Proceedings of the Pakistan Academy of Sciences 52 (2): 135-144 (2015)
Copyright © Pakistan Academy of Sciences
ISSN: 0377 - 2969 (print), 2306 - 1448 (online)

Pakistan Academy of Sciences

Research Article

Numerical Solution for Hydromagnetic Viscous Fluids Flow Between Two Horizontal Stretching Sheets

S. Hussain[1, *], M. A. Kamal[2], and F. Ahmad[1]

[1]Department of Mathematics, College of Science Al Zulfi, Majmaah University, Kingdom of Saudi Arabia
[2] Department of Mathematics, Suleman Bin Abdul Aziz University, Al-Kharj, Kingdom of Saudi Arabia

Abstract: The numerical solution has been obtained for the hydromagnetic flow of an incompressible, steady and viscous fluid between two stretching sheets in the presence of variable surface temperature. The governing highly nonlinear partial differential equations are reduced to ordinary differential form by using similarity functions. The resulting equations are solved by using Successive Over Relaxation (SOR) method and Simpson's (1/3) rule. The results have been improved by Richardson extrapolation. The effects of flow parameters namely magnetic parameter M, the Reynolds number R, the suction parameter λ, Prandtl number P_r and temperature index parameter n have been observed on velocity and temperature distribution.

Keywords: Flow between plates, hydromagnetic fluid, Reynolds number, suction parameter, Richardson extrapolation

1. INTRODUCTION

Studies of fluids flow induced by stretching boundaries and heat transfer have relevance to some important applications such as extrusion process and production of plastic sheets. The quality of final product can be improved with proper adjustment of cooling and heat transfer. Hence, for slower rate of solidification, the application of magnetic field is useful because of its easy implementation. The effect of magnetic field in different engineering applications such as the cooling of reactors and many metallurgical processes involve the cooling of continuous tiles has been under more considerable attention. Ganji et al [1] reported the analytical solution of the magneto-hydrodynamic flow over a nonlinearly stretching sheet. Ahmad et al [2] obtained closed form solution for a viscous, incompressible, MHD flow over a porous stretching sheet. Baag et al [3] analyzed MHD flow on a stretching sheet embedded in a porous medium. The fluid flow problems about stretching surface have been studied extensively in various topics such as porous medium, MHD flows, heat transfer and non-Newtonian fluids.

Several engineering processes, such as materials manufactured by extrusion processes and heat-treated materials traveling between a feed roll and a wind-up roll on convey belts possess the characteristics of a moving continuous surface. Sakiadis [4, 5] examined the boundary layer flow on a continuously stretching surface with a constant speed. Erickson et al [6] extended the work of Sakiadis to include blowingorsuction at the stretched sheet surface on a continuous moving surface with constant speed and investigated its effects on the heat and mass transfer in the boundary layer. Crane [7] found an exact solution of two-dimensional Navier-Stokes equation for a stretching plate. Chiam [8] analyzed steady two dimensional oblique stagnation point flow of a viscous fluid. The magneto-hydrodynamic flow

Received, September 2013; Accepted, March 2015
*Corresponding author: S. Hussain; E-mail: sajjad_h96@yahoo.com

over a stretching surface has been studied extensively [9-12] for both permeable and impermeable surfaces. Flow of an electrically conducting non- Newtonian fluid past a stretching surface was studied by Able et al [13], when a uniform magnetic field acts transverse to the surface. Kumaran et al [14] obtained an exact solution for a boundary layer flow of an electrically conducting fluid past a quadratically stretching and linearly permeable sheet. Sheikholeslami et al [15] analyzed hydromagnetic flow between two horizontal plates in a rotating system, where the lower plate is a stretching sheet and the upper is a porous solid plate and studied heat transfer in an electrically conducting fluid bounded by two parallel plates is in the presence of viscous dissipation using Homotopy perturbation method. Pandya [16] considered hydro-magnetic flow between two horizontal plates in a rotating system where the lower plate is a stretching sheet and obtained numerical solution for the governing coupled ordinary differential equations by employing quantic spline collocation method. Chakrabarti and Gupta [17] studied the hydromagnetic flow and heat transfer in a fluid, initially at rest and at uniform temperature, over a stretching sheet at a different uniform temperature. Banerjee [18] studied the effect of rotation on the hydromagnetic flow between two parallel plates where the upper plate is porous and solid, and the lower plate is a stretching sheet. Arthur and Seini [19] investigated the hydromagnetic stagnation point flow of an incompressible viscous electrically conducting fluid towards a stretching sheet in the presence of radiation and viscous dissipation.

In this research paper, hydromagnetic fluid flow between two horizontal stretching plates has been examined. The non linear coupled equations governing the flow and heat transfer have been solved numerically by using a very simple and efficient numerical scheme. The effects of the physical parameters of the study namely *M*, *R*, λ, Pr and *n* have been noticed on velocity and temperature function.

