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Research Article

Generalized Ostrowski Type Inequalities on Time Scales with Applications

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Abstract: Generalized Ostrowski type inequalities for function of three independent variables on time scales are derived that generalized some existing and classical inequalities with some applications for generalized polynomials.

Keywords: Ostrowski inequality; generalized polynomial; Time scales; rd-continuous function

1. INTRODUCTION

The role of mathematical inequalities within the mathematical branches as well as in its enormous applications should not be underestimated. The appearance of the new mathematical inequality often puts on firm foundation for the heuristic algorithms and procedures used in applied sciences. Among others one of the main inequality, which stipulates a bound between a function evaluated at an interior point x and the integral mean of f over an interval, called Ostrowski inequality is defined as [9]:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \sup_{a<x<b}|f'(x)|\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a), \tag{1}$$

where $f:[a,b]\to \boldsymbol{R}$ is a differentiable function. The constant factor $\frac{1}{4}$ is the best possible one. It has a lot of applications in numerical analysis, probability theory and in special means. P. Cerone et al. [3] proved an Ostrowski type inequality for n-times differentiable function. For other similar results for n-times differentiable functions, see [1, 2, 4, 5]. Sofo determined an Ostrowski type inequality in three independent variables [14]. For more results about Ostrowski consults various prior publications [7, 10, 11, 15, 16, 17, 18, 19, 20, 21].

Our main purpose in this paper is to prove some new results related to an Ostrowski inequality in three variables on time scales, generalizing some existing and classical results. Some applications to generalized polynomial are also given.

In Section 2, some time scales essentials are given. In Section 3, some new results are given and in the last Section 4, some applications for generalized polynomials are given.

2. TIME SCALES ESSENTIALS

A time scale (or measure chain) is a non-empty closed subset of the real $\boldsymbol{R}$, together with the topology of subspace of $\boldsymbol{R}$ and we usually denote it by the symbol $\mathbb{T}$. The two most popular examples are $\mathbb{T}=\boldsymbol{R}$ and $\mathbb{T}=\mathbb{Z}$. For any interval $\boldsymbol{I}$ of $\boldsymbol{R}$ (open or closed) $\boldsymbol{I}_{\mathbb{T}}=\boldsymbol{I}\cap\mathbb{T}$ is called a time scales interval. We define the forward and backward jump operators $\sigma,\rho:\mathbb{T}\to\mathbb{T}$ by:

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$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \text{and} \rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

(supplemented by $\sup\mathbb{T} = \inf\phi$ and $\sup\phi = \inf\mathbb{T}$ where ϕ denotes the empty set).The set $\mathbb{T}^k$is defined to be $\mathbb{T}$ if $\mathbb{T}$ does not have a left scattered maximum; otherwiseit is $\mathbb{T}$ without this left scattered maximum. The graininess $\mu: \mathbb{T} \to [0, \infty)$ defined by

$$\mu(t) = \sigma(t) - t.$$

Hence the graininess function is constant 0 if $\mathbb{T} = \mathbb{R}$ while it is constant 1 if $\mathbb{T} = \mathbb{Z}$.

However, a time scale $\mathbb{T}$ could have nonconstant graininess. Let $f: \mathbb{T} \to \mathbf{R}$, be a function then $f^{\sigma}: \mathbb{T} \to \mathbf{R}$ defined by $f^{\sigma}(t) = f(\sigma(t))$ for $t \in \mathbb{T}$,where $\sigma(t)$ is defined above. We also, say that f is delta differentiable (or simply: differentiable) at $t \in \mathbb{T}^k$ provided there exists an α such that for all $\epsilon > 0$ there is a neighborhood $\aleph$ of t with

$$\left|\left[f(\sigma(t)) - f(s)\right] - \alpha[\sigma(t) - s]\right| \le \epsilon|\sigma(t) - s| \qquad \text{for all } s \in \aleph.$$

