Proceedings of the Pakistan Academy of Sciences 52 (2): 151-158 (2015)

ISSN: 0377 - 2969 (print), 2306 - 1448 (online)

Pakistan Academy of Sciences

Research Article

Review of Numerical Schemes for Two Point Second Order Non-Linear Boundary Value Problems

Abdul Manaf[1*], Mustafa Habib[1], and M. O. Ahmad[2]

[1]Department of Mathematics, University of Engineering and Technology, Lahore, Pakistan
[2]Department of Mathematics, University of Lahore, Lahore, Pakistan

Abstract: In this paper, numerical solution of two points 2nd order nonlinear boundary-value problems was considered. The numerical solution was reviewed with nonlinear shooting method, finite-difference method and fourth order compact method. The results were compared to check the accuracy of numerical schemes with exact solution. It was found that the nonlinear shooting method is more accurate than finite-difference method and fourth order compact method.

Keywords: BVPs, ODEs, Nonlinear Shooting method, finite-difference method, fourth order compact method

1. INTRODUCTION

Consider a two point 2nd order nonlinear boundary value problem (BVP) of the type

$$y'' = f(x, y, y'), x \in [a,b]; \; a,b \in R \tag{1}$$

jointly among the boundary conditions

$$y(a) = \alpha \text{ and } y(b) = \beta. \tag{2}$$

Where α and β are constants.

The approach for solving this problem has been projected by a number of researchers such as Roberts and Shipman [1], Malathi [2], Ha [3], Auzinger et al [4] and Attili and Syam [5].

In this research paper, we considered nonlinear shooting method (NLSM), finite-difference method (FDM) and fourth order compact method (FOCM) for the solution of above two points 2nd order nonlinear boundary value problems (BVPs).

2. NONLINEAR SHOOTING METHOD (NLSM)

Consider a two point's 2nd order non-linear boundary-value problem (BVP)

$$y'' = f(x, y, y'), \; y(a) = \alpha, \; y(b) = \beta \tag{3}$$

where $a \le x \le b$ and α, β are constants.

Here, we used the solutions to a sequence of initial value problems (IVPs) of [3, 5, 6]

$$y'' = f(x, y, y'), \; y(a) = \alpha, \; y'(a) = t \tag{4}$$

Concerning a parameter t, and $a \le x \le b$, to approximate the solution to our BVP (3).

By choosing parameters $t = t_k$ in such a way that

$$\lim_{k \to \infty} y(b, t_k) = y(b) = \beta \tag{5}$$

Where $y(x, t_k)$ is the solution to the IVP (4) with $t = t_k$ and y(x) is the solution to the BVP (3).

This procedure is called a Shooting method.

We begin with a parameter t_0 that find out the initial elevation at which the object is excited from the point (a, α) and beside the curve described by the solution to the IVP [6].

$$y'' = f(x, y, y'), \; y(a) = \alpha, \; y'(a) = t_0 \tag{6}$$

Received, November 2014; Accepted, February 2015
*Corresponding author: Abdul Manaf; Email: manaf192@yahoo.com

If $y(b,t_0)$ is not satisfactorily close to β, we try to correct our approximation by choosing another elevation t_1 and so on, until $y(b,t_k)$ is satisfactorily close to striking β [3].

To decide how the parameter t_k can be chosen, suppose a BVP (3) has a single solution. If y(x, t) is the solution to the IVP (4), then we need to determine t so that

$$y(b,t)-\beta = 0 \tag{7}$$

Since this is a nonlinear equation, we use Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

to solve the problem. We need to choose initial approximation t_0 and then generate sequence by

$$t_k = t_{k-1} - \frac{y(b,t_{k-1})-\beta}{\frac{dy}{dt}(b,t_{k-1})} \tag{8}$$

this requires the knowledge of $\frac{dy}{dt}(b,t_{k-1})$. This presents a complexity, since an explicit illustration for y (b, t) is not identified; we only know the values

$$y(b,t_0),\ y(b,t_1),\ \dots\dots\dots\dots,\ y(b,t_{k-1}).$$

Suppose we modify the IVP (4), emphasizing that the solution depends on together x and t.

$$y''(x,t) = f\left(x, y(x,t), y'(x,t)\right), a \le x \le b,$$

$$y(a,t) = \alpha,\ y'(a,t) = t \tag{9}$$

retaining the prime notation to indicate differentiation with respect to x.