2. MATHEMATICAL ANALYSIS

Consider steady, incompressible fluid flow between two horizontal parallel non-conducting plates, the lower one is a stretching sheet and the upper one is a porous stretching. Cartesian coordinate system is used where the y-axis is perpendicular to the plates located at y=*h*, y=-*h*. The fluid with constant velocity V_0 is injected through the upper porous plate. Two equal and opposite forces are introduced to stretch the lower and the upper plates in a way that the position of the points (0, *h*), (0, -*h*) remains fixed. A uniform magnetic field of strength B_0 is applied in the positive y-direction normal to the stretching sheet. The external electric field is zero and the electric field due to polarization of charges is negligible. The induced magnetic field is neglected which is valid for small magnetic Reynolds number. The temperature of the fluid at the surface of the lower sheet is $T_w(x) = T_\infty + cx^n$. The heat flux boundary conditions are used and viscous dissipation is neglected. Here *c* and *n* are constants with *c*>0 stands for heated surface and T_∞ is ambient fluid temperature. The geometry of the problem is depicted in Fig. 1.

Under the above assumptions the equations of motion become as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{\sigma B_0^2}{\rho}u \tag{2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{3}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial y^2}\right) \tag{4}$$

where u, v are velocity components and T is fluid temperature, ν and α are kinematic viscosity and thermal diffusivity, respectively and the relation for electromagnetic body force as given by Rossow [20] is $\underline{J} \times \underline{B} = \underline{E}\sigma(\underline{V} \times \underline{B}) \times \underline{B} = -\sigma B^2 \underline{V}$

The boundary conditions are:

$$u = cx, v = 0, \; -\frac{\partial T}{\partial y} = \frac{T_w - T_\infty}{h}$$

at y=-h , $c > 0$

$$u=cx,\; v=V_0 \;,\; T = T_\infty \qquad \text{at } y=h\,,\; c > 0 \tag{5}$$

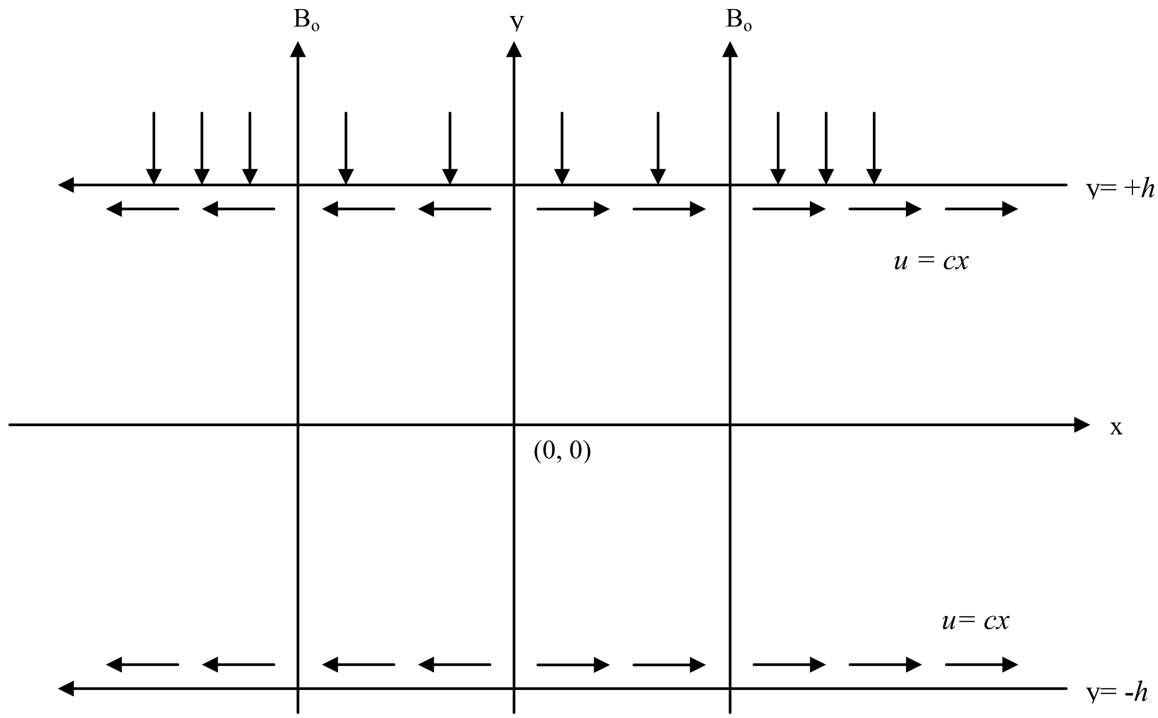

Fig. 1. Geometry of the problem.

