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Research Article

On the Existence of Periodic Solutions to Non-Linear Neutral Differential Equations of First Order with Multiple Delays

Emel Biçer[1], and Cemil Tunç[2]*

[1]Department of Mathematics, Faculty of Arts and Sciences,
Bingöl University, 12000, Bingöl, -Turkey
[2]Department of Mathematics,Faculty of Sciences,
Yüzüncü Yıl University, 65080, Van, –Turkey

Abstract: In this paper, on the basis of Krasnoselskii's fixed point theorem, sufficient conditions are derived for the existence of periodic solutions to certain nonlinear neutral differential equations of first order with multiple retarded arguments. Our results extend and improve some related ones in the literature.

Keywords: Existence of periodic solution, fixed point, neutral differential equation, first order.

2010 Mathematics Subject Classifications: 34K15, 34K40

1. INTRODUCTION

In the recent years, there has been a noticeable interest in the study of neutral functional differential equations of first order with delay due to their importance of applications in applied mathematics (see, for example Adıvar and Raffoul [1], Ardjouni and Djoudi [2, 3], Burton [4, 5, 6], Kaufmann [7], Kaufmann and Raffoul [8], Raffoul [9, 10, 11], Yankson [12] and the references cited in these sources). To the best of our knowledge from the literature, although there are many works concerned with the existence of periodic solutions for various neutral differential equations of first order, no works have been done to investigate the existence of periodic solutions of nonlinear neutral differential equations of first order with multiple variable delays. Therefore, it is worth to work on the existence of periodic solutions of neutral differential equations of first order with multiple variable delays.

We begin with summarizing a few relative results done in the literature on the existence of periodic solutions for neutral differential equations of first order.

In 2003, Raffoul [9] considered the first order nonlinear neutral differential equation with functional delay.

$$x'(t) = -a(t)x(t) + c(t)x'(t-g(t)) + q(t, x(t), x(t-g(t))).$$

Raffoul [9] benefited from Krasnoselskii's fixed point theorem and established sufficient conditions, which guarantee that this nonlinear neutral differential equation with functional delay has a periodic solution.

Later, in 2010, Ardjouni and Djoudi [2] concerned with the existence of periodic solutions for a nonlinear dynamic equation on a time scale T with functional delay $\sigma(t)$ of the form

$$x^{\Delta}(t) = -a(t)x^3(\sigma(t)) + G(t, x^3(t), x^3(t-r(t))), \quad t \in T$$

Ardjouni and Djoudi [2] constructed a suitable Banach space and a bounded convex subset, then convert the existence of periodic solutions to a fixed point problem for a map, that is ,the sum of a compact map and a large contraction, and the authors used a modification of Krasnoselskii's fixed point theorem due to Burton ([4], [6, Theorem 3]) to show the existence of a periodic solution of this equation.

More recently, Yankson [12] used a variant of

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* Corresponding author: Cemil Tunç; E-mail: cemtunc@yahoo.com

Krasnoselskii's fixed point theorem by Burton [6, Theorem 3] to show the existence of periodic solutions for the totally nonlinear neutral differential equation of first order with functional delay

$$x'(t) = -a(t)h(x((t))) + c(t)x'(t-g(t)) + q(t, x(t), x(t-g(t))).$$

Motivated by the above discussion, the aim of this paper is to give some new sufficient conditions which guarantee the existence of periodic solutions for the following nonlinear neutral differential equation of first order with two variable delays

$$x'(t) = -a(t)x((t)) + \sum_{j=1}^{2} b_j(t) f(t, x(t), x(t-\tau_j(t))) + \sum_{j=1}^{2} c_j(t)x'(t-\tau_j(t))g'(x(t-\tau_j(t))), \tag{1}$$

where $a(t)$, f and g' are continuous functions, $b_j(t)$ and $c_j(t)$ are continuously differentiable functions, and $\tau_j(t)\,(\geq 0)$, are twice continuously differentiable functions for $t \in \Re^+$, $\Re^+ = [0, \infty)$. We benefit from the fixed point theorem of Krasnoselskii's to prove the existence of periodic solutions of equation (1). First, we transform equation (1) into an integral equation written as a sum of the two mappings; one of them is compact and the other is contraction. Later, we use the Krasnoselskii's fixed point theorem to prove the existence of periodic solutions of equation (1).