Since we need to determine $\frac{dy}{dt}(b,t)$, when $t = t_{k-1}$, we take partial derivative of (9) with respect to t.

$$\frac{\partial y''(x,t)}{\partial t} = \frac{\partial f\left(x, y(x,t), y'(x,t)\right)}{\partial t}$$

$$= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y(x,t)}{\partial t} + \frac{\partial f}{\partial y'}\frac{\partial y'(x,t)}{\partial t}$$

Since x and t are independent, $\frac{\partial x}{\partial t} = 0$, so

$$\frac{\partial y''}{\partial t}(x,t) = \frac{\partial f}{\partial y}\frac{\partial y(x,t)}{\partial t} + \frac{\partial f}{\partial y'}\frac{\partial y'(x,t)}{\partial t} \tag{10}$$

for $a \le x \le b$. The initial conditions give

$$\frac{\partial y}{\partial t}(a,t) = 0, \text{ and } \frac{\partial y'}{\partial t}(a,t) = 1$$

If we make simpler the notation by using $z(x,t)$ to denote $\frac{\partial y}{\partial t}(x,t)$ and suppose that the order of differentiation of x & t can be reversed, Eq. (10) with initial conditions becomes IVP [6]

$$z''(x,t) = \frac{\partial f}{\partial y} z(x,t) + \frac{\partial f}{\partial y'} z'(x,t),$$

$$a \le x \le b,\ z(a,t) = 0,\ z'(a,t) = 1 \tag{11}$$

Therefore, one requires that two IVPs be solved for each iteration, (4) and (11).

Then from Eq. (8),

$$t_k = t_{k-1} - \frac{y(b, t_{k-1}) - \beta}{Z(b, t_{k-1})} \tag{12}$$

In practice, none of these IVPs is solved exactly; instead, the solutions are approximated by one of the IVP solvers [3].

Hence, in shooting method for nonlinear BVPs, we use classical Runge-Kutta fourth-order method to approximate both solutions required by Newton's method.

3. FINITE-DIFFERENCE M ETHOD (FDM)

Methods involving finite differences for solving boundary value problems (BVPs) replace each of the derivatives in the differential equation by an appropriate difference-quotient approximation. The difference quotient is chosen to maintain a specified order of truncation error [7].

The nonlinear second order BVP of the form (3) requires that difference–quotient approximations be used to approximate both y' *and* y''. First, we take N>0, which is an integer, and break up $[a,b]$ into N+1 equal subintervals, whose end points are mesh points $x_i = a + ih, for\ i = 0,1,2,...,N+1\ where\ h = \frac{b-a}{N+1}$.

At interior mesh points, x_i $for\ i = 0,1,2,...,N,$ following differential-equation is to be approximated

$$y''(x_i) = f\left(x_i, y(x_i), y'(x_i)\right) \quad (13)$$

Expanding y, in the third Taylor-polynomial, about x_i evaluated at x_{i+1} and x_{i-1}, we get

$$y(x_{i+1}) = y(x_i) + hy'(x_i) + \frac{h^2}{2!}y''(x_i) + \frac{h^3}{6}y'''(x_i) + \frac{h^4}{24}y^4(\xi_i^+)$$

for some ξ_i^+ in $\left(x_i, x_{i+1}\right)$, and

$$y(x_{i-1}) = y(x_i) - hy'(x_i) + \frac{h^2}{2}y''(x_i) - \frac{h^3}{6}y'''(x_i) + \frac{h^4}{24}y^4(\xi_i^-)$$

for some ξ_i^- in $\left(x_{i-1}, x_i\right)$, assuming $y \in C^4\left[x_{i-1}, x_{i+1}\right]$. By adding the above two equations, we have

$$y''(x_i) = \frac{1}{h^2}\left[y(x_{i+1}) - 2y(x_i) + y(x_{i-1})\right] - \frac{h^2}{24}\left[y^4(\xi_i^+) + y^4(\xi_i^-)\right]$$

Using intermediate value theorem, this can be simplified even further as

$$y''(x_i) = \frac{1}{h^2}\left[y(x_{i+1}) - 2y(x_i) + y(x_{i-1})\right] - \frac{h^2}{12}y^4(\xi_i) \quad (14)$$

for some $\xi_i\ in\left(x_{i-1}, x_{i+1}\right)$. This is called the centered-difference formula for $y''(x_i)$.