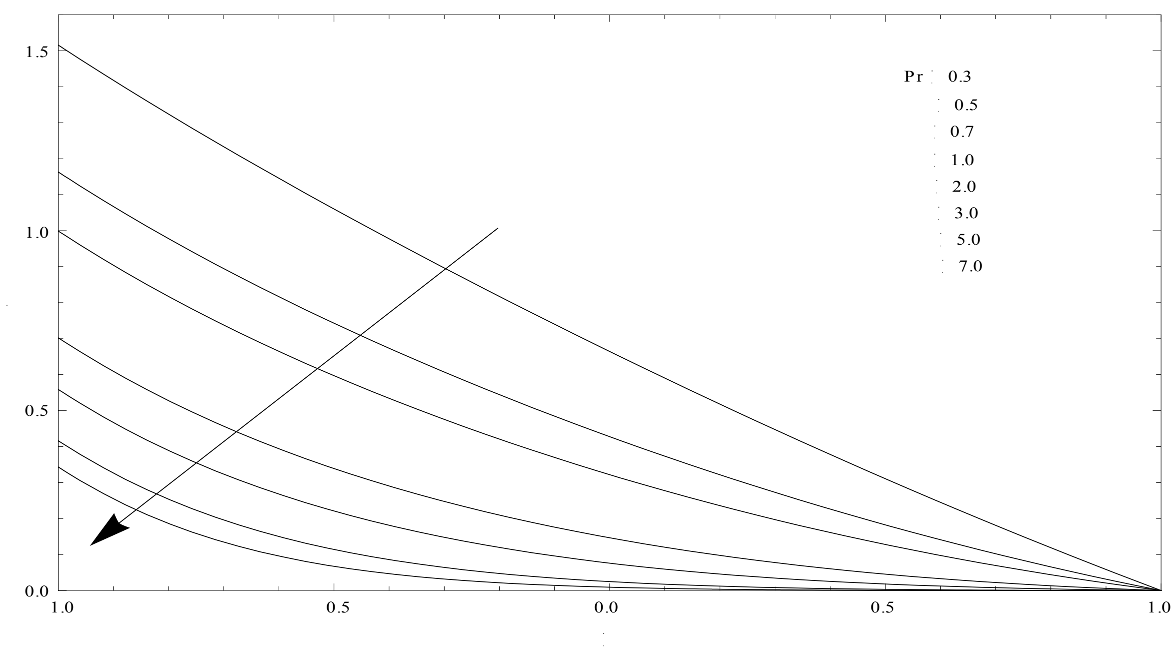

Fig. 2. Graph of θ for different values of Pr when M=1, R=0.05, λ=1, n=1.

Using similarity transformations:

$$u = cxf'(\eta), v = -chf(\eta), \eta = \frac{y}{h},$$

$$\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty} \quad (6)$$

where the primes denote differentiation with respect to η and $u = cx$ represents the velocity of both the plates. Equation of continuity (1) is identically satisfied.

Substituting (6) in to equations (2) and (3), we have

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = c^2 x(f'^2 - ff'' - \frac{1}{R}f''' + \frac{M^2}{R}f') \quad (7)$$

$$-\frac{1}{\rho h}\frac{\partial p}{\partial \eta} = c^2 h(ff' + \frac{1}{R}f'') \quad (8)$$

Now, differentiating equation (7) with respect to η and equation (8) with respect to *x*, we obtain

$$f^{iv} - R(f'f'' - ff''') - M^2 f'' = 0 \quad (9)$$

The integration of equation(9) yields,

$$f''' - R(f'^2 - ff'') - M^2 f' = A \quad (10)$$

Where A is the constant of integration and equation(4) becomes as:

$$\theta'' + \Pr(f\theta' - nf'\theta) = 0 \quad (11)$$

The corresponding boundary conditions are:

$$\begin{aligned} &f(-1) = 0, \quad f'(-1) = 1, \quad \theta'(-1) = -1 \\ &f(1) = \lambda, \quad f'(1) = 1, \quad \theta(1) = 0 \end{aligned} \quad (12)$$

3. COMPUTATIONAL PROCEDURE

For numerical purpose, we reduce the order of ODE (10) and let,

$$f' = g \quad (13)$$

The equation(10) becomes:

$$g'' - R(g^2 - fg') - M^2 g = A \quad (14)$$

Equation (14) and equation (11) are discretized by central difference approximation.

The resulting finite difference equations are solved by using SOR method, Smith [21] and the first order ordinary differential equation (13) is solved by Simpson's (1/3) rule, Gerald [22] with the formula given in Milne [23] subject to the appropriate boundary conditions.

The steps for the sequence of iterations are as follows:

1. The corresponding difference equations for the solution of g and θ are solved subject to the following boundary conditions:

   $f = 0, g = 1, \theta' = -1$ when $\eta = -1$

   $f = \lambda, g = 1, \theta = 0$ when $\eta = 1$.

2. For the solution of f, we use the computed values of g from above step in to equations (13) and integrate by Simpson's (1/3) rule.

3. The optimum value of the relaxation parameter ω_{opt} is estimated to accelerate the convergence of the SOR method.

4. The SOR procedure is terminated when the following criterion is satisfied for q:

$$\max_{i=1}^{q}\left|U_i^{q+1} - U_i^q\right| < 10^{-6},$$

Where q denotes the number of iterations and U stands for g and θ.

