{2h}+\frac{2a_5(x_j)}{h}+\frac{3a_5(x_{j+1})}{2h}+a_6(x_{j+1})\right)w_{j+1}$$
$$+\left(-\frac{3a_7(x_{j-1})}{2h}-\frac{2a_7(x_j)}{h}+\frac{a_7(x_{j+1})}{2h}+a_8(x_{j-1})\right)v_{j-1}+\left(\frac{2a_7(x_{j-1})}{h}-\frac{2a_7(x_{j+1})}{h}+4a_8(x_j)\right)v_j+$$
$$\left(-\frac{a_7(x_{j-1})}{2h}+\frac{2a_7(x_j)}{h}+\frac{3a_7(x_{j+1})}{2h}+a_8(x_{j+1})\right)v_{j+1}+$$
$$\left(-\frac{3a_9(x_{j-1})}{2h}-\frac{2a_9(x_j)}{h}+\frac{a_9(x_{j+1})}{2h}+a_{10}(x_{j-1})\right)u_{j-1}+\left(\frac{2a_9(x_{j-1})}{h}-\frac{2a_9(x_{j+1})}{h}+4a_{10}(x_j)\right)u_j+$$
$$\left(-\frac{a_9(x_{j-1})}{2h}+\frac{2a_9(x_j)}{h}+\frac{3a_9(x_{j+1})}{2h}+a_{10}(x_{j+1})\right)u_{j+1}=f(x_{j-1})+4f(x_j)+f(x_{j-1}) \quad (42)$$

$$\frac{6}{h^2}(u_{j+1}-2u_j+u_{j-1})=(v_{j+1}+4v_j+v_{j-1}), \quad (43)$$

$$\frac{6}{h^2}(v_{j+1}-2v_j+v_{j-1})=(w_{j+1}+4w_j+w_{j-1}), \quad (44)$$

$$\frac{6}{h^2}(w_{j+1}-2w_j+w_{j-1})=(r_{j+1}+4r_j+r_{j-1}), \quad (45)$$

$$\frac{6}{h^2}(r_{j+1}-2r_j+r_{j-1})=(t_{j+1}+4t_j+t_{j-1}). \quad (46)$$

The equations (42-46) associated with boundary conditions in equations (12-16) form a complete system of the $5(n+1)$ linear equations in the $5(n+1)$ unknowns. This system can be solved by simple numerical algorithms.

5. RESULTS AND DISCUSSIONS

To observe the computational efficiency of the above developed schemes, we considered the following two test problems.

Test Problem 1

We considered the following tenth-order equation

$$u^{10}(x)=(1-x)\sin x+10\cos x, \quad 0\le x\le 1,$$

with boundary conditions:

$$u(0)=0, \qquad u(1)=0,$$
$$u^2(0)=2, \qquad u^2(1)=2\cos 1,$$
$$u^4(0)=-4, \qquad u^4(1)=-4\cos 1,$$
$$u^6(0)=6, \qquad u^6(1)=6\cos 1,$$
$$u^8(0)=-8, \qquad u^8(1)=-8\cos 1.$$

The analytical solution to the above tenth-order BVP is

$$u(x)=(x-1)\sin x.$$

To quantify the quality of the developed schemes, the first set of experiments was performed to observe the absolute error while comparing the numerical approximations obtained by NPCSM and PCSM with the exact solutions applied to the test problem 1. The associated absolute errors for $h=\frac{1}{15}$ and $h=\frac{1}{20}$ had been showed in Table 1 and Table 2, respectively.