A centered difference formula for $y'(x_i)$ is obtained in a related manner resulting in

$$y'(x_i) = \frac{1}{2h}\left[y(x_{i+1}) - 2y(x_{i-1})\right] - \frac{h^2}{6}y'''(\eta_i) \quad (15)$$

for some η_i in (x_{i-1}, x_{i+1}).

The use of the centered difference formulas in equation (13) gives

$$\frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2}$$
$$= f\left(x_i, y(x_i), \frac{y(x_{i+1}) - 2y(x_{i-1})}{2h} - \frac{h^2}{6}y'''(\eta_i)\right) + \frac{h^2}{12}y^{(4)}(\xi_i)$$

for some $\xi_i\ and\ \eta_i$ in the interval (x_{i-1}, x_{i+1}).

As in linear case, the difference method results when the error terms are deleted and boundary conditions employed [7]:

$$w_0 = \alpha, \quad w_{N+1} = \beta,\ and$$

$$-\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} + f\left(x_i, w_i, \frac{w_{i+1} - w_{i-1}}{2h}\right) = 0$$

for each $i = 1,2,...,N$.

The $N \times N$ nonlinear system obtained from this method is

$$2w_1 - w_2 + h^2 f\left(x_1, w_1, \frac{w_2 - \alpha}{2h}\right) - \alpha = 0,$$

$$-w_1 + 2w_2 - w_3 + h^2 f\left(x_2, w_2, \frac{w_3 - w_1}{2h}\right) = 0,$$

....

....

$$-w_{N-2} + 2w_{N-1} - w_N + h^2 f\left(x_{N-1}, w_{N-1}, \frac{w_N - w_{N-2}}{2h}\right) = 0,$$

$$-w_{N-1} + 2w_N + h^2 f\left(x_N, w_N, \frac{\beta - w_{N-1}}{2h}\right) - \beta = 0.$$

To approximate the solution to this system, we use Newton's method for nonlinear system. A sequence of iterates $\left(w_1^{(k)}, w_2^{(k)}, ..., w_N^{(k)}\right)^t$ is generated that converges to the solution of system, provided that the initial approximation $\left(w_1^{(0)}, w_2^{(0)}, ..., w_N^{(0)}\right)^t$ is sufficiently close to the solution, $\left(w_1, w_2, ..., w_N\right)^t$, and that the Jacobian matrix for the system is nonsingular. For the system, the Jacobian matrix is tridiagonal.

Newton's method for nonlinear systems requires that at each iteration, the $N \times N$ linear system [8]

$$J\left(w_1,w_2,...,w_N\right)^t\left(v_1,v_2,...,v_N\right)^t$$
$$=-\left(2w_1-w_2-\alpha+h^2f\left(x_1,w_1,\frac{w_2-\alpha}{2h}\right),\right.$$
$$-w_1+2w_2-w_3+h^2f\left(x_2,w_2,\frac{w_3-w_1}{2h}\right),..........,$$
$$-w_{N-2}+2w_{N-1}-w_N+h^2f\left(x_{N-1},w_{N-1},\frac{w_N-w_{N-2}}{2h}\right),$$
$$\left.-w_{N-1}+2w_N+h^2f\left(x_N,w_N,\frac{\beta-w_{N-1}}{2h}\right)-\beta\right)^t$$

be solved for $v_1,v_2,........,v_N$, since $w_i^{(k)}=w_i^{(k-1)}+v_i$, $for\,each\; i=1,2,.....,N$ (16)

4. FOURTH ORDER COMPACT METHOD (FOCM)

A fourth order general differencing scheme, proposed by Kreiss [9], is developed and it is applied on the three test problems to confirm accuracy and applicability of this technique. Burger's equation, the Howarth's retarded boundary flow, and the incompressible driven cavity, are the solved test problems [6, 8, 9]. A number of researchers have contributed in this method, some of them are Ahmad [8], Orsazag and Israeli [10], Richard [11], Leventhal and Ciment [12] and Pettigrew and Rasmussen [13]. The detailed work on compact method was done by Ahmad [8]. We follow the procedure of Pettigrew and Rasmussen [13] to derive the compact method for nonlinear ODEs [14, 15].

