Proceedings of the Pakistan Academy of Sciences 52 (3): 227–232 (2015)

ISSN: 0377 - 2969 (print), 2306 - 1448 (online)

Pakistan Academy of Sciences

Research Article

Certain Properties of an Operator Involving the Generalized Hypergeometric Functions

Khadeejah R. Alhindi, and Maslina Darus*

School of Mathematical Sciences, Faculty of Science and Technology,
Universiti Kebangsaan Malaysia, 43600 UKM,
Selangor Darul Ehsan, Malaysia

Abstract: In this work, based on the generalized derivative operator $\mathrm{K}_{\lambda_1,\lambda_2}^{m}(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s)f(z)$ and by making use of the notion of subordination, two new subclasses of functions are derived. With regards to these two subclasses, some properties are discussed briefly.

Keywords: Analytic function, Hadamard product, differential operator, subordination, coefficient estimate.

1. INTRODUCTION

Historically, we were informed that John Wallis was the first ever mathematician who used hypergeometric functions and this can be found in his book entitled "Arithmetica Infinitorum" [1]. Euler also found to be in the lists of those who used hypergeometric functions as mentioned in the book entitled "Theory of hypergeometric functions" [2]. However, the first full systematic treatment was given by Carl Friedrich Gauss, and thereafter by Ernst Kummer [3]. The fundamental characterization was addressed by Bernhard Riemann for solving hypergeometric function by means of differential equation where it satisfied [4]. The importance of the hypergeometric theory is stemmed from its applications in many subjects such as, numerical analysis, dynamical system and mathematical physics.

Definition 1.1 [11]: Denote by A the class of analytic functions of the form

$$f(z)=z+\sum_{n=2}^{n=\infty}a_nz^n;\quad z\in(U=\{z\in C:|z|<1\}) \tag{1}$$

and $\mathcal{S}$ the subclass of A consisting of univalent functions, and $S(\alpha)$, $(0<\alpha\leq 1)$ denotes the subclasse of A consisting of functions that are starlike of order α in U.

Definition 1.2 [10]: For two analytic functions $f(z)=z+\sum_{n=2}^{n=\infty}a_nz^n$ and $g(z)=z+\sum_{n=2}^{n=\infty}b_nz^n$ in the open unit disk $U=\{z\in C:|z|<1\}$. The Hadamard product (or convolution) $f*g$ of f and g is defined by

$$f(z)*g(z)=(f*g)(z)=z+\sum_{n=2}^{n=\infty}a_nb_nz^n. \tag{2}$$

Definition 1.3 [11]: Let $p(z)$ and $q(z)$ be analytic in U. Then the function $p(z)$ is said to be subordinate to $q(z)$ in U, written by

$$p(z)\prec q(z);\qquad (z\in U), \tag{3}$$

if there exists a function $w(z)$ which is analytic in U with $w(0)=0$ and $|w(z)|<1$ with $z\in U$, and such that $p(z)=q(w(z))$ for $z\in U$. From the definition of the subordinations, it is easy to show that the subordination (3) implies that

$$p(0)=q(0)\qquad and\qquad p(U)\subset q(U) \tag{4}$$

For complex parameters $\alpha_1,...\alpha_r$ and $\beta_1,...\beta_s$ $(\beta_j\neq 0,-1,-2,...;\ j=1...s)$, Dziok and

Received, March 2014; Accepted, May 2015
*Corresponding author: Maslina Darus; Email: maslina@ukm.edu.my

Srivastava [5] defined the generalized hypergeometric function ${}_rF_s(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s;z)$ by

$$ {}_rF_s(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s;z)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n...(\alpha_r)_n}{(\beta_1)_n...(\beta_s)_n}\frac{z^n}{n!}; $$
$$ (r\le s+1;r,s\in N_0;z\in U), \qquad (5) $$

where $(x)_n$ is the Pochhammer symbol defined, in terms of Gamma function Γ, by

$$ (x)_n=\frac{\Gamma(x+n)}{\Gamma(x)}=\begin{cases}1 & \text{if} \quad n=0,\\ x(x+1)...(x+n-1) & \text{if} \quad n\in N.\end{cases} \qquad (6) $$