It is clear that equation (1) includes and improves the equation discussed by Raffoul [9]. Further, when $h(x(t)) = x(t)$ in Yankson [12], then equation (1) also includes and improves the equation by Yankson [12].

2. EXISTENCE OF PERIODIC SOLUTIONS

Let

$$C_T = \{\varphi : \varphi \in C(\Re, \Re) \text{ and } \varphi(t+T) = \varphi(t)\}$$

for $T > 0$. Here C denotes the set of all real valued continuous functions. Then C_T is a Banach space when it is endowed with the supremum norm

$$\|x\| = \max_{t\in[0,T]} |x(t)|.$$

Throughout this paper, we assume that

$(C1)$ $a(t+T) = a(t)$, $b_j(t+T) = b_j(t)$, $c_j(t+T) = c_j(t)$,

$\tau_j(t+T) = \tau_j(t)$, $\tau_j(t) \geq \tau_j^* > 0$,

$\int_0^T a(s)ds > 0$, $(j = 1, 2)$,

$(C2)$ $f(t, x, y)$ is continuous, periodic in t and Lipschitz continuous in x and y, $g(x)$ is continuous and Lipschitz continuous in x. That is,

$$f(t+T, x, y) = f(t, x, y),$$

and for some positive constants k_1, k_2, k_3, we have

$$|g(x) - g(y)| \leq k_1 \|x - y\|,$$
$$|f(t, x, y) - f(t, z, w)| \leq k_2 \|x - z\| + k_3 \|y - w\|,$$

$(C3)$ $\tau_j'(t) \neq 1$ for all $t \in [0, T]$.

Lemma 1. Assume that conditions $(C1)-(C3)$ hold. If $x(t) \in C_T$, then $x(t)$ is solution of equation (1) if and only if

$$x(t) = \sum_{j=1}^{2} \frac{c_j(t)}{1-\tau_j'(t)} g(x(t-\tau_j(t))) + (1 - e^{-\int_{t-T}^{t} a(s)ds})^{-1} \times \int_{t-T}^{t} [\sum_{j=1}^{2} b_j(u) f(u, x(u), x(u-\tau_j(u))) - \sum_{j=1}^{2} r_j(u) g(x(u-\tau_j(u)))] e^{-\int_u^t a(s)ds} du, \tag{2}$$

where

$$r_j(u) = \frac{(c_j'(u) + a(u)c_j(u)(1-\tau_j'(u)) + c_j(u)\tau_j''(u)}{(1-\tau_j'(u))^2}. \tag{3}$$

Proof. Let $x(t) \in C_T$ be a solution of (1). Multiplying both sides of equation (1) by $e^{\int_0^t a(s)ds}$ and then integrating from $t-T$ to t, it follows that

$$\int_{t-T}^{t}\left[x(u)\exp\left(\int_{0}^{u}a(s)ds\right)\right]' du =$$

$$\int_{t-T}^{t}\sum_{j=1}^{2}b_j(u)f(u,x(u),x(u-\tau_j(u)))e^{\int_0^u a(s)ds}\,du$$

$$+\int_{t-T}^{t}\sum_{j=1}^{2}c_j(u)x'(u-\tau_j(u))g'(x(u-\tau_j(u)))e^{\int_0^u a(s)ds}\,du.$$

Then, we have

$$x(t)e^{\int_0^t a(s)ds}-x(t-T)e^{\int_0^{t-T} a(s)ds}=$$

$$\int_{t-T}^{t}\sum_{j=1}^{2}b_j(u)f(u,x(u),x(u-\tau_j(u)))e^{\int_0^u a(s)ds}\,du$$

$$+\int_{t-T}^{t}\sum_{j=1}^{2}c_j(u)x'(u-\tau_j(u))g'(x(u-\tau_j(u)))e^{\int_0^u a(s)ds}\,du.$$