Consider the nonlinear two point second order BVP of the form (3). Let us denote first and second derivatives with respect to x of y by F and S respectively, i.e.

$$y'=F,\; and\;\; y''=S \tag{17}$$
$$\Rightarrow F'=S$$

Consider $y'=F$

Integrating both sides from $x_{i-1}\; to\; x_{i+1}$, we get

$$y_{i+1}-y_{i-1}=\int_{x_{i-1}}^{x_{i+1}}F(\xi)d\xi \tag{18}$$

$$y_{i+1}=y_{i-1}+\int_{x_{i-1}}^{x_{i+1}}F(\xi)d\xi$$

Approximating integral by Simpson's rule [8]

$$y_{i+1}=y_{i-1}+\frac{h}{3}\left[F_{i-1}+4F_i+F_{i+1}\right]-\frac{1}{90}h^5F^4(\xi)$$
$$\Rightarrow\frac{h}{3}\left[F_{i-1}+4F_i+F_{i+1}\right]+y_{i-1}-y_{i+1}=\frac{1}{90}h^5F^4(\xi)$$

multiplying both sides by $\frac{3}{h}$, we get

$$F_{i-1}+4F_i+F_{i+1}+\frac{3}{h}\left[y_{i-1}-y_{i+1}\right]=\frac{1}{30}h^4F^4(\xi)$$

and to the fourth order, we have

$$F_{i-1}+4F_i+F_{i+1}+\frac{3}{h}\left[y_{i-1}-y_{i+1}\right]=0 \tag{19}$$

Thus we have a relationship between y and F and it is the first difference-equation.

Now for the second equation, we start by evaluating equation (eq) (1) at the midpoint i. Thus equation (1) becomes [7]

$$y_i''=f\left(x_i,y(x_i),y'(x_i)\right)$$

Since $y''=S$, so we have

$$S_i=f\left(x_i,y(x_i),F_i\right) \tag{20}$$

Now, we require an expression for S_i.

If we expand y in the 5th Taylor's polynomial on x_i evaluated at x_{i+1} and at the x_{i-1}, we get

$$y_{i+1}=y_i+hy_i'+\frac{h^2}{2!}y_i''+\frac{h^3}{3!}y_i'''+\frac{h^4}{4!}y_i^{(4)}+\frac{h^5}{5!}y_i^{(5)}+\frac{h^6}{6!}y_i^{(6)}(\xi_i^+)$$

for some ξ_i^+ in $\left(x_i,x_{i+1}\right)$, and

$$y_{i-1}=y_i-hy_i'+\frac{h^2}{2!}y_i''-\frac{h^3}{3!}y_i'''+\frac{h^4}{4!}y_i^{(4)}-\frac{h^5}{5!}y_i^{(5)}+\frac{h^6}{6!}y_i^{(6)}(\xi_i^-)$$

for some ξ_i^- in $\left(x_{i-1},x_i\right)$, assuming $y\in C^6\left[x_{i-1},x_{i+1}\right]$. If we add these equations, terms involving $y_i', y_i''\, and\; y_i^{(5)}$ are eliminated and we get

$$y_{i+1}+y_{i-1}=2y_i+h^2y_i''+\frac{h^4}{12}y_i^4+\frac{h^6}{720}\left[y^{(6)}(\xi_i^+)+y^{(6)}(\xi_i^-)\right]$$
$$\Rightarrow y_{i+1}+y_{i-1}=2y_i+h^2S_i+\frac{h^4}{12}y_i^4+\frac{h^6}{720}\left[y^{(6)}(\xi_i^+)+y^{(6)}(\xi_i^-)\right]$$

Here intermediate-value theorem is used to further simplify this as

$$y_{i+1}+y_{i-1}=2y_i+h^2S_i+\frac{h^4}{12}y_i^4+\frac{h^6}{360}y^{(6)}(\xi_i) \tag{21}$$

for some ξ_i in $\left(x_{i-1}, x_{i+1}\right)$.