In this case we denote the α by$f^{\Delta}(t)$, and if f is differentiable for every $t \in \mathbb{T}^k$, then f is said to be differentiable on $\mathbb{T}$ and f^{Δ} is a new function on $\mathbb{T}^k$. If f is differentiable at $t \in \mathbb{T}^k$, then it is easy to see that

$$f^{\Delta}(t) = \begin{cases} \lim\limits_{s \to t (s \in \mathbb{T})} \dfrac{f(t) - f(s)}{t - s} & if \;\; \mu(t) = 0 \\ \dfrac{f(\sigma(t)) - f(t)}{\mu(t)} & if \;\; \mu(t) > 0 \end{cases}$$

Several useful delta derivative formulae can be recorded in [13, Lemma 1,2] (see also [12]).

A continuous function $f: \mathbb{T} \to \boldsymbol{R}$is said to be pre-differentiable with $D \subset \mathbb{T}^k$ (where $\mathbb{T}^k \backslash D$ is countable) as differentiation region, contains no right scattered elements of $\mathbb{T}$ and f is differentiable at $t \in D$. A function f is called regulated if its right sided limit exist at all right dense points in $\mathbb{T}$ and its left sided limit exist at all left dense points in $\mathbb{T}$.If there exist a pre-differentiable function F such that

$$F^{\Delta}(t) = f(t) \quad \forall\, t \in D$$

Then F is called pre-antiderivative of f and indefinite integral of f is defined by

$$\int f(t)\Delta t = F(t) + C$$

Where C is a constant and

$$\int_s^t f(\tau)\Delta\tau = F(t) - F(s), \qquad \text{for } s, t \in \mathbb{T}.$$

A function $f: \mathbb{T} \to \mathbf{R}$ is said to be rd-continuous, provided it is continuous at every right-dense point and its left sided limit exists at every left dense pointin $\mathbb{T}$. We denote the set of functions $f: \mathbb{T} \to \mathbf{R}$ whose nth order derivative is rd-continuous by $C_{rd}^n(\mathbb{T}, \boldsymbol{R}), n \in \mathbb{N}$. The importance of rd-continuous functions is revealed by the following existence result by Hilger [8]. Every rd-continuous function possesses an antiderivative.

The generalized polynomial is the function $h_k : \mathbb{T}^2 \to \mathbf{R}$, $k \in \mathbb{N}_0$, defined recursively as follows $h_0(t, s) = 1$ for all $s, t \in \mathbb{T}$, and given $h_k$fork $\in \mathbb{N}_0$, the functionh_{k+1} is given by:

$$h_{k+1}(t, s) = \int_s^t h_k(\tau, s)\, \Delta\tau \qquad \text{for all } s, t \in \mathbb{T}.$$

3. MAIN RESULTS

To make the presentation compact and easier to understand, we make some symbolical representations. Here, $n, m, q \in \mathbb{N}$and k a non-negative integer.

$$P_n(x,r)=\begin{cases}h_n(r,a_1), & a_1\le r\le x\\ h_n(r,b_1), & x<r\le b_1\end{cases}. \quad Q_m(y,s)=\begin{cases}h_m(s,a_2), & a_2\le s\le y\\ h_m(s,b_2), & y<s\le b_2\end{cases}.$$

$$S_q(z,t)=\begin{cases}h_q(t,a_3), & a_3\le t\le z\\ h_q(t,b_3), & z<t\le b_3\end{cases}. \quad w_{m,k}(x)=X_m(x)f^{\Delta^{m-1}}\left(\sigma^k(x)\right).$$

$$X_{k+1}(x)=(-1)^k h_{k+1}(x,a_1)+(-1)^{k+1}h_{k+1}(x,b_1)$$

$$Y_{k+1}(y)=(-1)^k h_{k+1}(y,a_2)+(-1)^{k+1}h_{k+1}(y,b_2)$$

$$Z_{k+1}(z)=(-1)^k h_{k+1}(z,a_3)+(-1)^{k+1}h_{k+1}(z,b_3)$$

Lemma 1. Let $f:[a_1,b_1]_{\mathbb{T}}\to\boldsymbol{R}$be a function such that $f\in C_{rd}^k([a_1,b_1]_{\mathbb{T}},\boldsymbol{R}), 0\le k\le n-1$. If graininess, μ, is constant. Then, for $x\in[a_1,b_1]$, the following identity holds:

