Certain Properties of an Operator Involving the Generalized Hypergeometric Functions

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Abstract: In this work, based on the generalized derivative operator $K_{\lambda_1,\lambda_2}^{m}(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_s)f(z)$ and by making use of the notion of subordination, two new subclasses of functions are derived. With regards to these two subclasses, some properties are discussed briefly.

Keywords: Analytic function, Hadamard product, differential operator, subordination, coefficient estimate.

1. INTRODUCTION

Historically, we were informed that John Wallis was the first ever mathematician who used hypergeometric functions and this can be found in his book entitled "Arithmetica Infinitorum" [1]. Euler also found to be in the lists of those who used hypergeometric functions as mentioned in the book entitled “Theory of hypergeometric functions” [2]. However, the first full systematic treatment was given by Carl Friedrich Gauss, and thereafter by Ernst Kummer [3]. The fundamental characterization was addressed by Bernhard Riemann for solving hypergeometric function by means of differential equation where it satisfied [4]. The importance of the hypergeometric theory is stemmed from its applications in many subjects such as, numerical analysis, dynamical system and mathematical physics.

Definition 1.1 [11]: Denote by $A$ the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n; \quad z \in (U = \{z \in C : |z| < 1\})$$

(1)

and $S$ the subclass of $A$ consisting of univalent functions, and $S(\alpha), (0 < \alpha \leq 1)$ denotes the subclass of $A$ consisting of functions that are starlike of order $\alpha$ in $U$.

Definition 1.2 [10]: For two analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ in the open unit disk $U = \{z \in C : |z| < 1\}$. The Hadamard product (or convolution) $f * g$ of $f$ and $g$ is defined by

$$f(z) * g(z) = (f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$  

(2)

Definition 1.3 [11]: Let $p(z)$ and $q(z)$ be analytic in $U$. Then the function $p(z)$ is said to be subordinate to $q(z)$ in $U$, written by

$$p(z) \prec q(z); \quad (z \in U),$$

(3)

if there exists a function $w(z)$ which is analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1$ with $z \in U$, and such that $p(z) = q(w(z))$ for $z \in U$. From the definition of the subordinations, it is easy to show that the subordination (3) implies that

$$p(0) = q(0) \quad \text{and} \quad p(U) \subset q(U).$$

(4)