Dziok and Srivastava [5] defined also the linear operator

$$ H(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s)f(z)=z+\sum_{n=2}^{\infty}\Gamma_n a_n z^n, \qquad (7) $$

where

$$ \Gamma_n=\frac{(\alpha_1)_{n-1}...(\alpha_r)_{n-1}}{(\beta_1)_{n-1}...(\beta_s)_{n-1}(n-1)!}. \qquad (8) $$

Abbadi and Darus [6] defined the analytic function

$$ \Phi^m_{\lambda_1,\lambda_2}=z+\sum_{n=2}^{\infty}\frac{(1+\lambda_1(n-1))^{m-1}}{(1+\lambda_2(n-1))^m}z^n, \qquad (9) $$

where $m\in\mathbb{N}_0=\{0,1,2,....\}$ and $\lambda_2\ge\lambda_1\ge 0$.

Using the Hadamard product (2), Alhindi and Darus [8, 9] has derived the generalized derivative operator $K^m_{\lambda_1,\lambda_2}(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s)$ as follows

$$ \varphi^m_{\lambda_1,\lambda_2}(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s)f(z)= $$
$$ z+\sum_{n=2}^{\infty}\frac{(1+\lambda_1(n-1))^{m-1}}{(1+\lambda_2(n-1))^m}\Gamma_n a_n z^n, \qquad (10) $$

where Γ_n is as given in (8).

Now, after some calculations we obtain the following equation:

$$ z(K^m_{\lambda_1,\lambda_2}(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s)f(z))'= $$
$$ \alpha_1K^m_{\lambda_1,\lambda_2}(\alpha_1+1,...,\alpha_r;\beta_1,...,\beta_s)f(z) $$
$$ -\alpha_1K^m_{\lambda_1,\lambda_2}(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s)f(z). \qquad (11) $$

The linear operator $\mathcal{K}^m_{\lambda_1,\lambda_2}(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s)$ includes many other operators which were mentioned earlier in [8, 9].

If we recall the generalized Bernardi-Libera-Livingston integral operator $j_c:A\to A$ (see [13, 14, 15]), defined by

$$ j_cf(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt; \qquad (v>-1;f\in A). $$

One can easily observe that

$$ j_cf(z)=\mathrm{K}^0_{0,\lambda_2}(1+c,1;c+2) $$
$$ =\mathrm{K}^1_{\lambda_1,0}(1+c,1;c+2) $$
$$ =\mathrm{K}^2_{0,0}(1+c,1;c+2). $$

Owa [16] introduced the fractional derivative operator by these definitions (see also [17]).

Definition 1.4 [12]: The fractional integral operator of order μ is defined, for a function f, by

$$ D_z^{-\mu}f(z)=\frac{1}{\Gamma(\mu)}\int_0^z\frac{f(\eta)}{(z-\eta)^{1-\mu}}d\eta; \qquad (\mu<0), \qquad (12) $$

where *f(z)* is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-\eta)^{\mu-1}$ is removed by requiring $\log(z-\eta)$ to be real when $z-\eta>0$.

Definition 1.5 [16]: The fractional derivative operator of order μ is defined, for a function f, by

$$ D_z^{\mu}f(z)=\frac{1}{\Gamma(1-\mu)}\frac{d}{dz}\int_0^z\frac{f(\eta)}{(z-\eta)^{\mu}}d\eta; \quad (0\le\mu<1), \qquad (13) $$

where *f(z)* is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-\eta)^{-\mu}$ is removed same as the previous definition.