By dividing both sides of the last estimate by $\exp(\int_0^t a(s)ds)$ and using the estimate $x(t) = x(t-T)$, we get

$$x(t)=[1-\exp(\int_{t-T}^{t}a(s)ds)]^{-1}\times$$

$$[\int_{t-T}^{t}\{\sum_{j=1}^{2}b_j(u)f(u,x(u),x(u-\tau_j(u)))$$

$$+\sum_{j=1}^{2}c_j(u)x'(u-\tau_j(u))g'(x(u-\tau_j(u)))]\}e^{-\int_u^t a(s)ds}\,du. \quad (4)$$

Rewriting the last term and applying integration by parts, it follows that

$$\int_{t-T}^{t}\sum_{j=1}^{2}c_j(u)x'(u-\tau_j(u))g'(x(u-\tau_j(u)))e^{-\int_u^t a(s)ds}\,du$$

$$=\int_{t-T}^{t}\sum_{j=1}^{2}\frac{c_j(u)}{1-\tau_j'(u)}g'(x(u-\tau_j(u)))x'$$

$$(u-\tau_j(u))(1-\tau_j'(u))e^{-\int_u^t a(s)ds}\,du$$

$$=\sum_{j=1}^{2}\frac{c_j(u)}{1-\tau_j'(u)}g(x(u-\tau_j(u)))e^{-\int_u^t a(s)ds}\Big|_{t-T}^{t}$$

$$-\int_{t-T}^{t}\sum_{j=1}^{2}r_j(u)g(x(u-\tau_j(u)))e^{-\int_u^t a(s)ds}\,du$$

$$=\sum_{j=1}^{2}\frac{c_j(u)}{1-\tau_j'(u)}g(x(u-\tau_j(u)))$$

$$(1-\exp(\int_{t-T}^{t}a(s)ds))$$

$$-\int_{t-T}^{t}\sum_{j=1}^{2}r_j(u)g(x(u-\tau_j(u)))e^{-\int_u^t a(s)ds}\,du, \quad (5)$$

where $r_j(u)$ is given by (3). Hence, substituting estimate (5) into (4), we obtain (2).

Conversely, let

$$x(t)=\sum_{j=1}^{2}\frac{c_j(t)}{1-\tau_j'(t)}g(x(t-\tau_j(t)))+$$

$$(1-e^{-\int_{t-T}^{t}a(s)ds})^{-1}$$

$$\times\int_{t-T}^{t}[\sum_{j=1}^{2}b_j(u)f(u,x(u),x(u-\tau_j(u)))$$

$$-\sum_{j=1}^{2}r_j(u)g(x(u-\tau_j(u)))]e^{-\int_u^t a(s)ds}\,du.$$

It is clear that, $x(t)$ is a solution of equation (1). This completes the proof.

We now give Krasnoselskii's fixed point theorem to prove the existence of a periodic solution for equation (1).

Theorem A (Krasnoselskii). Let M be a closed convex nonempty subset of a Banach space $(B,\|\,.\,\|)$. Suppose that A and B map M into B such that

$(K1)$ $x, y \in M$ implies $Ax+By\in M$,

$(K2)$ A is compact and continuous,

$(K3)$ B is a contraction mapping.