For complex parameters $\alpha_1,\ldots,\alpha_r$ and $\beta_1,\ldots,\beta_s$ ($\beta_j \neq 0, -1, -2, \ldots ; j = 1,\ldots,s$), Dziok and
Srivastava [5] defined the generalized hypergeometric function \( rF_s(\alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s; z) \) by
\[
rF_s(\alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n} \frac{z^n}{n!};
\]
where \((x)_n = (x)(x+1)\cdots(x+n-1)\) is the Pochhammer symbol defined, in terms of Gamma function \(\Gamma\), by
\[
(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1 & \text{if } n = 0, \\ x(x+1)\cdots(x+n-1) & \text{if } n \in \mathbb{N}. \end{cases}
\]
Dziok and Srivastava [5] defined also the linear operator
\[
H(\alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s; f(z)) = f(z) + \sum_{n=2}^{\infty} \frac{\Gamma(x+n)}{\Gamma(x)} \alpha_n z^n,
\]
where
\[
\Gamma_n = \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n (n-1)!}.
\]
Abbadi and Darus [6] defined the analytic function
\[
\phi_{\lambda_1, \lambda_2}^m = z + \sum_{n=2}^{\infty} \frac{(1+\lambda_1(n-1))^{m-1}}{(1+\lambda_2(n-1))^{m}} \alpha_n z^n,
\]
where \(m \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}\) and \(\lambda_1, \lambda_2 \geq 0\).
Using the Hadamard product (2), Alhindi and Darus [8, 9] has derived the generalized derivative operator \(K_{\lambda_1, \lambda_2}^m(\alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s)\) as follows
\[
\phi_{\lambda_1, \lambda_2}^m(\alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s; f(z)) = f(z) + \sum_{n=2}^{\infty} \frac{(1+\lambda_1(n-1))^{m-1}}{(1+\lambda_2(n-1))^{m}} \alpha_n z^n,
\]
where \(\Gamma_n\) is as given in (8).
Now, after some calculations we obtain the following equation:
\[
z(K_{\lambda_1, \lambda_2}^m(\alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s; f(z)))' = \alpha_1 K_{\lambda_1, \lambda_2}^m(\alpha_1+1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s; f(z))
\]
\[-\alpha_1 K_{\lambda_1, \lambda_2}^m(\alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s; f(z)).
\]
The linear operator \(K_{\lambda_1, \lambda_2}^m(\alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s)\) includes many other operators which were mentioned earlier in [8, 9].
If we recall the generalized Bernardi-Libera-Livingston integral operator \(j_c : A \to A\) (see [13, 14, 15]), defined by
\[
j_c.f(z) = \frac{c^{v-1} f(t)dt}{z^{c} v}; \quad (v > -1; f \in A).
\]
One can easily observe that
\[
j_c.f(z) = K_{0,\lambda_2}^1(1+c,1;c+2)
\]
\[
= K_{0,0}^1(1+c,1;c+2).
\]
Owa [16] introduced the fractional derivative operator by these definitions (see also [17]).
**Definition 1.4** [12]: The fractional integral operator of order \(\mu\) is defined, for a function \(f\), by
\[
D_{z}^{-\mu}f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(\eta)}{(z-\eta)^{1-\mu}}d\eta; \quad (\mu < 0),
\]
where \(f(z)\) is an analytic function in a simply connected region of the \(z\)-plane containing the origin, and the multiplicity of \((z-\eta)^{1-\mu}\) is removed by requiring \(\log(z-\eta)\) to be real when \(z-\eta > 0\).
**Definition 1.5** [16]: The fractional derivative operator of order \(\mu\) is defined, for a function \(f\), by
\[
D_{z}^{\mu}f(z) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \int_0^z \frac{f(\eta)}{(z-\eta)^{\mu}}d\eta; \quad (0 \leq \mu < 1),
\]
where \(f(z)\) is an analytic function in a simply connected region of the \(z\)-plane containing the origin, and the multiplicity of \((z-\eta)^{-\mu}\) is removed same as the previous definition.
**Definition 1.6** [16]: Using the assumption of Definition 1.5, the fractional derivative of order \(n + \mu\) is defined, for a function \(f\), by
\[
D_{z}^{n+\mu}f(z) = \frac{d^n}{dz^n} D_{z}^{\mu}f(z); \quad (0 \leq \mu < 1; n \in \mathbb{N}_0).
\]
these definitions of fractional calculus to define the linear operator $\Omega^\mu : A \to A$ as follows

$$\Omega^\mu f(z) = \Gamma(2-\mu) z^\mu D^\mu_z f(z);$$

$$\mu \neq 2, 3, 4, \ldots; f \in A).$$

By some calculations, we can find that

$$f(z) = K_0(2,1;2,0),$$

$$f(z) = K_1(2,1;2,1),$$

$$f(z) = K_2(2,1;2,2).$$

Kim and Srivastava [23] investigated the class of functions $f \in A$ such that

$$f(z) = \sum_{n=1}^{\infty} a_n z^n; \quad (a_n \geq 0; n \in \mathbb{N} \setminus 1).$$

with the normalization

$$f(0) = f'(0) - 1 = 0,$$

which also satisfy the following condition:

$$K^m_{A_1, A_2} (a_1 + 1, a_2, \ldots a_r, \beta_1, \ldots, \beta_s) f(z) f(z)$$

$$\alpha_1 - K^m_{A_1, A_2} (a_1, \ldots, a_r, \beta_1, \ldots, \beta_s) f(z)$$

$$+ 1 - \alpha_1 < \frac{1 + A}{1 + B z}.$$  \hspace{1cm} (19)

in terms of subordination, where $0 \leq B \leq -1$ and $-B \leq A < B.$

In this section, the coefficient estimate for the new class $W_{A_1, A_2}^m (r, s; A, B)$ is investigated. For this purpose, two lemmas are listed. Going back to (11), for a function of the form (17) and by considering $A = 1, B = -1$, one can notice that the condition (19) is equivalent to

$$K^m_{A_1, A_2} (a_1 + 1, a_2, \ldots, a_r; \beta_1, \ldots, \beta_s) f(z) \in S(0).$$

Thus we can get the following Lemma.