The above steps 1 to 4 are repeated for higher grid levels *h/2* and *h/4*. The SOR procedure gives the solution of $f' = g$ in the order of accuracy $O(h^2)$ due to second order finite differences used for derivatives and Simpson's (1/3) rule gives the order of accuracy $O(h^5)$ in the solution of *f*. Higher order accuracy $O(h^6)$ in the solution of $f' =$ g is achieved by using Richardson's extrapolation [24].

4. RESULTS AND DISCUSSION

The nonlinear ordinary differential equations (11) and (14) are solved numerically subject to the appropriate boundary conditions. Our numerical scheme is straight forward and very efficient as

Table 1. Results show the effects of the parameters *R*, *M* and λ on f' at η=0.

R	*M*	λ	*A*	f'
0.05	1.0	1.0	1.093626	0.261939
0.05	3.0	1.0	-2.267975	0.326008
0.20	3.0	3.0	-16.3264	1.675328
0.40	2.0	1.0	-0.16873	0.291429
0.40	4.0	1.0	-5.36558	0.357061
0.40	2.0	3.0	-9.081195	1.709340
0.80	2.0	3.0	-10.33248	1.707032
0.80	4.0	3.0	-29.01695	1.647703

Table 2. Numerical results using Richardson Extrapolation Method.

R=0.25, *M* =1, λ =1, *Pr*=0.7, *n*=1					**R=0.05, *M* =4, λ =3, *Pr*=0.7, *n*=1**				
***h*=0.2**	***h*=0.1**	***h*=0.05**	**Extrapolated**		***h*=0.2**	***h*=0.1**	***h*=0.05**	**Extrapolated**	
η	f'	f'	f'	f'	η	f'	f'	f'	f'
0.000	1.000000	1.000000	1.000000	1.000000	0.000	1.000000	1.000000	1.000000	1.000000
0.200	0.875013	0.874960	0.874946	0.874941	0.200	1.362455	1.366961	1.368149	1.368550
0.400	0.767668	0.767608	0.767591	0.767586	0.400	1.527394	1.531557	1.532644	1.533010
0.600	0.672598	0.672559	0.672548	0.672544	0.600	1.601332	1.604336	1.605114	1.605376
0.800	0.585113	0.585109	0.585107	0.585105	0.800	1.632242	1.634416	1.634972	1.635159
1.000	0.501016	0.501052	0.501059	0.501061	1.000	1.640323	1.642228	1.642715	1.642878
1.200	0.416443	0.416513	0.416529	0.416534	1.200	1.631037	1.633260	1.633830	1.634021
1.400	0.327717	0.327809	0.327831	0.327838	1.400	1.598749	1.601842	1.602642	1.602911
1.600	0.231219	0.231311	0.231334	0.231341	1.600	1.523363	1.527618	1.528729	1.529104
1.800	0.123266	0.123331	0.123346	0.123352	1.800	1.357970	1.362519	1.363719	1.364125
2.000	0.000000	0.000000	0.000000	0.000000	2.000	1.000000	1.000000	1.000000	1.000000

Table 3. Numerical results using Richardson Extrapolation Method.

R=0.05, *M* =3, λ =3, *Pr*=0.7, *n*=1					**R=0.2, *M*=3, λ =3, *Pr*=0.7, *n*=1**				
***h*=0.2**	***h*=0.1**	***h*=0.05**	**Extrapolated**		***h*=0.2**	***h*=0.1**	***h*=0.05**	**Extrapolated**	
η	f'	f'	f'	f'	η	f'	f'	f'	f'
0.000	1.000000	1.000000	1.000000	1.000000	0.000	1.000000	1.000000	1.000000	1.000000
0.200	1.334084	1.336862	1.337578	1.337819	0.200	1.341450	1.344216	1.344923	1.345160
0.400	1.516063	1.519251	1.520069	1.520343	0.400	1.524671	1.527783	1.528582	1.528850
0.600	1.612367	1.615250	1.615985	1.616231	0.600	1.619392	1.622161	1.622871	1.623109
0.800	1.658401	1.660943	1.661589	1.661805	0.800	1.662694	1.665125	1.665747	1.665955
1.000	1.671292	1.673711	1.674324	1.674529	1.000	1.672386	1.674730	1.675328	1.675527
1.200	1.656125	1.658697	1.659351	1.659570	1.200	1.653726	1.656269	1.656920	1.657138
1.400	1.607904	1.610832	1.611580	1.611831	1.400	1.601822	1.604762	1.605511	1.605761
1.600	1.509951	1.513178	1.514006	1.514284	1.600	1.500667	1.503903	1.504732	1.505011
1.800	1.328270	1.331056	1.331774	1.332016	1.800	1.318725	1.321479	1.322189	1.322428
2.000	1.000000	1.000000	1.000000	1.000000	2.000	1.000000	1.000000	1.000000	1.000000