$$\int_{a_1}^{b_1} f\left(\sigma^n(t)\right)\Delta t-\sum_{k=0}^{n-1}X_{n-k}(x)\,f^{\Delta^{n-k-1}}\left(\sigma^k(x)\right)=(-1)^n\int_{a_1}^{b_1}P_n(x,t)\,f^{\Delta^n}(t)\Delta t \qquad (2)$$

Proof. Consider

$$I_{n,0}(x)=(-1)^n J_{n,0}(a_1,x,b_1)=(-1)^n\int_{a_1}^{b_1}P_n(x,t)\,f^{\Delta^n}\left(\sigma^0(t)\right)\Delta t.$$

$$J_{n,0}(a_1,x,x)=\int_{a_1}^{x}h_n(t,a_1)\,f^{\Delta^n}\left(\sigma^0(t)\right)\Delta t=\left|h_n(t,a_1)f^{\Delta^{n-1}}\left(\sigma^0(t)\right)\right|_{a_1}^{x}-\int_{a_1}^{x}h_n(t,a_1)\,f^{\Delta^{n-1}}\left(\sigma^1(t)\right)\Delta t$$

$$=h_n(x,a_1)f^{\Delta^{n-1}}\left(\sigma^0(x)\right)-J_{n-1,1}(a_1,x,x)(3)$$

Similarly:

$$J_{n,0}(x,x,b_1)=\int_{x}^{b_1}h_n(t,b_1)\,f^{\Delta^n}\left(\sigma^0(t)\right)\Delta t=-h_n(x,b_1)f^{\Delta^{n-1}}\left(\sigma^0(x)\right)-J_{n-1,1}(x,x,b_1) \qquad (4)$$

Addition of (3) and (4) yields:

$$I_{n,0}(x)-I_{n-1,1}(x)=-w_{n,0}(x). \qquad (5)$$

Similarly:

$$I_{n-1,1}(x)-I_{n-2,2}(x)=-w_{n-1,1}(x)$$
$$\vdots$$
$$I_{1,n-1}(x)-I_{0,n}(x)=-w_{1,n-1}(x).$$

Addition of these relations yields:

$$I_{n,0}(x)=I_{0,n}(x)-\sum_{k=0}^{n-1}w_{n-k,k}(x).$$

And hence (2) is proved.

Remark 1. The above identity (2) generalizes the identity in [6]. In particular for $\mathbb{T}=\boldsymbol{R}$ it is re-captured. Moreover by using the properties of modulus and supremum norm, for $\mathbb{T}=\boldsymbol{R}$,we get [6, Theorem 1] and for $n=1$ we recapture the classical Ostrowski inequality (1).

Theorem 1. Let $a_i,b_i\in\mathbb{T}, 1\le i\le 3$; let $g:\prod_{i=1}^{3}[a_i,b_i]_{\mathbb{T}}\to\boldsymbol{R}$be rd-continuous function such that the partial derivatives $\frac{\partial^{k+l+p}g(.,.,.)}{\Delta x^k\Delta y^l\Delta z^p}, 0\le k\le n-1; 0\le l\le m-1; 0\le p\le q-1$ exist and are continuous on $\prod_{i=1}^{3}[a_i,b_i]_{\mathbb{T}}$.Then, for $(x,y,z)\in\Pi_{i=1}^{3}[a_i,b_i]$, the following inequality holds:

$$\Bigg|\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} g\big(\sigma^n(r),\sigma^m(s),\sigma^q(t)\big)\Delta t\Delta s\Delta r-\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\sum_{p=0}^{q-1}X_{k+1}(x)Y_{l+1}(y)Z_{p+1}(z)$$