Definition 1.6 [16]: Using the assumption of Definition 1.5, the fractional derivative of order $n+\mu$ is defined, for a function f, by

$$ D_z^{n+\mu}f(z)=\frac{d^n}{dz^n}D_z^{\mu}f(z); \quad (0\le\mu<1;n\in\mathbb{N}_0), \qquad (14) $$

Srivastava and Owa [18] (see also [19-22]) used these definitions of fractional calculus to define the linear operator $\Omega^{\mu}: A \to A$ as follows

$$\Omega^{\mu} f(z) = \Gamma(2-\mu) z^{\mu} D_z^{\mu} f(z); \quad (\mu \neq 2,3,4,...; f \in A). \tag{15}$$

By some calculations, we can find that

$$\Omega^{\mu} f(z) = \mathrm{K}^{0}_{0,\lambda_2}(2,1;2-\mu)$$
$$= \mathrm{K}^{1}_{\lambda_1,0}(2,1;2-\mu)$$
$$= \mathrm{K}^{2}_{0,0}(2,1;2-\mu).$$

Kim and Srivastava [23] investigated the class of functions $f \in A$ such that $\mathcal{L}(a,c)f(z) \in S^*(\alpha)$,

$$a\frac{\ell(a+1,c)f(z)}{\ell(a,c)f(z)} + 1 - a \prec \frac{1+(1-2\alpha)z}{1-z}. \tag{16}$$

After that, Dziok and Srivastava [5] introduced the class $V(r,s;A,B)$ of function f with some conditions, and studied its properties.

2. THE NEW CLASS $W^m_{\lambda_1,\lambda_2}(r,s;A,B)$

Let us denote by $W^m_{\lambda_1,\lambda_2}(r,s;A,B)$ the class of functions f of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n; \qquad (a_n \geq 0; n \in \mathbb{N}\setminus 1). \tag{17}$$

with the normalization

$$f(0) = f'(0) - 1 = 0, \tag{18}$$

which also satisfy the following condition:

$$\alpha_1 \frac{K^m_{\lambda_1,\lambda_2}(\alpha_1+1,\alpha_2,...,\alpha_r;\beta_1,...,\beta_s)f(z)}{K^m_{\lambda_1,\lambda_2}(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s)f(z)} + 1 - \alpha_1 \prec \frac{1+Az}{1+Bz}. \tag{19}$$

in terms of subordination, where $0 \leq B \leq -1$ and $-B \leq A < B$.

In this section, the coefficient estimate for the new class $W^m_{\lambda_1,\lambda_2}(r,s;A,B)$ is investigated. For this purpose, two lemmas are listed. Going back to(11), for a function of the form (17) and by considering $A=1$, $B=-1$, one can notice that the condition (19) is equivalent to

$$K^m_{\lambda_1,\lambda_2}(\alpha_1+,\alpha_2,...,\alpha_r;\beta_1,...,\beta_s)f(z) \in S(0). \tag{20}$$

Thus we can get the following Lemma.

Lemma 2.1 If $\alpha_i = \beta_j (j=1,...,s)$ then $W^m_{\lambda_1,\lambda_2}(s;1,-1) \subset S(0)$.

By the definition of the class $W^m_{\lambda_1,\lambda_2}(r,s;A,B)$, we can get the following lemma.

Lemma 2.2 If $A_1 \leq A_2$ and $B_1 \geq B_2$, then

$$W^m_{\lambda_1,\lambda_2}(r,s;A_1,B_1) \subset W^m_{\lambda_1,\lambda_2}(r,s;A_2,B_2) \subset W^m_{\lambda_1,\lambda_2}(r,s;1,-1). \tag{21}$$

Theorem 2.3 Let f of the form), then $f \in W^m_{\lambda_1,\lambda_2}(r,s;A,B)$ if and only if

$$\sum_{n=2}^{\infty} ((B+1)n-(A+1)) \frac{(1+\lambda_1(n-1))^{m-1}}{(1+\lambda_2(n-1))^m} \Gamma_n a_n \leq (B-A), \tag{22}$$

where Γ_n is is defined by (8).