Now expanding F in Taylor series

$$F_{i+1} = F_i + hF_i' + \frac{h^2}{2!}F_i'' + \frac{h^3}{3!}F_i''' + \frac{h^4}{4!}F_i^{(4)} + \frac{h^{(5)}}{5!}F_i^{(5)}(\xi_i^+) \quad (22)$$

$$F_{i-1} = F_i - hF_i' + \frac{h^2}{2!}F_i'' - \frac{h^3}{3!}F_i''' + \frac{h^4}{4!}F_i^{(4)} - \frac{h^{(5)}}{5!}F_i^{(5)}(\xi_i^+) \quad (23)$$

On subtracting eq(23) from eq(22), we get

$$F_{i+1} - F_{i-1} = 2hF_i' + \frac{h^3}{3}F_i''' + \frac{h^{(5)}}{120}\left[F^{(5)}(\xi_i^+) + F^{(5)}(\xi_i^-)\right]$$

By using intermediate value theorem, we get

$$F_{i+1} - F_{i-1} = 2hF_i' + \frac{h^3}{3}F_i''' + \frac{h^{(5)}}{60}F^{(5)}(\xi_i^+)$$

For a few ξ_i in $\left(x_{i-1}, x_{i+1}\right)$.

Since, $F' = S$, $F''' = y^{(4)}$, $F^{(5)} = y^{(6)}$, So,

$$F_{i+1} - F_{i-1} = 2hS_i + \frac{h^3}{3}y_i^{(4)} + \frac{h^{(5)}}{60}y^{(6)}(\xi_i) \quad (24)$$

Eliminating $y^{(4)}$ from eq (21) and (24), we get

$$S_i = \frac{2}{h^2}\left[y_{i-1} - 2y_i + y_{i+1}\right] - \frac{1}{2h}\left[F_{i+1} - F_{i-1}\right] + \frac{h^{(4)}}{360}y^{(6)}(\xi)$$

By a similar procedure, we get

$$S_{i-1} = \frac{1}{2h^2}\left[-23y_{i-1} + 16y_i + 7y_{i+1}\right] - \frac{1}{h}\left[6F_{i-1} + 8F_i + F_{i+1}\right] + \frac{h^{(4)}}{90}y^{(6)}(\xi_i)$$

$$S_{i-1} = \frac{1}{2h^2}\left[7y_{i-1} + 16y_i - 23y_{i+1}\right] + \frac{1}{h}\left[F_{i-1} + 8F_i + 6F_{i+1}\right] + \frac{h^{(4)}}{90}y^{(6)}(\xi_i)$$

and to the fourth order, we have

$$S_i = \frac{2}{h^2}\left[y_{i-1} - 2y_i + y_{i+1}\right] - \frac{1}{2h}\left[F_{i+1} - F_{i-1}\right] \quad (25)$$

$$S_{i-1} = \frac{1}{2h^2}\left[-23y_{i-1} + 16y_i + 7y_{i+1}\right] - \frac{1}{h}\left[6F_{i-1} + 8F_i + F_{i+1}\right] \quad (26)$$

$$S_{i+1} = \frac{1}{2h^2}\left[7y_{i-1} + 16y_i - 23y_{i+1}\right] + \frac{1}{h}\left[F_{i-1} + 8F_i + 6F_{i+1}\right] \quad (27)$$

From eq (25), putting the value of S_i in eq (20), we get

$$\frac{2}{h^2}\left[y_{i-1} - 2y_i + y_{i+1}\right] - \frac{1}{2h}\left[F_{i+1} - F_{i-1}\right] = f\left(x_i, y(x_i), F_i\right) \quad (28)$$

We have replaced now differential equation (3) by two difference-equations (19) and (28).