$$\times\frac{\partial^{k+l+p}g\left(\sigma^{n-k-1}(x),\sigma^m(y),\sigma^q(z)\right)}{\Delta x^k\Delta y^l\Delta z^p}+(-1)^{q+1}\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}X_{k+1}(x)Y_{l+1}(y)\int_{a_3}^{b_3}S_q(z,t)$$

$$\times\frac{\partial^{k+l+q}g\left(\sigma^{n-k-1}(x),\sigma^m(y),\sigma^q(t)\right)}{\Delta x^k\Delta y^l\Delta t^q}\Delta t+(-1)^{m+1}\sum_{k=0}^{n-1}\sum_{p=0}^{q-1}X_{k+1}(x)Z_{p+1}(z)\int_{a_2}^{b_2}Q_m(y,s)$$

$$\times\frac{\partial^{k+m+p}g\left(\sigma^{n-k-1}(x),\sigma^m(s),\sigma^q(z)\right)}{\Delta x^k\Delta s^m\Delta z^p}\Delta s+(-1)^{m+q+1}\sum_{k=0}^{n-1}X_{k+1}(x)$$

$$\times\int_{a_2}^{b_2}\int_{a_3}^{b_3}Q_m(y,s)S_q(z,t)\frac{\partial^{k+m+q}g\left(\sigma^{n-k-1}(x),\sigma^m(s),\sigma^q(t)\right)}{\Delta x^k\Delta s^m\Delta t^q}\Delta t\Delta s+(-1)^{n+1}$$

$$\times\sum_{l=0}^{m-1}\sum_{p=0}^{q-1}Y_{l+1}(y)Z_{p+1}(z)\int_{a_1}^{b_1}P_n(x,r)\frac{\partial^{n+l+p}g\big(r,\sigma^m(y),\sigma^q(z)\big)}{\Delta r^n\Delta y^l\Delta z^p}\Delta r+(-1)^{n+q+1}$$

$$\times\sum_{l=0}^{m-1}Y_{l+1}(y)\int_{a_1}^{b_1}\int_{a_3}^{b_3}P_n(x,r)S_q(z,t)\frac{\partial^{n+l+q}g\big(r,\sigma^m(y),\sigma^q(t)\big)}{\Delta r^n\Delta y^l\Delta t^q}\Delta t\Delta r+(-1)^{m+n+1}$$

$$\times\sum_{p=0}^{q-1}Z_{P+1}(z)\int_{a_1}^{b_1}\int_{a_2}^{b_2}P_n(x,r)Q_m(y,s)\frac{\partial^{n+m+p}g\big(r,\sigma^m(s),\sigma^q(z)\big)}{\Delta r^n\Delta s^m\Delta z^p}\Delta s\Delta r\Bigg|$$

$$\leq\left\|\frac{\partial^{n+m+q}g}{\Delta r^n\Delta s^m\Delta t^q}\right\|_\infty\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\big|P_n(x,r)Q_m(y,s)S_q(z,t)\big|\Delta t\Delta s\Delta r. \tag{6}$$

Proof. Consider the following identity by lemma 1:

$$\int_{a_1}^{b_1}f\big(\sigma^n(r)\big)\Delta r=\sum_{k=0}^{n-1}X_{k+1}(x)\,f^{\Delta^k}\left(\sigma^{n-k-1}(x)\right)+(-1)^n\int_{a_1}^{b_1}P_n(x,r)f^{\Delta^n}(r)\Delta r. \tag{7}$$