Proof. Firstly, Let a function f be of the form (17) belongs to the class $W^m_{\lambda_1,\lambda_2}(r,s;A,B)$. Using the definition of subordination and by equation (19), we can write

$$\alpha_1 \frac{K^m_{\lambda_1,\lambda_2}(\alpha_1+1,\alpha_2,...,\alpha_r;\beta_1,...,\beta_s)f(z)}{K^m_{\lambda_1,\lambda_2}(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s)f(z)} + 1 - \alpha_1 = \frac{1+Aw(z)}{1+Bw(z)}.$$

After some calculation, and by consider that $w(0)=0$ and $|w(z)|<1$ we can write

$$\left| \frac{\alpha_1\{K^{m,r,s}_{\lambda_1,\lambda_2}(\alpha_1+1)f(z) - K^{m,r,s}_{\lambda_1,\lambda_2}(\alpha_1)f(z)\}}{\alpha_1 B K^{m,r,s}_{\lambda_1,\lambda_2}(\alpha_1+1)f(z) - (A+(\alpha_1-1)B)K^{m,r,s}_{\lambda_1,\lambda_2}(\alpha_1)f(z)} \right| < 1, \tag{23}$$

where, for convenience, we write

$$K^{m,r,s}_{\lambda_1,\lambda_2}(\alpha_1)f(z) = K^m_{\lambda_1,\lambda_2}(\alpha_1,\alpha_2,...,\alpha_r;\beta_1,...,\beta_s)f(z),$$

and

$$K^{m,r,s}_{\lambda_1,\lambda_2}(\alpha_1+1)f(z) = K^m_{\lambda_1,\lambda_2}(\alpha_1+1,\alpha_2,...,\alpha_r;\beta_1,...,\beta_s)f(z).$$

Thus, by equation (13), one can write

$$\left|\frac{\sum_{n=2}^{\infty}(n-1)\frac{(1+\lambda_1(n-1))^{m-1}}{(1+\lambda_2(n-1))^m}\Gamma_n a_n z^{n-1}}{(B-A)-\sum_{n=2}^{\infty}(Bn-A)\frac{(1+\lambda_1(n-1))^{m-1}}{(1+\lambda_2(n-1))^m}\Gamma_n a_n z^{n-1}}\right|<1; \quad (z\in U),$$

where Γ_n is is defined by (8). If we put $z=r$ for $0\le r<1$, we conclude that

$$\sum_{n=2}^{\infty}(n-1)\frac{(1+\lambda_1(n-1))^{m-1}}{(1+\lambda_2(n-1))^m}\Gamma_n a_n r^{n-1}$$
$$<(B-A)-\sum_{n=2}^{\infty}(Bn-A)\frac{(1+\lambda_1(n-1))^{m-1}}{(1+\lambda_2(n-1))^m}\Gamma_n a_n r^{n-1}$$

which yields the assertion (22) by letting $r\to 1$.

Secondly, if the function f is of the form (17) and satisfying the condition (22). Then, we are supposed to prove that $f\in W^m_{\lambda_1,\lambda_2}(r,s;A,B)$.

Using the relation (23), then it is sufficient to prove that

$$\left|\alpha_1\left\{K^{m,r,s}_{\lambda_1,\lambda_2}(\alpha_1+1)f(z)-K^{m,r,s}_{\lambda_1,\lambda_2}(\alpha_1)f(z)\right\}\right|$$
$$-\left|\alpha_1 BK^{m,r,s}_{\lambda_1,\lambda_2}(\alpha_1+1)f(z)-(A+(\alpha_1-1)B)K^{m,r,s}_{\lambda_1,\lambda_2}(\alpha_1)f(z)\right|. \quad (24)$$