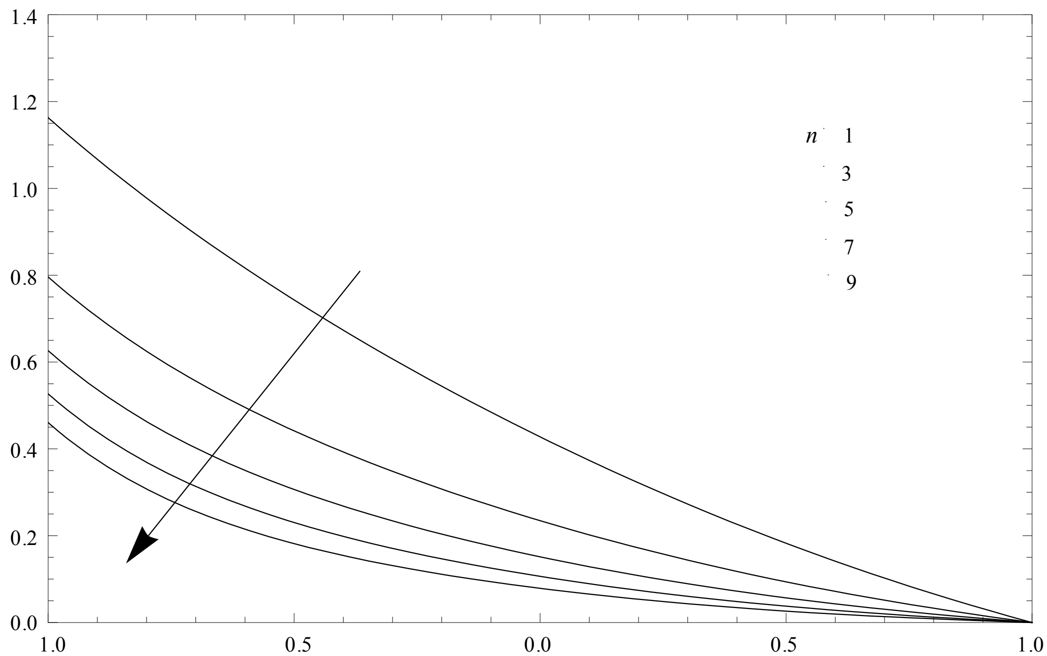

Fig. 3. Graph of θ for different values of n when M=1, R=0.05, λ=1, Pr=0.7.

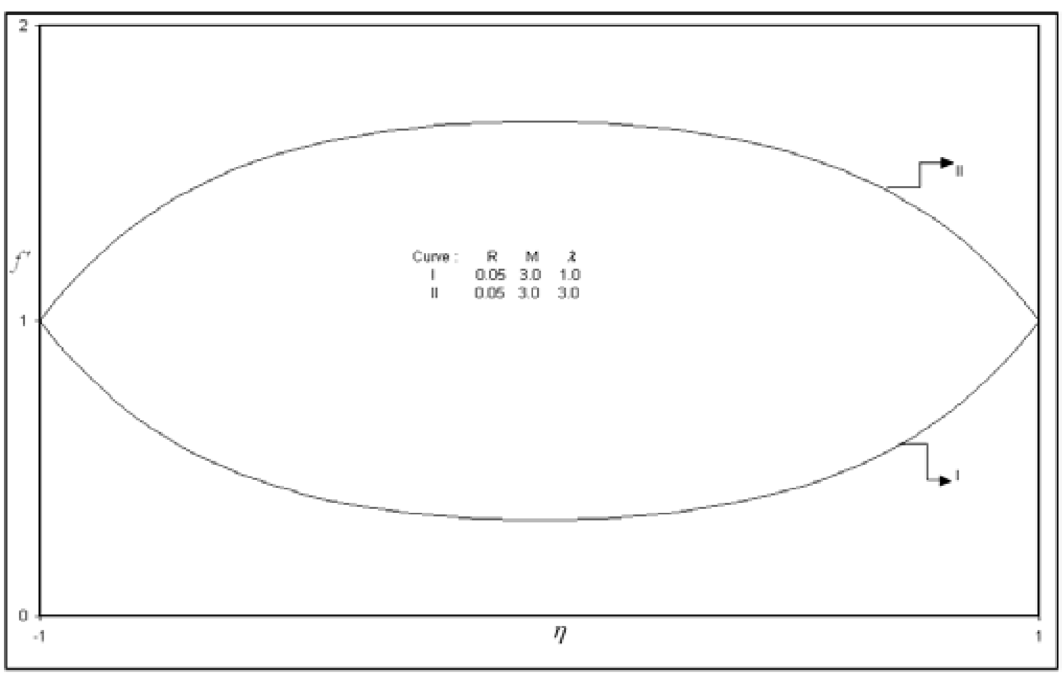

Fig. 4. Graph of f' when both the plates are stretching sheets.

Table 4. Optimum value of relaxation parameter in SOR method and number of iterations when both the plates are being stretching sheets.