For the partial mapping $g\big(.\,,\sigma^m(s),\sigma^q(t)\big)$ for $(s,t)\in\prod_{i=2}^{3}[a_i,b_i]$

$$\int_{a_1}^{b_1}g\big(\sigma^n(r),\sigma^m(s),\sigma^q(t)\big)\Delta r=\sum_{k=0}^{n-1}X_{k+1}(x)\frac{\partial^k g\left(\sigma^{n-k-1}(x),\sigma^m(s),\sigma^q(t)\right)}{\Delta x^k}$$

$$+(-1)^n\int_{a_1}^{b_1}P_n(x,r)\frac{\partial^n g\big(r,\sigma^m(s),\sigma^q(t)\big)}{\Delta r^n}\Delta r. \tag{8}$$

Δ −integrating over $s\in[a_2,b_2]$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}g\big(\sigma^n(r),\sigma^m(s),\sigma^q(t)\big)\Delta s\Delta r=\sum_{k=0}^{n-1}X_{k+1}(x)\int_{a_2}^{b_2}\frac{\partial^k g\left(\sigma^{n-k-1}(x),\sigma^m(s),\sigma^q(t)\right)}{\Delta x^k}\Delta s$$

$$+(-1)^n\int_{a_1}^{b_1}\int_{a_2}^{b_2}P_n(x,r)\frac{\partial^n g\big(r,\sigma^m(s),\sigma^q(t)\big)}{\Delta r^n}\Delta s\Delta r. \tag{9}$$

For the partial mapping $\frac{\partial^k g\left(\sigma^{n-k-1}(x), \;.\;, \sigma^q(t)\right)}{\Delta x^k}$ on $[a_2, b_2]$ for $(x,t) \in [a_1, b_1] \times [a_3, b_3]$

$$\int_{a_2}^{b_2} \frac{\partial^k g\left(\sigma^{n-k-1}(x), \sigma^m(s), \sigma^q(t)\right)}{\Delta x^k} \Delta s = \sum_{l=0}^{m-1} \frac{\partial^{k+1} g\left(\sigma^{n-k-1}(x), \sigma^m(y), \sigma^q(t)\right)}{\Delta x^k \Delta y^l}$$
$$\times Y_{l+1}(y) + (-1)^m \int_{a_2}^{b_2} Q_m(y,s) \frac{\partial^{k+m} g\left(\sigma^{n-k-1}(x), \sigma^m(s), \sigma^q(t)\right)}{\Delta x^k \Delta s^m} \Delta s. \tag{10}$$

Similarly:

$$\int_{a_2}^{b_2} \frac{\partial^n g\left(r, \sigma^m(s), \sigma^q(t)\right)}{\Delta r} \Delta s = \sum_{l=0}^{m-1} \frac{\partial^{n+1} g\left(r, \sigma^m(y), \sigma^q(t)\right)}{\Delta r^n \Delta y^l} Y_{l+1}(y)$$
$$+(-1)^m \int_{a_2}^{b_2} Q_m(y,s) \frac{\partial^{n+m} g\left(r, \sigma^m(s), \sigma^q(t)\right)}{\Delta r^n \Delta s^m} \Delta s. \tag{11}$$

From (9) – (11)

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} g\left(\sigma^n(r), \sigma^m(s), \sigma^q(t)\right) \Delta s \Delta r$$
$$= \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} X_{k+1}(x) Y_{l+1}(y) \frac{\partial^{k+1} g\left(\sigma^{n-k-1}(x), \sigma^m(y), \sigma^q(t)\right)}{\Delta x^k \Delta y^l}$$
$$+ (-1)^m \sum_{k=0}^{n-1} X_{k+1}(x) \int_{a_2}^{b_2} Q_m(y,s) \frac{\partial^{k+m} g\left(\sigma^{n-k-1}(x), \sigma^m(s), \sigma^q(t)\right)}{\Delta x^k \Delta s^m} \Delta s$$
$$+ (-1)^n \sum_{l=0}^{m-1} Y_{l+1}(y) \int_{a_1}^{b_1} P_n(x,r) \frac{\partial^{n+l} g\left(r, \sigma^m(y), \sigma^q(t)\right)}{\Delta r^n \Delta y^l} \Delta r$$
$$+ (-1)^{m+n} \int_{a_1}^{b_1} \int_{a_2}^{b_2} P_n(x,r) Q_m(y,s) \frac{\partial^{n+m} g\left(r, \sigma^m(s), \sigma^q(t)\right)}{\Delta r^n \Delta s^m} \Delta s \Delta r. \tag{12}$$