If we put $|z|=r$ for $0\le r<1$, then we can write

$$\left|\alpha_1\{K^{m,r,s}_{\lambda_1,\lambda_2}(\alpha_1+1)f(z)-K^{m,r,s}_{\lambda_1,\lambda_2}(\alpha_1)f(z)\}\right|-$$
$$\left|\alpha_1 BK^{m,r,s}_{\lambda_1,\lambda_2}(\alpha_1+1)f(z)-(A+(\alpha_1-1)B)K^{m,r,s}_{\lambda_1,\lambda_2}(\alpha_1)f(z)\right|$$
$$=\left|\sum_{n=2}^{\infty}(n-1)\frac{(1+\lambda_1(n-1))^{m-1}}{(1+\lambda_2(n-1))^m}\Gamma_n a_n z^{n}\right|-$$
$$\left|(A-B)-\sum_{n=2}^{\infty}(Bn-A)\frac{(1+\lambda_1(n-1))^{m-1}}{(1+\lambda_2(n-1))^m}\Gamma_n a_n z^{n}\right|$$
$$\le\sum_{n=2}^{\infty}(n-1)\frac{(1+\lambda_1(n-1))^{m-1}}{(1+\lambda_2(n-1))^m}\Gamma_n a_n r^{n}$$
$$-\left((A-B)-\sum_{n=2}^{\infty}(Bn-A)\frac{(1+\lambda_1(n-1))^{m-1}}{(1+\lambda_2(n-1))^m}\Gamma_n a_n r^{n}\right)$$

$$=r(\sum_{n=2}^{\infty}((B+1)n-(A+1))\frac{(1+\lambda_1(n-1))^{m-1}}{(1+\lambda_2(n-1))^m}\Gamma_n r^{n-1}-(B-A))$$

$$<\sum_{n=2}^{\infty}((B+1)n-(A+1))\frac{(1+\lambda_1(n-1))^{m-1}}{(1+\lambda_2(n-1))^m}\Gamma_n-(B-A)\le 0. \quad (25)$$

Thus, $f\in W^m_{\lambda_1,\lambda_2}(r,s;A,B)$ and the proof is complete.

Based on Theorem 2.3, the following corollary can be derived.

Corollary 2.4 If a function f is of the form (17)and $f\in W^m_{\lambda_1,\lambda_2}(r,s;A,B)$, then we can write

$$a_n\le\frac{(B-A)}{C_n}; \qquad (n=2,3,4,...),$$

where

$$C_n=((B+1)n-(A+1))\frac{(1+\lambda_1(n-1))^{m-1}}{(1+\lambda_2(n-1))^m}\Gamma_n; \quad (n=2,3,4,...).$$

The result is sharp, the functions f_n of the form:

$$f_n(z)=z-\frac{A-B}{C_n}z^n; \qquad (n=2,3,4,...),$$

are the extremal functions.

3. THE NEW CLASS $S^*(A,B)$

In this section,a new subclass $S^*(A,B)$ of analytic functionssatisfying the following condition is defined.

Let $f\in A$, then $f\in S^*(A,B)$ if and only if

$$\frac{z\left[K^m_{\lambda_1,\lambda_2}(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s)f(z)\right]'}{K^m_{\lambda_1,\lambda_2}(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s)f(z)}\prec\frac{1+Az}{1-Bz}; \quad (26)$$

where $0\le A\le 1$ and $0\le B\le 1$.

In the proceeding theorem we will study the sufficient condition for functions *f*to be in the class $S^*(A,B)$, by applying the following lemma.

Lemma 3.1 [24] Let $w(z)$ be analytic in U with $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at a point z_0, then

$$z_0 w'(z_0)=kw(z_0),$$

where k is a real number and $k\ge 1$.

Theorem 3.2 *Suppose* $f \in A$ *which satisfying*

$$\Re\left(1+\frac{z\left[K^m_{\lambda_1,\lambda_2}(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s)f(z)\right]''}{\left[K^m_{\lambda_1,\lambda_2}(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s)f(z)\right]'}\right) < \frac{(1+A)^2+(A+B)}{(1+A)(1-B)}; \quad (z \in U), \tag{27}$$

for some $0 \le A \le 1$ and $0 \le B \le 1$, then $f \in S^*(A,B)$.