**Lemma 2.1** If $\alpha_j = \beta_j (j = 1, \ldots, s)$ then

$$W^m_{A_1, A_2} (s; 1, -1) \subset S(0).$$

By the definition of the class $W_{A_1, A_2}^m (r, s; A, B)$, we can get the following lemma.

**Lemma 2.2** If $A_1 \leq A_2$ and $B_1 \geq B_2$, then

$$W^m_{A_1, A_2} (r, s; A_1, B_1) \subset W^m_{A_1, A_2} (r, s; A_2, B_2)$$

$$W^m_{A_1, A_2} (r, s; A, B) \subset W^m_{A_1, A_2} (r, s; 1, -1).$$

**Theorem 2.3** Let $f$ of the form (17) belongs to the class $f \in W_{A_1, A_2}^m (r, s; A, B)$ if and only if

$$\sum_{n=2}^{\infty} \left( \frac{(B + 1)(B)(A + 1)}{1 + A z} \right)^{m-1} f(z)$$

$$\sum_{n=2}^{\infty} \left( \frac{(B + 1)(B)(A + 1)}{1 + A z} \right)^{-1} f(z)$$

$$\Gamma_n a_n \leq (B - A),$$

where $\Gamma_n$ is defined by (8).

**Proof.** Firstly, Let a function $f$ be of the form (17) belongs to the class $W_{A_1, A_2}^m (r, s; A, B)$ using the definition of subordination and by equation (19), we can write

$$K^m_{A_1, A_2} (a_1 + 1, a_2, \ldots, a_r; \beta_1, \ldots, \beta_s) f(z)$$

$$K^m_{A_1, A_2} (a_1, \ldots, a_r; \beta_1, \ldots, \beta_s) f(z)$$

$$+ 1 - \alpha_1 < \frac{1 + A}{1 + B z}.$$  \hspace{1cm} (23)

where, for convenience, we write

$$K^m_{A_1, A_2} (a_1, \ldots, a_r; \beta_1, \ldots, \beta_s) f(z)$$

and

$$K^m_{A_1, A_2} (a_1 + 1, a_2, \ldots, a_r; \beta_1, \ldots, \beta_s) f(z).$$
Thus, by equation (13), one can write
\[
\left| \sum_{n=2}^{\infty} \frac{(n-1)(1+\lambda_1(n-1))^{m-1}}{(1+\lambda_2(n-1))^m} \Gamma_n a_n z^{n-1} \right| < 1, \quad (z \in U),
\]

where \( \Gamma_n \) is defined by (8). If we put \( z = r \) for \( 0 \leq r < 1 \), we conclude that
\[
\sum_{n=2}^{\infty} \left( (Bn-A) - \sum_{n=2}^{\infty} (Bn-A) \frac{(1+\lambda_1(n-1))^{m-1}}{(1+\lambda_2(n-1))^m} \Gamma_n a_n r^{n-1} \right) < (B-A),
\]

which yields the assertion (22) by letting \( r \to 1 \).

Secondly, if the function \( f \) is of the form (17) and satisfying the condition (22). Then, we are supposed to prove that \( f \in W^m_{\lambda_1,\lambda_2}(r,s;A,B) \).

Using the relation (23), then it is sufficient to prove that
\[
\left| a_n \right| \leq \frac{(B-A)}{C_n}; \quad (n = 2,3,4,...),
\]

where
\[
C_n = (B+1)n-(A+1)) \frac{(1+\lambda_1(n-1))^{m-1}}{(1+\lambda_2(n-1))^m} \Gamma_n; \quad (n = 2,3,4,...).
\]

The result is sharp, the functions \( f_n \) of the form:
\[
f_n(z) = z - \frac{A-B}{C_n} z^n; \quad (n = 2,3,4,...),
\]

are the extremal functions.

3. THE NEW CLASS \( S^*(A,B) \)

In this section, a new subclass \( S^*(A,B) \) of analytic functions satisfying the following condition is defined.