Then there exists a $z\in M$ with $z = Az + Bz.$

To apply Theorem A, we define a bounded convex subset of M of C_T, $M=\{\varphi\in C_T:\|\varphi\|\le L\}$, where L is a positive constant, and the mapping $S:C_T\to C_T$ by

$$(S\varphi)(t)=\sum_{j=1}^{2}\frac{c_j(t)}{1-\tau_j'(t)}g(\varphi(t-\tau_j(t)))+$$

$$(1-e^{-\int_{t-T}^{t}a(s)ds})^{-1}$$

$$\times \int_{t-T}^{t} [\sum_{j=1}^{2} b_j(u) f(u, \varphi(u), \varphi(u - \tau_j(u)))$$

$$- \sum_{j=1}^{2} r_j(u) g(\varphi(u - \tau_j(u)))] e^{-\int_u^t a(s)ds} du. \quad (6)$$

To apply Krasnoselskii's theorem we need to construct two mappings, one is a contraction and the other is compact. Therefore, we state (6) as

$$(S\varphi)(t) = (B\varphi)(t) + (A\varphi)(t),$$

where $A, B : C_T \to C_T$ are given by

$$(A\varphi)(t) = (1 - e^{-\int_{t-T}^{t} a(s)ds})^{-1}$$

$$\times \int_{t-T}^{t} [\sum_{j=1}^{2} b_j(u) f(u, \varphi(u), \varphi(u - \tau_j(u)))$$

$$- \sum_{j=1}^{2} r_j(u) g(\varphi(u - \tau_j(u)))] e^{-\int_u^t a(s)ds} du \quad (7)$$

and

$$(B\varphi)(t) = \sum_{j=1}^{2} \frac{c_j(t)}{1 - \tau_j'(t)} g(\varphi(t - \tau_j(t))). \quad (8)$$

Lemma 2. If $(C1)-(C3)$ hold, then $A : M \to C_T$, as defined by (7), is continuous and compact.

Proof. A change of variable in (7) shows that $(A\varphi)(t+T) = (A\varphi)(t)$. To show that A is continuous, we assume $\varphi, \psi \in M$ with $\|\varphi\| \le C$ and $\|\psi\| \le C$. Let

$$\sigma = \max_{t\in[0.T]} \left| (1 - \exp(-\int_{t-T}^{t} a(s)ds)^{-1} \right|,$$

$$\delta = \max_{t\in[0,T]} \sum_{j=1}^{2} |b_j(t)|, \quad \rho_1 = |g(0)|,$$

$$\gamma = \max_{u\in[t-T,t]} e^{-\int_u^t a(s)ds}, \quad \mu = \max_{t\in[0,T]} \sum_{j=1}^{2} |r_j(t)|,$$

$$\rho_2 = \max_{t\in[0,T]} |f(t,0,0)|, \quad (9)$$

$$|g(x)| \le k_1|x| + |g(0)|$$

and

$$|f(t,x,y)| \le |f(t,x,y) - f(t,0,0)| + |f(t,0,0)|$$

$$\le k_2|x| + k_3|y| + |f(t,0,0)|.$$

For the constants α, β with $\alpha + \beta < 1$, we suppose the following assumptions:

$$\sum_{j=1}^{2} |b_j(t)| [(k_2 + k_3)L + |q(t,0,0,0)|] \le \alpha L a(t),$$

$$\sum_{j=1}^{2} |r_j(t)| [k_1 L + |g(0)|] \le \beta L a(t). \quad (10)$$

For any $\phi \in C_T$, we will show that $|(A\phi)(t)| \le L$. In view of the above estimates, we have

$$|(A\varphi)(t)| = \left| (1 - e^{-\int_{t-T}^{t} a(s)ds})^{-1} \times \int_{t-T}^{t} \begin{matrix} [\sum_{j=1}^{2} b_j(u) f(u, \phi(u), \phi(u - \tau_j(u))) \\ -\sum_{j=1}^{2} r_j(u) g(\phi(u - \tau_j(u)))] e^{-\int_u^t a(s)ds} \end{matrix} du \right|$$

$$\le (1 - e^{-\int_{t-T}^{t} a(s)ds})^{-1}$$

$$\times \int_{t-T}^{t} [\sum_{j=1}^{2} |b_j(u)| \; |f(u, \phi(u), \phi(u - \tau_j(u)))|$$