Proof. Let $w(z)$ is defined by

$$\frac{z\left[K^m_{\lambda_1,\lambda_2}(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s)f(z)\right]'}{K^m_{\lambda_1,\lambda_2}(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s)f(z)} = \frac{1+Aw(z)}{1-Bw(z)}; \quad (Bw(z)\neq 1).$$

It follows that $w(0)=0$. Moreover, $w(z)$ is analytic and after some calculations we can write

$$1+\frac{z\left[K^m_{\lambda_1,\lambda_2}(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s)f(z)\right]''}{\left[K^m_{\lambda_1,\lambda_2}(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s)f(z)\right]'} = \frac{(1+Aw(z))^2+zw'(z)(A+B)}{(1-Bw(z))(1+Aw(z))}.$$

Thus

$$\Re\left(1+\frac{z\left[K^m_{\lambda_1,\lambda_2}(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s)f(z)\right]''}{\left[K^m_{\lambda_1,\lambda_2}(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s)f(z)\right]'}\right) = \Re\left(\frac{(1+Aw(z))^2+zw'(z)(A+B)}{(1-Bw(z))(1+Aw(z))}\right)$$

$$< \frac{(1+A)^2+(A+B)}{(1+A)(1-B)}.$$

Next, we prove that $|w(z)|<1$. Suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z|\le|z_0|} |w(z)| = |w(z_0)| = 1.$$

Suppose $w(z_0)=e^{i\theta}$ and $z_0 w'(z_0)=ke^{i\theta}; k\ge 1$, then by applying Lemma 3.1 we can get

$$\Re\left(1+\frac{z\left[K^m_{\lambda_1,\lambda_2}(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s)f(z)\right]''}{\left[K^m_{\lambda_1,\lambda_2}(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s)f(z)\right]'}\right)$$

$$= \Re\left(\frac{(1+Aw(z_0))^2+z_0w'(z_0)(A+B)}{(1+Aw(z_0))(1-Bw(z_0))}\right)$$

$$= \Re\left(\frac{(1+Ae^{i\theta})^2+ke^{i\theta}(A+B)}{(1+Ae^{i\theta})(1-Be^{i\theta})}\right)$$

$$= \Re\left(\frac{(1+A)^2+k(A+B)}{(1+A)(1-B)}\right) \ge \frac{(1+A)^2+(A+B)}{(1+A)(1-B)}.$$

We conclude that

$$\Re\left(1+\frac{z\left[K^m_{\lambda_1,\lambda_2}(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s)f(z)\right]''}{\left[K^m_{\lambda_1,\lambda_2}(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s)f(z)\right]'}\right) \ge \frac{(1+A)^2+(A+B)}{(1+A)(1-B)}; \quad (z\in U),$$

which contradicts our assumption. Therefore, we can obtain that $|w(z)|<1$ for all $(z\in U)$ implies

$$\frac{z\left[K^m_{\lambda_1,\lambda_2}(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s)f(z)\right]'}{K^m_{\lambda_1,\lambda_2}(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s)f(z)} \prec \frac{1+Az}{1-Bz};$$

where $0\le A\le 1$ and $0 \le B \le 1$. Thus, the proof is complete.

Corollary 3.3 Suppose that $f \in S^*(A,0)$ then we can write

$$\left|\frac{z\left[K^m_{\lambda_1,\lambda_2}(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s)f(z)\right]'}{K^m_{\lambda_1,\lambda_2}(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s)f(z)}-1\right| < A.$$

Putting $A=1$ implies that $K^m_{\lambda_1,\lambda_2}(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s)$ is starlike.

4. CONCLUSIONS

In this paper, two new subclasses

$W^{m}_{\lambda_1,\lambda_2}(r,s;A,B)$ and $S^*(A,B)$ were introduced involving the operator $K^{m}_{\lambda_1,\lambda_2}(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s)$. Moreover, by considering the subordination notion, certain properties of the two subclasses were investigated.