Now, consider 1st boundary condition i.e. at x=a, and denote the points $x = a,\ a+h,\ a+2h$ *by* $0,1,2$. the first difference-equation, we obtain from the boundary condition, is [7]

$$y_0 = \alpha \quad (29)$$

Now to obtain the second equation, we begin with differential equation at points 0 and1.

$$S_0 = f\left(x_0, y_0, F_0\right) \quad (30)$$

$$S_1 = f\left(x_1, y_1, F_1\right) \quad (31)$$

For $i = 1$, eq(26) implies

$$S_0 = \frac{1}{2h^2}\left[-23y_0 + 16y_1 + 7y_2\right] - \frac{1}{h}\left[6F_0 + 8F_1 + F_2\right] \quad (32)$$

Eq (25) implies

$$S_1 = \frac{2}{h^2}\left[y_0 - 2y_1 + y_2\right] - \frac{1}{2h}\left[F_2 - F_0\right] \quad (33)$$

Eq (19) implies

$$F_0 + 4F_1 + F_2 + \frac{3}{h}\left[y_0 - y_2\right] = 0 \quad (34)$$

Now, we have five equations from (30) to (34). We eliminate S_0, S_1, y_2, F_2 from these equations.

From eq (30) and (32), we have

$$f\left(x_0, y_0, F_0\right) = -\frac{23}{2h^2}y_0 + \frac{8}{h^2}y_1 + \frac{7}{2h^2}y_2 - \frac{6}{h}F_0 - \frac{8}{h}F_1 - \frac{1}{h}F_2 \quad (35)$$

From eq (31) and (33), we have

$$f\left(x_1, y_1, F_1\right) = \frac{2}{h^2}y_0 - \frac{4}{h^2}y_1 + \frac{2}{h^2}y_2 - \frac{1}{2h}F_2 + \frac{1}{2h}F_0 \quad (36)$$

From eq (34), we get

$$y_2 = y_0 + \frac{h}{3}F_0 + \frac{4h}{3}F_1 + \frac{h}{3}F_2 \quad (37)$$

Putting this value of y_2 in equations (35) and (36), we get

$$f\left(x_0, y_0, F_0\right) = -\frac{8}{h^2}y_0 + \frac{8}{h^2}y_1 - \frac{29}{6h}F_0 - \frac{10}{3h}F_1 + \frac{1}{6h}F_2 \quad (38)$$

$$f\left(x_1, y_1, F_1\right) = -\frac{4}{h^2}y_0 - \frac{4}{h^2}y_1 + \frac{7}{6h}F_0 + \frac{8}{3h}F_1 + \frac{1}{6h}F_2 \quad (39)$$

Subtracting eq (38) from eq (39), we get

$$f\left(x_1, y_1, F_1\right) - f\left(x_0, y_0, F_0\right) = \frac{12}{h^2}y_0 - \frac{12}{h^2}y_1 + \frac{6}{h}F_0 + \frac{6}{h}F_1 \quad (40)$$

In a similar approach, we derive the following two difference-equations for y and F at x=m. i.e. at the right boundary point [7]

$$y_m = \beta \quad (41)$$

$$f\left(x_m, y_m, F_m\right) - f\left(x_{m-1}, y_{m-1}, F_{m-1}\right) = \frac{12}{h^2}\left(y_{m-1} - y_m\right) + \frac{6}{h}\left(F_{m-1} + F_m\right) \quad (42)$$

Thus for each point we have two difference equations. If we write them all together, we have the following "Fourth Order Compact Scheme".

$$y_0=\alpha$$
$$\frac{12}{h^2}y_0-\frac{12}{h^2}y_1+\frac{6}{h}F_0+\frac{6}{h}F_1=f\left(x_1,y_1,F_1\right)-f\left(x_0,y_0,F_0\right)$$
$$\frac{3}{h}y_{i-1}-\frac{3}{h}y_{i+1}+F_{i-1}+4F_i+F_{i+1}=0$$
$$\frac{2}{h^2}y_{i-1}-\frac{4}{h^2}y_i+\frac{2}{h^2}y_{i+1}+\frac{1}{2h}F_{i-1}-\frac{1}{2h}F_{i+1}=f\left(x_i,y_i,F_i\right)$$
$$y_m=\beta$$
$$\frac{12}{h^2}y_{m-1}-\frac{12}{h^2}y_m+\frac{6}{h}F_{m-1}+\frac{6}{h}F_m=f\left(x_m,y_m,F_m\right)-f\left(x_{m-1},y_{m-1},F_{m-1}\right)$$

5. RESULTS AND DISCUSSION

Test Problem-1:

Consider the two points 2nd order nonlinear BVP of the form

$$y''(x)=2y^3,\ 1\le x\le 5,\ y(1)=\frac{1}{4},\ y(5)=\frac{1}{8}$$

and its exact solution is $y(x)=\dfrac{1}{x+3}$.