$$+ \sum_{j=1}^{2} |r_j(u)| |g(\phi(u - \tau_j(u)))|] e^{-\int_u^t a(s)ds} du$$

$$\le (1 - e^{-\int_{t-T}^{t} a(s)ds})^{-1}$$

$$\times \int_{t-T}^{t} [\sum_{j=1}^{2} |b_j(u)| \; ((k_2 + k_3)L + |f(u,0,0)|$$

$$+ \sum_{j=1}^{2} |r_j(u)| \; (k_1 L + |g(0)|)] e^{-\int_u^t a(s)ds} du$$

$$\le (1 - e^{-\int_{t-T}^{t} a(s)ds})^{-1} \times (\alpha + \beta) L \int_{t-T}^{t} a(u) e^{-\int_u^t a(s)ds} du$$

$$\le (\alpha + \beta) L < L.$$

Hence, we have $A\varphi \in M$.

We will now show that A is continuous in the supremum norm.

Given $\varepsilon > 0,$ take $\delta = \frac{\varepsilon}{K}$ such that $K = T\sigma\gamma[\delta(k_2 + k_3) + \mu k_1].$ In view of $(C2)$ and the last estimate, then, for $\varphi, \psi \in M$, it follows that

$$|(A\varphi)(t) - (A\psi)(t)| \le (1 - e^{-\int_{t-T}^{t} a(s)ds})^{-1} \times \int_{t-T}^{t} \{\sum_{j=1}^{2} |b_j(u)|$$
$$|f(u, \phi(u), \phi(u - \tau_j(u))) - f(u, \psi(u), \psi(u - \tau_j(u)))|$$
$$+\sum_{j=1}^{2} |r_j(u)| \begin{vmatrix} g(\phi(u - \tau_j(u))) - \\ g(\psi(u - \tau_j(u))) \end{vmatrix} \} e^{-\int_u^t a(s)ds} du$$
$$\le \sigma\gamma \int_{t-T}^{t} [\sum_{j=1}^{2} |b_j(u)| (k_2 + k_3) \, \|\phi - \psi\| +$$
$$\sum_{j=1}^{2} |r_j(u)| k_1 \|\phi - \psi\|] du$$
$$\le T\sigma\gamma[\delta(k_2 + k_3) + \mu k_1] \|\phi - \psi\|.$$

Then, for $\|\phi - \psi\| < \delta,$ we get

$$\|A\phi - A\psi\| \le K \|\phi - \psi\| < \varepsilon.$$

This result proves A is continuous.

We now have to show that A is compact. For $n \in Z^+$, let $\phi_n \in M$. Then, as the above, we can see that

$$\|A\phi_n\| \le L.$$

If we calculate $(A\varphi_n)'(t)$, then

$$(A\phi_n)'(t) = \sum_{j=1}^{2} b_j(t) f(t, \phi_n(t), \phi_n(t - \tau_j(t)))$$
$$-\sum_{j=1}^{2} r_j(t) g(\phi_n(t - \tau_j(t)))$$
$$-a(t)(1 - e^{-\int_{t-T}^{t} a(s)ds})^{-1} \times$$
$$\int_{t-T}^{t} [\sum_{j=1}^{2} b_j(u) f(u, \phi_n(u), \phi_n(u - \tau_j(u)))$$
$$-\sum_{j=1}^{2} r_j(u) g(\phi_n(u - \tau_j(u)))] e^{-\int_u^t a(s)ds} du.$$

Hence, for some positive constant D, we obtain

$$|(A\phi_n)'(t)| \le \sum_{j=1}^{2} |b_j(t)| \, |f(t, \phi_n(t), \phi_n(t - \tau_j(t)))| +$$
$$\sum_{j=1}^{2} |r_j(t)| \, |g(\phi_n(t - \tau_j(t)))|$$
$$+a(t)(1 - e^{-\int_{t-T}^{t} a(s)ds})^{-1} \times$$
$$\int_{t-T}^{t} [\sum_{j=1}^{2} |b_j(u)| \, |f(u, \phi_n(u), \phi_n(u - \tau_j(u)))|$$
$$+ \sum_{j=1}^{2} |r_j(u)| |g(\varphi_n(u - \tau_j(u)))|] e^{-\int_u^t a(s)ds} du \le D.$$

Thus, the sequence $(A\varphi_n)$ is uniformly bounded and equi-continuous. The Arzela-Ascoli theorem implies that there exists a subsequence $(A\varphi_{n_k})$ of $(A\varphi_n)$ converges uniformly to a continuous $T-$ periodic function $\varphi*$. Thus, A is compact.