5. ACKNOWLEDGEMENTS

The above study was supported by UKM's grant: AP-2013-009 and DIP-2013-001.

6. REFERENCES

1. Stedall, J. A. *The Arithmetic of Infinitesimals*: John Wallis 1656. New York. (2004).
2. Aomoto, K. et al. *Theory of hypergeometric functions*. Springer, Berlin (2011).
3. Kummer, E. E. "Über die hypergeometrische Reihe....(Fortsetzung)." *Journal für die reine und angewandte Mathematik* 15: 127-172 (1836).
4. Riemann, B. *Theorie der Abel'schen functionen*. Georg Reimer, (1857).
5. Dziok, J. & H.M. Srivastava. Classes of analytic functions associated with the generalized hypergeometric function. *Applied Mathematics and Computation* 103: 1-13 (1999).
6. Al-Abbadi, M.H. & M. Darus. Differential subordination for new generalized derivative operator. *Acta Universitatis Apulensis* 20: 265-280 (2009).
7. Hohlov, Yu.E. Operators and operations in the class of univalent functions. *Izvestiya Vysshikh Uchebnykh ZavedeniiMatematik*10:83-89 (1978).
8. Alhindi, K.R. & M. Darus. Certain properties for a class of analytic functions associated with hypergeometric functions."*Revue D'analyse Numerique Et De Th´ Eorie De L'approximation´* 43(2): 93–102 (2014).
9. Alhindi, K.R. & M. Darus. New class of analytic functions associated with the generalized hypergeometric functions.*Acta Universitatis Apulensis* 37: 83-91 (2014).
10. Srivastava, H.M. & S. Owa. *Univalent Functions, Fractional Calculus, and Their Applications*. Ellis Horwood, New York, USA (1989).
11. Goodman, A.W. *Univalent functions*. Vol. 2. Mariner Publishing Company (1983).
12. Duren, P.L. *Univalent Functions*. Springer-Verlag, New York, USA (1983).
13. Bernardi, S.D. Convex and starlike univalent functions. *Transactions of the American Mathematical Society* 135: 429-446 (1969).
14. Libera, R.J. Some classes of regular univalent functions.*Proceedings of the American Mathematical Society* 16: 755-758 (1965).
15. Livingston, A.E. On the radius of univalence of certain analytic functions. *Proceedings of the American Mathematical Society* 17:352-357 (1966).
16. Owa, S. On the distortion theorems. *Kyungpook Mathematical Journal* 18: 53-59 (1978).
17. Srivastava, H.M. & S. Owa (Ed.).*Univalent Functions, Fractional Calculus, and Their Applications*. Wiley, New York, USA (1989).
18. Srivastava, H.M. & S. Owa. Some characterization and distortion theorems involving fractional calculus, generalized hypergeometric functions, Hadamard products, linear operators, and certain subclasses of analytic functions. *Nagoya Mathematical Journal* 106: 1-28 (1987).
19. Srivastava, H.M. & M.K. Aouf. A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions with negative coeficients. *Journal of Mathematical Analysis and Applications* 171(1): 1-13 (1992).
20. Srivastava, H.M. & M.K. Aouf. A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions with negative coeficients, II. *Journal of Mathematical Analysis and Applications* 192: 673-688 (1995).
21. Srivastava, H.M. & S. Owa (Ed.).*Current Topics in Analytic Function Theory*. World Scientific, Singapore (1992).
22. Dziok, J. Classes of analytic functions involving some integral operator.*Folia Fac Sci Nat Univ Purkyn Brun. Tech. Resoviensis* 20: 21-39 (1995).
23. Kim, Y.C. & H.M. Srivastava. Fractional integral and other linear operators associated with the Gaussian hypergeometric function.*Complex Variables, Theory and Application* 34: 293-312 (1997).
24. Jack, I.S. Functions starlike and convex of order K.*Journal of the London Mathematical Society* 3: 469–474 (1971).