M	λ	R	NI is Number of Iterations in SOR method with ω_{opt}					
			h=0.1		h=0.05		h=0.25	
			NI	ω_{opt}	NI	ω_{opt}	NI	ω_{opt}
1.0	1.0	0.05	41	1.65	74	1.67	235	1.65
3.0	3.0	0.05	87	1.60	82	1.65	102	1.70
2.0	3.0	0.05	37	1.65	173	1.70	397	1.75
4.0	1.0	0.40	46	1.55	38	1.66	52	1.60
2.0	3.0	0.80	52	1.70	54	1.70	877	1.72
4.0	3.0	0.80	67	1.75	61	1.70	65	1.75

compared to the numerical methods used in previous studies [15, 16]. The results have been computed for different values of the parameters M, R, λ, Pr and n in order to observe their effect on velocity and temperature function. The results are presented in tabular and graphical forms. Table 1 presents the values of the unknown constant A are chosen by hit and trial technique to satisfy the boundary conditions. The results for f' in the higher order accuracy $O(h^6)$ are given in Table 2 and Table 3. Table 4 shows that higher order accuracy solution is obtained with very small number of iterations by choosing ω_{opt}.

Fig. 2 and Fig. 3 depict the curves of heat distribution under the effect of Prandtl number P_r and heat index parameter n, respectively. The non-dimensional temperature function θ decreases with increase in the values of P_r and n.

Fig. 4 and Fig. 5 demonstrate the behavior of horizontal velocity component f', when both the plates are stretched. Fig. 6 depicts f', when the lower plate stretches. It is noticed that f' attains maximum value in the midway of the channel. When the two sheets are stretching there is almost symmetry of the curves of f' about the line $(\eta = 0)$. The magnetic parameter M decreases f' around the lower sheet and increases it around the upper sheet for small $\lambda = 1$.

The increase in Reynold number R (small R<1) decreases f' in the upper half of the channel and increases it in the lower half of the channel. The suction parameter λ has increasing effect on f'. This effect is maximum at the mid of the channel. Fig. 7 shows that the normal velocity component f increases with the increase in the values of λ when M is constant.

5. CONCLUSIONS

Numerical solution has been sought for hydromagnetic fluid flow between two horizontal stretching plates with heat flux. Results have been computed to notice the effects of the physical parameters (M, R, λ, P_r and n) on velocity and temperature distributions. The main results of this study are summarized as follows:

- The horizontal velocity f' attains maximum value in the midway of the channel.
- There is almost asymmetry of the curves of f' about the line $(\eta = 0)$, when the two sheets are stretching.
- The magnetic parameter M decreases f' around the lower sheet and increases it around the upper sheet for small $\lambda = 1$.
- The increase in Reynolds number R (small R<1) decreases f' in the upper half of the channel and increases it in the lower half of the channel.
- The suction parameter λ has increasing effect on f'. This effect is maximum at the mid of the channel.
- The normal velocity component f increases with the increase in the values of λ when M is constant.

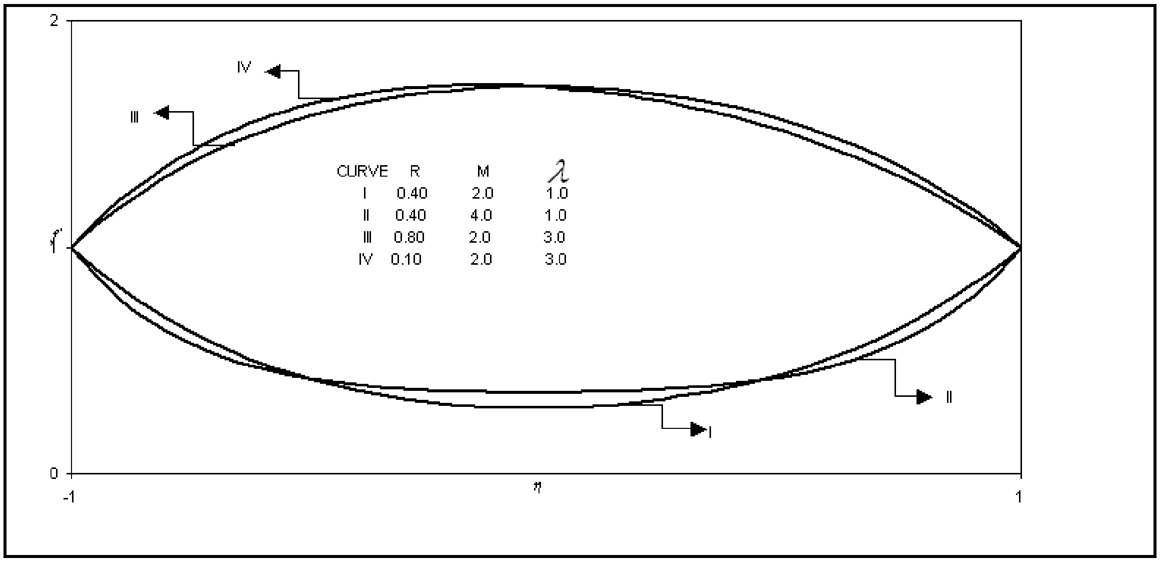

Fig. 5. Graph of f' when both the plates are stretching sheets.