Lemma 3. Let B defined by (8) and

$$\left| \frac{c_j(t)}{1 - g_j'(t)} \right| \le \zeta_j, \quad k_1(\zeta_1 + \zeta_2) < 1. \tag{11}$$

Then B is a contraction.

Proof. For $\phi, \psi \in C_T$, we have

$$\|(B\varphi) - (B\psi)\| = \max_{t\in[0,T]} |(B\varphi)(t) - (B\psi)(t)|$$
$$= \max_{t\in[0,T]} \sum_{j=1}^{2} \left| \frac{c_j(t)}{1 - \tau_j'(t)} \right| |g(\varphi(t - \tau_j(t))) - g(\psi(t - \tau_j(t)))|$$
$$\le (\zeta_1 + \zeta_2) k_1 \|\phi - \psi\|.$$

Thus, B is a contraction.

Theorem 2. If $(C1)-(C3)$, (11) and the inequality

$$L[\sigma\gamma T(\delta(k_2 + k_3) + \mu k_1) + (\zeta_1 + \zeta_2) k_1] +$$
$$\sigma\gamma T(\delta\rho_2 + \mu\rho_1) + (\zeta_1 + \zeta_2)\rho_1 \le L$$

hold, then equation (1) has a T-periodic solution.

Proof. From Lemma 1, we know that A is compact and continuous. Also, from Lemma 2, we know that B is a contraction mapping. Now, for $\phi,\psi \in M$, we will show that

$A\phi + B\psi \in M$. From (7) and (8) we have

$$\|(A\phi)+(B\psi)\| \le$$

$$\sigma\gamma \int_{t-T}^{t} [(k_2+k_3+\mu)\|\phi\|+\rho]du + (\zeta_1+\zeta_2)\|\psi\|$$

$$\le L[\sigma\gamma T(\delta(k_2+k_3)+\mu k_1)+(\zeta_1+\zeta_2)k_1]$$

$$+\sigma\gamma T(\delta\rho_2+\mu\rho_1)+(\zeta_1+\zeta_2)\rho_1 \le L.$$

Then, it follows that all the conditions of Krasnoselskii's theorem hold on the set M. Thus, there exist a fixed point z in M such that $z = Az + Bz$. By Lemma 1 this fixed point is a solution of equation (1). Hence equation (1) has a T-periodic solution.

Theorem 3. Suppose assumptions (2)-(6) and (9)-(11) hold. If

$$(\zeta_1+\zeta_2)+T\gamma\sigma(k_1+k_2+k_3)<1,$$

then equation (1) has a unique $T-$periodic solution.

Proof. Let the mapping S be given by (6). For $\phi,\psi \in M$, we have from (6) that

$$\|(S\varphi(t))-(S\psi(t))\| \le (\zeta_1+\zeta_2)\|\varphi-\psi\|+$$

$$\sigma\gamma \int_{t-T}^{t} \delta(k_2+k_3)\|\varphi-\psi\|+\mu k_1\|\varphi-\psi\|$$

$$\le [(\zeta_1+\zeta_2)+T\gamma\sigma(k_1+k_2+k_3)]\|\varphi-\psi\|.$$

This completes the proof.

3. CONCLUSIONS

A kind of non-linear neutral differential equations of first order has been considered. On the basis of Krasnoselskii's fixed point theorem, two new results have been proved on the existence of periodic solutions that equation. The obtained results extend and improve some recent results in the literature.

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