Table 1. Absolute errors at $h = \frac{1}{15}$.

x	Exact	Absolute Error (NPCSM)	Absolute Error (PCSM)
0.2	-0.158938872	4.198E-08	1.299E-04
0.4	-0.233656041	6.206E-08	1.925E-04
0.6	-0.225861866	6.010E-08	1.865E-04
0.8	-0.143471218	3.810E-08	1.182E-04

Table 2. Absolute errors at $h = \frac{1}{20}$.

x	Exact	Absolute Error (NPCSM)	Absolute Error (PCSM)
0.2	-0.158938872	1.328E-08	7.309E-05
0.4	-0.233656041	1.963E-08	1.083E-04
0.6	-0.225861866	1.901E-08	1.050E-04
0.8	-0.143471218	1.205E-08	6.649E-05

Table 1 showed the absolute errors associated with $h = \frac{1}{15}$ and $x = 0.2\ (0.2)\ 0.8$. Here we observed that at spatial displacement $x = 0.4$, for NPCSM and PCSM, the maximum absolute error between the numerical approximation and the exact solution was not more than 6.21×10^{-08} and 1.92×10^{-04}, respectively. Whereas, the best observed numerical accuracy was obtained by the NPCSM at $x = 0.8$ with absolute error approximation 3.81×10^{-08}.

Similar set of experiments for $h = \frac{1}{20}$ and $x = 0.2\ (0.2)\ 0.8$ were performed in Table 2. For all the experiments performed in Table 2, we observed very much the same trend as of the results obtained in Table1. The best observed accuracy was again obtained by the NPCSM with absolute error approximately 1.20×10^{-08}.

Test Problem 2

For the second set of experiments, we considered the following tenth-order equation

$$u^{10}(x) = -(80 + 19x + x^2)e^x, \quad 0 \le x \le 1,$$

with boundary conditions:

$$u(0) = 0, \qquad u(1) = 0,$$

$$u^2(0) = 0, \qquad u^2(1) = -4e,$$

$$u^4(0) = -8, \qquad u^4(1) = -16e,$$

$$u^6(0) = -24, \qquad u^6(1) = -36e,$$

$$u^8(0) = -48, \qquad u^8(1) = -64e.$$

The true solution to the above BVP is given by

$u(x) = x(1 - x)e^x$.

To illustrate the performance of NPCSM and PCSM schemes, we repeated the previous sets of experiments for the test problem 2 as shown in Tables 3-4. The NPCSM again achieved the best observed accuracy compared to PCSM. It has been observed that using NPCSM the best observed accuracy for $h = \frac{1}{15}$ and $h = \frac{1}{20}$ was obtained at $x = 0.2$. The corresponding absolute errors were approximately 2.43×10^{-07} and 7.70×10^{-08} as shown in Table 3 and Table 4, respectively.

Table 3. Absolute errors at $h = \frac{1}{15}$.

x	Exact	Absolute Error(NPCSM)	Absolute Error(PCSM)
0.2	0.195424441	2.433E-07	3.982E-04
0.4	0.358037927	3.986E-07	6.663E-04
0.6	0.437308512	4.428E-07	7.598E-04
0.8	0.356086549	3.328E-07	5.885E-04

Table 4 Absolute errors at $h = \frac{1}{20}$.

x	Exact	Absolute Error (NPCSM)	Absolute Error (PCSM)
0.2	0.195424441	7.698E-08	2.238E-04
0.4	0.358037927	1.261E-07	3.745E-04
0.6	0.437308512	1.401E-07	4.271E-04
0.8	0.356086549	1.053E-07	3.308E-04

In this research work, the NPCSM and PCSM along with decomposition procedure were used for the spatial derivatives. Siddiqi et al [11] used eleventh degree spline for the numerical approximation of tenth-order linear special case BVPs. The absolute errors were calculated by Shahid S. Siddiqi at different step sizes, $\frac{1}{7}, \frac{1}{14}, \frac{1}{21}$, and $\frac{1}{28}$. The maximum accuracy of approximately 2.13×10^{-8} was obtained at a step size of $\frac{1}{28}$. On the other hand, in this paper we obtained an accuracy of approximately 1.20×10^{-08} was obtained at a step size of $\frac{1}{20}$, using cubic spline method which is much better accuracy with small step size. We further investigate with reduced step size and obtained an accuracy of approximately 3.40×10^{-12} at a step size of $\frac{1}{80}$.

The overall conclusion was that the performance of the developed schemes was remarkably good when implemented on linear tenth-order BVPs and produced encouraging results which were very much close to the exact solution.

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