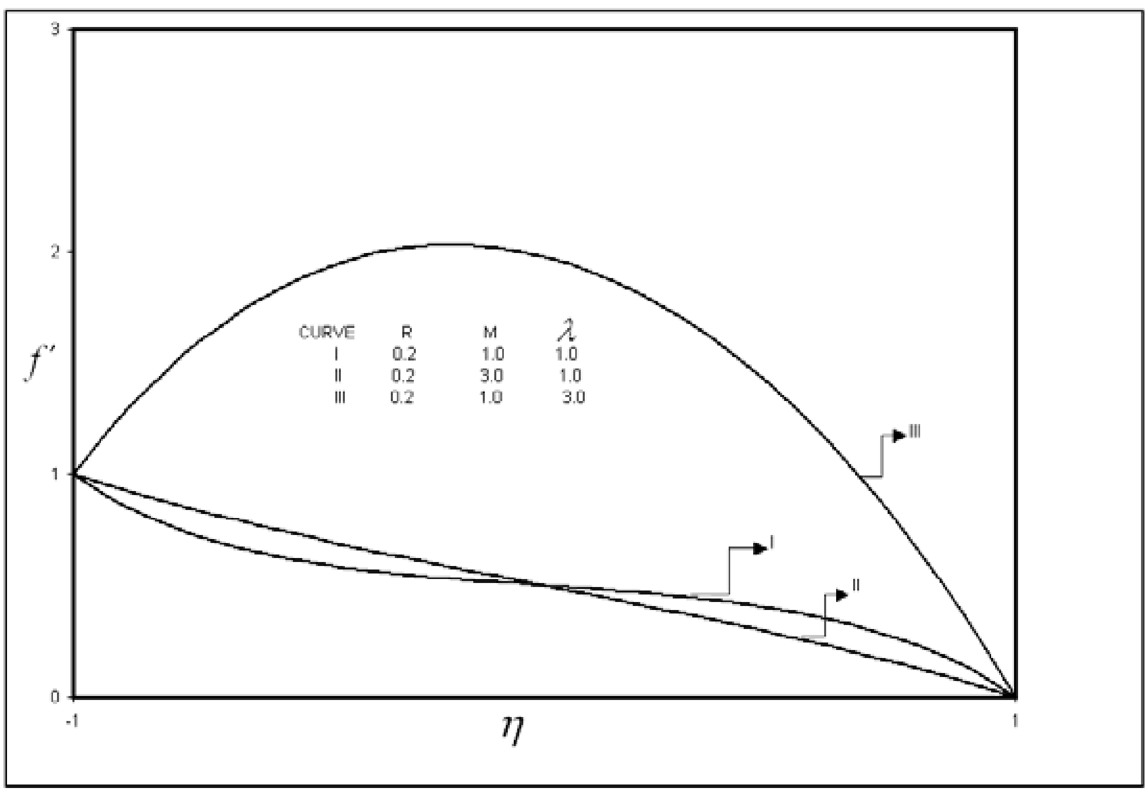

Fig. 6. Graph of f' when lower Plate being a stretching sheet.

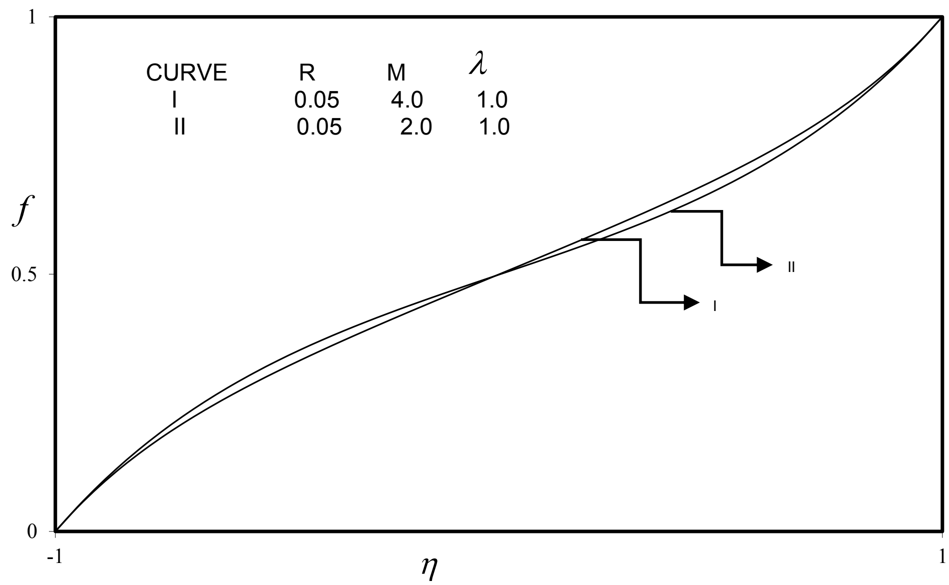

Fig. 7. Graph for different values of M, R and λ .