Table 1. Comparison of results.

x	Exact values	FDM results	FOCM results	NLSM results
1.00000000	0.25000000	0.25000000	0.25000000	0.25000000
2.00000000	0.20000000	0.15976405	0.20019856	0.20002568
3.00000000	0.16666667	0.12060503	0.16719549	0.16668811
4.00000000	0.14285714	0.11056759	0.14333433	0.14286864
5.00000000	0.12500000	0.12500000	0.12500000	0.12500000

The above Table shows that results of NLSM are more accurate to the exact solution as compared with the results of FDM and FOCM.

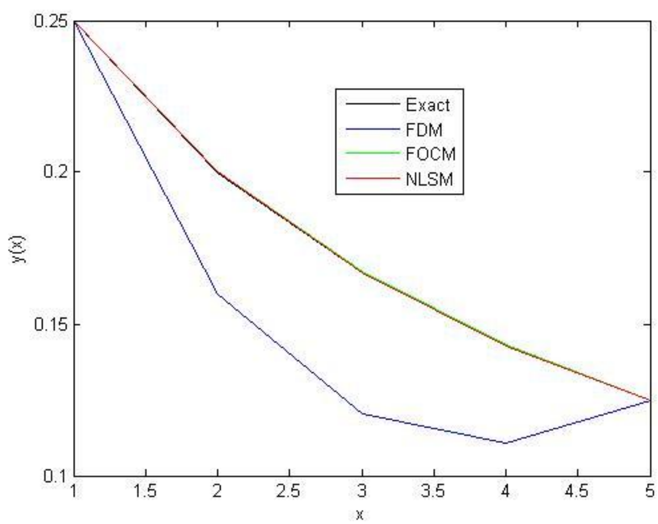

Fig. 1. Comparison of results.

Table 2. Comparison of absolute errors.

x	FDM results	FOCM results	NLSM results
1.00000000	0.00000000	0.00000000	0.00000000
2.00000000	0.04023595	0.00019856	0.00002568
3.00000000	0.04606164	0.00052882	0.00002144
4.00000000	0.03228955	0.00047719	0.00000115
5.00000000	0.00000000	0.00000000	0.00000000

The above table shows the comparison of absolute errors for FDM, FOCM and NLSM. It is found that an absolute error for NLSM is less as compared to the other two methods.

Test Problem-2:

Consider the two points 2nd order nonlinear BVP of the form $y''(x) = \frac{1}{2}y^3$, $1 \le x \le 2,\ y(1) = \frac{-2}{3},\ y(2) = -1$ and its actual solution is $y(x) = \frac{2}{x-4}$.

Table 3. Comparison of results.

x	Exact values	FDM results	FOCM results	NLSM results
1.00000000	-0.66666667	-0.66666667	-0.66666667	-0.66666667
1.25000000	-0.72727272	-0.78365013	-0.72732785	-0.72727455
1.50000000	-0.80000000	-0.88144644	-0.80010259	-0.80000299
1.75000000	-0.88888888	-0.95496986	-0.88999967	-0.88889178
2.00000000	-1.00000000	-1.00000000	-1.00000000	-1.00000000

The above table shows that results of NLSM are more accurate to the exact solution as compared with the results of FDM and FOCM.