Let \( f \in A \), then \( f \in S^*(A,B) \) if and only if
\[
\sum_{n=2}^{\infty} \left( (B+1)n-(A+1)) \frac{(1+\lambda_1(n-1))^{m-1}}{(1+\lambda_2(n-1))^m} \Gamma_n a_n z^{n-1} \right) < \infty, \quad 0 \leq A \leq 1 \quad 0 \leq B \leq 1.
\]

In the proceeding theorem we will study the sufficient condition for functions \( f \) to be in the class \( S^*(A,B) \), by applying the following lemma.

**Lemma 3.1** [24] Let \( w(z) \) be analytic in \( U \) with \( w(0) = 0 \). If \( |w(z)| \) attains its maximum value on the circle \( |z| = r < 1 \) at a point \( z_0 \), then
\[
z_0 w'(z_0) = k w(z_0),
\]
where \( k \) is a real number and \( k \geq 1. \)
Theorem 3.2 Suppose \( f \in A \) which satisfying
\[
1 + \frac{z \left( K_{\alpha_1,\alpha_2}^m (a_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s) f(z) \right)^n}{K_{\alpha_1,\alpha_2}^m (a_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s) f(z)} \leq \frac{(1 + A)^2 + (A + B)}{(1 + A)(1 - B)} ; \quad (z \in U),
\]
for some \( 0 \leq A \leq 1 \) and \( 0 \leq B \leq 1 \), then \( f \in S^*(A, B) \).

Proof. Let \( w(z) \) is defined by
\[
z \left( K_{\alpha_1,\alpha_2}^m (a_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s) f(z) \right)
\]
\[
= 1 + \frac{A w(z)}{1 - B w(z)} ; \quad (B w(z) \neq 1).
\]
It follows that \( w(0) = 0 \). Moreover, \( w(z) \) is analytic and after some calculations we can write
\[
z \left( K_{\alpha_1,\alpha_2}^m (a_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s) f(z) \right)^n
\]
\[
= \frac{(1 + A w(z))^2 + z w'(z)(A + B)}{(1 - B w(z))(1 + A w(z))}.
\]
Thus
\[
1 + \frac{z \left( K_{\alpha_1,\alpha_2}^m (a_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s) f(z) \right)^n}{K_{\alpha_1,\alpha_2}^m (a_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s) f(z)}
\]
\[
= \Re \left( 1 + \frac{z \left( K_{\alpha_1,\alpha_2}^m (a_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s) f(z) \right)^n}{K_{\alpha_1,\alpha_2}^m (a_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s) f(z)} \right)
\]
\[
\leq \frac{(1 + A)^2 + (A + B)}{(1 + A)(1 - B)} ; \quad (z \in U),
\]
which contradicts our assumption. Therefore, we can obtain that \( |w(z)| < 1 \) for all \( (z \in U) \) implies
\[
z \left( K_{\alpha_1,\alpha_2}^m (a_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s) f(z) \right)^n
\]
\[
< \frac{1 + A z}{1 - B z}.
\]
Next, we prove that \( |w(z)| < 1 \). Suppose that there exists a point \( z_0 \in U \) such that
\[
\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1.
\]
Suppose \( w(z_0) = e^{i\theta} \) and \( z_0 w'(z_0) = k e^{i\theta} ; k \geq 1 \), then by applying Lemma 3.1 we can get

Corollary 3.3 Suppose that \( f \in S^*(A,0) \) then we can write
\[
z \left( K_{\alpha_1,\alpha_2}^m (a_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s) f(z) \right)^n
\]
\[
< \frac{1 + A}{1 - B}.
\]
Putting \( A = 1 \) implies that \( K_{\alpha_1,\alpha_2}^m (a_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s) \) is starlike.

4. CONCLUSIONS
In this paper, two new subclasses
\( w_{\lambda_1, \lambda_2}^m (r, s; A, B) \) and \( S^*(A, B) \) were introduced involving the operator \( K_{\lambda_1, \lambda_2}^m (\alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s) \).
Moreover, by considering the subordination notion, certain properties of the two subclasses were investigated.

5. ACKNOWLEDGEMENTS

The above study was supported by UKM's grant: AP-2013-009 and DIP-2013-001.

6. REFERENCES