Δ −integrating over $t \in [a_3, b_3]$

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} g\left(\sigma^n(r), \sigma^m(s), \sigma^q(t)\right) \Delta t \Delta s \Delta r = \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} X_{k+1}(x) Y_{l+1}(y)$$
$$\times \int_{a_3}^{b_3} \frac{\partial^{k+1} g\left(\sigma^{n-k-1}(x), \sigma^m(y), \sigma^q(t)\right)}{\Delta x^k \Delta y^l} \Delta t + (-1)^m \sum_{k=0}^{n-1} X_{k+1}(x) \int_{a_2}^{b_2} \int_{a_3}^{b_3} Q_m(y,s)$$
$$\times \frac{\partial^{k+m} g\left(\sigma^{n-k-1}(x), \sigma^m(s), \sigma^q(t)\right)}{\Delta x^k \Delta s^m} \Delta t \Delta s + (-1)^n \sum_{l=0}^{m-1} Y_{l+1}(y) \int_{a_1}^{b_1} \int_{a_3}^{b_3} P_n(x,r)$$
$$\times \frac{\partial^{n+1} g\left(r, \sigma^m(y), \sigma^q(t)\right)}{\Delta r^n \Delta y^l} \Delta t \Delta r + (-1)^{m+n} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} P_n(x,r) Q_m(y,s)$$
$$\times \frac{\partial^{n+m} g\left(r, \sigma^m(s), \sigma^q(t)\right)}{\Delta r^n \Delta s^m} \Delta t \Delta s \Delta r. \tag{13}$$

Where,

$$\int_{a_3}^{b_3} \frac{\partial^{k+1} g\left(\sigma^{n-k-1}(x), \sigma^m(y), \sigma^q(t)\right)}{\Delta x^k \Delta y^l} \Delta t = \sum_{p=0}^{q-1} Z_{p+1}(z) \frac{\partial^{k+l+p} g\left(\sigma^{n-k-1}(x), \sigma^m(y), \sigma^q(z)\right)}{\Delta x^k \Delta y^l \Delta z^p}$$

$$+(-1)^q \int_{a_3}^{b_3} S_q(z,t) \frac{\partial^{k+l+q} g\left(\sigma^{n-k-1}(x), \sigma^m(y), \sigma^q(t)\right)}{\Delta x^k \Delta y^l \Delta t^q} \Delta t. \tag{14}$$

$$\int_{a_3}^{b_3} \frac{\partial^{k+m} g\left(\sigma^{n-k-1}(x), \sigma^m(s), \sigma^q(t)\right)}{\Delta x^k \Delta s^m} \Delta t = \sum_{p=0}^{q-1} Z_{p+1}(z) \frac{\partial^{k+m+p} g\left(\sigma^{n-k-1}(x), \sigma^m(s), \sigma^q(z)\right)}{\Delta x^k \Delta s^m \Delta z^p}$$

$$+(-1)^q \int_{a_3}^{b_3} S_q(z,t) \frac{\partial^{k+m+q} g\left(\sigma^{n-k-1}(x), \sigma^m(s), \sigma^q(t)\right)}{\Delta x^k \Delta s^m \Delta t^q} \Delta t. \tag{15}$$

$$\int_{a_3}^{b_3} \frac{\partial^{n+l} g\left(r, \sigma^m(y), \sigma^q(t)\right)}{\Delta r^n \Delta y^l} \Delta t = \sum_{p=0}^{q-1} Z_{p+1}(z) \frac{\partial^{n+l+p} g\left(r, \sigma^m(y), \sigma^q(z)\right)}{\Delta r^n \Delta y^l \Delta z^p}$$

$$+(-1)^q \int_{a_3}^{b_3} S_q(z,t) \frac{\partial^{n+l+q} g\left(r, \sigma^m(y), \sigma^q(t)\right)}{\Delta r^n \Delta y^l \Delta t^q} \Delta t. \tag{16}$$