Nomenclature

$M^2 = \frac{\sigma}{\rho\nu} B_0^2 h^2$: Magnetic parameter

$\lambda = \frac{V_0}{ch}$: Suction parameter

$R = \frac{ch^2}{\nu}$: Rynold's number:

Pr: Prandtl number

B_0 : Strength of transverse magnetic field

σ : Electrical conductivity

ν : Coefficient of kinematic viscosity

c : Positive constant

ρ : Fluid density

V_0: Injection velocity

u, v : Velocity components along x-,y- axes, respectively

η : Dimensionless variable

- The temperature function θ decreases with increase in the values of Prandtl number P_r and heat index parameter n.

6. REFERENCES

1. Ganji, D.D., H. Bararnia, S. Soleimani & E. Ghasemi. Analytical solution of the magneto-hydrodynamic flow over a nonlinear stretching sheet. *Modern Physics Letters B* 23: 2541–2556 (2009).
2. Farooq, A., S. Hussain & M. Ali. An analytical solution of MHD flow over porous stretching sheet. *Journal of Basic and Applied Scientific Research* 4(3):160-167 (2014).
.3. Baag, S., M.R. Acharya & G.C. Dash. MHD Flow analysis using DTM-Pade' and numerical methods. *American Journal of Fluid Dynamics* 4(1): 6-15 (2014).
4. Sakiadas, B.C. Boundary layer behavior on continuous solid surface: I. Boundary-layer equations for two-dimensional and axisymmetric flow. *Journal AlChe* 7: 221-225 (1961).
5. Sakiadas, B.C. Boundary layer behavior on continuous solid surface: II. Boundary-layer equations for two-dimensional and axisymmetric flow. *Journal AlChe* 7: 226-228 (1961).
6. Erickson , L.E., L.T. Fan & V.G. Fox. Heat and mass transfer on a moving continuous flat plate with suction or injection. *Industrial and Engineering Chemistry Fundamentals* 5(1): 19–25 (1966).
7. Crane, I.J. Flow past a stretching plate. *Zeitschrift für angewandte Mathematik und Physik (Journal of Applied Mathematics and Physics)* 21: 645-647 (1970).
8. Chiam, T.C. Stagnation-point flow towards a stretching plate. *Journal of the Physical Society of Japan* 63: 2443-2444 (1994).
9. Chakrabarti, A. & A.S. Gupta. Hyderomagnetic flow and heat transfer over a stretching sheet. *The Quarterly of Applied Mathematics* 37: 73-78 (1979).
10. Andersson, H.I. An exact solution of Navier-Stokes equations for magneto-hydrodynamic flow. *Acta Mechnica* 113: 241-244 (1995).
11. Pop, I. & T.Y. Na. A note on MHD flow over a stretching permeable surface. *Mechanical Research Communication* 25(3): 263-269 (1998).
12. Liao, S.J. On the analytic solution of magnetohyderodynamic flows of non-Newtonian fluids over a stretching sheet. *Journal of Fluid Mechanics* 488: 189-212 (2003).
13. Able, S., P.H. Veena, K. Rajagopal & V.K. Pravin. Non-Newtonian magnetohydrodynamic flow over a stretching surface with heat and mass transfer. *International Journal of Nonlinear Mechanics* 39: 1067-1078 (2004).
14. Kumaran, V., A.K. Banerjee, A.V. Kumar & K. Vajravelu. MHD flow past a stretching permeable sheet. *Applied Mathematics and Computation* 210:26-32 (2009).
15. Sheikholeslami, M., H.R. Ashorynejad, D.D. Ganji & A. Kolahdooz. Investigation of Rotating MHD viscous flow and heat transfer between stretching and porous surfaces using analytical method. *Mathematical Problems in Engineering* 2011, doi:10.1155/2011/258734 (2011).
16. Pandya, J.U. The solution of a coupled nonlinear system arising in a three-dimensional rotating flow using spline method. *International Journal of Mathematics and Mathematical Sciences* 2012. doi: 10.1155/2012/702458.
17. Chakrabarti & A.S. Gupta. Hydromagnetic flow and heat transfer over a stretching sheet.*Quarterly of Applied Mathematics* 37(1): 73–78 (1979).
18. Banerjee, B. Magneto-hydrodynamic flow between two horizontal plates in a rotating system, the lower plate being a stretched sheet. *Transactions ASME Journal of Applied Mechanics* 50: 470 (1983).
19. Arthur, E.M. & I.Y. Seini. Hydromagnetic stagnation point flow over a porous stretching surface in the presence of radiation and viscous dissipation. *Applied and Computational Mathematics* 3(5): 191-196 (2014).
20. Rossow, V.J. On Flow of Electrically *Conducting Fluids Over a Flat Plate in the Presence of a Transverse Magnetic Field*.Tech Report 1358: NASA (1958).
21. Smith, G.D. *Numerical Solution of Partial Differential Equation*. Clarendon Press, Oxford (1979).
22. Gerald, C.F. *Applied Numerical Analysis*. Addison-Wesley Pub, NY (1989).
23. Milne, W.E. *Numerical Solution of Differential Equation*. Dover Pub (1970).
24. Burden, R.L. *Numerical Analysis*. Prindle, Weber & Schmidt, Boston (1985).