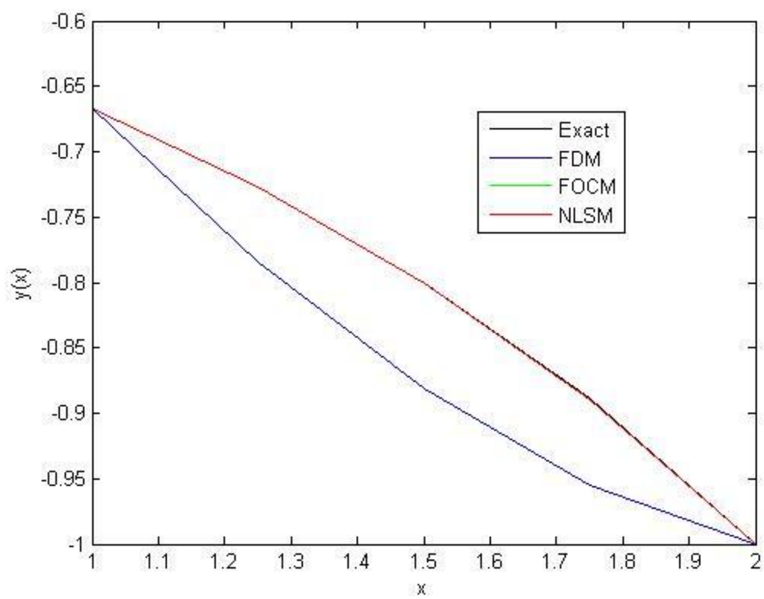

Fig. 2. Comparison of results.

Table 4. Comparison of absolute errors.

x	FDM results	FOCM results	NLSM results
1.00000000	0.00000000	0.00000000	0.00000000
1.25000000	0.05637741	0.00005513	0.00000183
1.50000000	0.08144644	0.00010259	0.00000299
1.75000000	0.06608098	0.00111079	0.00000029
2.00000000	0.00000000	0.00000000	0.00000000

The above table shows the comparison of absolute errors for FDM, FOCM and NLSM. It is found that an absolute error for NLSM is less as compared to the other two methods.

6. CONCLUSION

In this paper, we considered two different nonlinear 2^{nd} order two points BVPs for ordinary differential equations (ODEs) and solved these problems using NLSM, FDM and FOCM. All the above methods are appropriate for solving two points 2^{nd} order nonlinear BVPs, but it was found that NLSM is more accurate than FDM and FOCM.

7. REFERENCES

1. Roberts, S.M, & J.S. Shipman. *Two Point BVPs; Shooting Methods.* American Elsevier, New York, USA (1972).
2. Malathi, V. *Solving Boundary Value Problems for Ordinary Differential Equations Using Direct Integration and Shooting Techniques.* PhD thesis, Universiti Putra Malaysia, Malaysia (1999).
3. Ha, S.N. A nonlinear shooting method for two-point BVPs. *Computer and Mathematics Applications* 42: 1411-1420 (2001).
4. Auzinger. W., G. Kneisl, O. Koch, & E. Weinmuller. A collocation code for boundary value problems in ordinary differential equations. *Numerical Algorithms* 33: 27–39 (2003).
5. Attili, B.S. & M.I. Syam. Efficient shooting method for solving two point BVPs. *Chaos, Solitons and Fractals* 35: 895-903 (2008).
6. Faires, J.D. & R.L. Burden. *Numerical Analysis.* International Thomson Publishing Inc, New York, USA (1998).
7. Mustafa, H. *Fourth Order Compact Method for Nonlinear Second Order BVPs.* M.Phil thesis, University of Engineering and Technology, Lahore, Pakistan (2007).
8. Ahmad, M.O. An *Exploration of Compact Finite Difference-methods for Numerical Solution of PDE.* PhD thesis, University of Western, Ontario, Canada (1997).
9. Kreiss, H.O. *Methods for the Approximate Solution of Time Dependent Problems. GARP* Report No. 13 (1975).
10. Orszag, S.A. & M. Israeli. Numerical simulation of viscous incompressible flows. *Annual Reviews Inc* 6: 281-318 (1974).
11. Richard, S.H. & D.H. Rudy. The role of diagonal dominance and cell Reynolds number in implicit difference methods for fluid mechanics problems. *Journal of Computational Physics* 16: 304-310 (1974).
12. Leventhal, S.H. & M. Ciment. A note on the operator compact implicit method for the wave equation. *Mathematics of Computation* 32(1): 143-147 (1978).
13. Pettigrew, M.F. & H. Rasmussen. A compact method for second order boundary value on neon uniform gaids. *Computer and Mathematics Applications* 31: 1-16 (1996).
14. Schilling, R.J. & S.L. Harries. *Applied Numerical Methods for Engineers.* Thomson Business Information India, Bangalore, India (2006).
15. Sastry, S.S. *Introductory Methods of Numerical Analysis.* Prentice Hall of India, New Delhi, India (2003).