And

$$\int_{a_3}^{b_3} \frac{\partial^{n+m} g\left(r, \sigma^m(s), \sigma^q(t)\right)}{\Delta r^n \Delta s^m} \Delta t = \sum_{p=0}^{q-1} Z_{p+1}(z) \frac{\partial^{n+m+p} g\left(r, \sigma^m(s), \sigma^q(z)\right)}{\Delta r^n \Delta s^m \Delta z^p}$$

$$+(-1)^q \int_{a_3}^{b_3} S_q(z,t) \frac{\partial^{n+m+q} g\left(r, \sigma^m(s), \sigma^q(t)\right)}{\Delta r^n \Delta s^m \Delta t^q} \Delta t. \tag{17}$$

From (13) – (17), we have

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} g\left(\sigma^n(r), \sigma^m(s), \sigma^q(t)\right) \Delta t \Delta s \Delta r = \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \sum_{p=0}^{q-1} X_{k+1}(x) Y_{l+1}(y) Z_{p+1}(z)$$

$$\times \frac{\partial^{k+l+p} g\left(\sigma^{n-k-1}(x), \sigma^m(y), \sigma^q(z)\right)}{\Delta x^k \Delta y^l \Delta z^p} + (-1)^q \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} X_{k+1}(x) Y_{l+1}(y) \int_{a_3}^{b_3} S_q(z,t)$$

$$\times \frac{\partial^{k+l+q} g\left(\sigma^{n-k-1}(x), \sigma^m(y), \sigma^q(t)\right)}{\Delta x^k \Delta y^l \Delta t^q} \Delta t + (-1)^m \sum_{k=0}^{n-1} \sum_{p=0}^{q-1} X_{k+1}(x) Z_{p+1}(z) \int_{a_2}^{b_2} Q_m(y,s)$$

$$\times \frac{\partial^{k+m+p} g\left(\sigma^{n-k-1}(x), \sigma^m(s), \sigma^q(z)\right)}{\Delta x^k \Delta s^m \Delta z^p} \Delta s + (-1)^{m+q} \sum_{k=0}^{n-1} X_{k+1}(x)$$

$$\times \int_{a_2}^{b_2} \int_{a_3}^{b_3} Q_m(y,s) S_q(z,t) \frac{\partial^{k+m+q} g\left(\sigma^{n-k-1}(x), \sigma^m(s), \sigma^q(t)\right)}{\Delta x^k \Delta s^m \Delta t^q} \Delta t \Delta s + (-1)^n$$

$$\times \sum_{l=0}^{m-1} \sum_{p=0}^{q-1} Y_{l+1}(y) Z_{p+1}(z) \int_{a_1}^{b_1} P_n(x,r) \frac{\partial^{n+l+p} g\left(r, \sigma^m(y), \sigma^q(z)\right)}{\Delta r^n \Delta y^l \Delta z^p} \Delta r + (-1)^{n+q}$$

$$\times\sum_{l=0}^{m-1} Y_{l+1}(y)\int_{a_1}^{b_1}\int_{a_3}^{b_3} P_n(x,r)S_q(z,t)\frac{\partial^{n+l+q} g\left(r,\sigma^m(y),\sigma^q(t)\right)}{\Delta r^n \Delta y^l \Delta t^q}\Delta t\Delta r+(-1)^{m+n}$$

$$\times\sum_{p=0}^{q-1} Z_{P+1}(z)\int_{a_1}^{b_1}\int_{a_2}^{b_2} P_n(x,r)Q_m(y,s)\frac{\partial^{n+m+p} g\left(r,\sigma^m(s),\sigma^q(z)\right)}{\Delta r^n \Delta s^m \Delta z^p}\Delta s\Delta r$$

$$+(-1)^{n+m+q}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} P_n(x,r)Q_m(y,s)S_q(z,t)\frac{\partial^{n+m+q} g\left(r,\sigma^m(s),\sigma^q(t)\right)}{\Delta r^n \Delta s^m \Delta t^q}\Delta t\Delta s\Delta r. \quad (18)$$

The relation (6) follows from (18).

Remark 2. Relation (18) generalizes the relation (2.5) in [14], and for $\mathbb{T}=\boldsymbol{R}$it is re-captured. By setting $x\mapsto\frac{a_1+b_1}{2}$; $y\mapsto\frac{a_2+b_2}{2}$ and $z\mapsto\frac{a_3+b_3}{2}$in (18), we get the generalization of [14, Corollary 2.3] and for $\mathbb{T}=\boldsymbol{R}$[14, Corollary 2.3] is re-captured. For $\mathbb{T}=\boldsymbol{R}$ and by using the different norms such as L_1, L_∞ and L_α , $\alpha>1$, norms and some mathematical calculations, from (18) [14, Theorem 2.4] is re-captured. Similarly, by setting $\mathbb{T}=\boldsymbol{R}$, we re-capture [14, Corollaries 2.6, 2.7] at the respective boundary points.

Corollary 1. Let $a_i,b_i\in\mathbb{T},\ 1\le i\le 3$; let $g:\prod_{i=1}^{3}[a_i,b_i]_{\mathbb{T}}\to\boldsymbol{R}$be rd-continuous function such that the partial derivatives $\frac{\partial^{k+l+p} g(.,.,.)}{\Delta r^k\Delta s^t\Delta t^p}, 0\le k\le 1; 0\le l\le 1; 0\le p\le 1$ exist and are continuous on $\prod_{i=1}^{3}[a_i,b_i]_{\mathbb{T}}$. Then, for $(x,y,z)\in\prod_{i=1}^{3}[a_i,b_i]$, the following inequality holds:

$$\Bigg|\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{a_3} g\left(\sigma(r),\sigma(s),\sigma(t)\right)\Delta t\Delta s\Delta r - X_1(x)Y_1(y)Z_1(z)g\left(x,\sigma(y),\sigma(z)\right)$$

$$+X_1(x)Y_1(y)\int_{a_3}^{b_3} S_1(z,t)\frac{\partial g\left(x,\sigma(y),\sigma(t)\right)}{\Delta t}\Delta t$$

$$+X_1(x)Z_1(z)\int_{a_2}^{b_2} Q_1(y,s)\frac{\partial g\left(x,\sigma(s),\sigma(z)\right)}{\Delta s}\Delta s$$

$$-X_1(x)\int_{a_2}^{b_2}\int_{a_3}^{b_3} Q_1(y,s)S_1(z,t)\frac{\partial^2 g\left(x,\sigma(s),\sigma(t)\right)}{\Delta s\Delta t}\Delta s\Delta t$$

$$+Y_1(y)Z_1(z)\int_{a_1}^{b_1} P_1(x,r)\frac{\partial g\left(r,\sigma(y),\sigma(z)\right)}{\Delta r}\Delta r$$

$$-Y_1(y)\int_{a_1}^{b_1}\int_{a_3}^{b_3} P_1(x,r)S_1(z,t)\frac{\partial^2 g\left(r,\sigma(y),\sigma(t)\right)}{\Delta r\Delta t}\Delta t\Delta r -$$

$$-Z_1(z)\int_{a_1}^{b_1}\int_{a_2}^{b_2} P_1(x,r)Q_1(y,s)\frac{\partial^2 g\left(r,\sigma(s),\sigma(z)\right)}{\Delta r\Delta s}\Delta s\Delta r\Bigg|$$

$$\le\left\|\frac{\partial^3 g}{\Delta r\Delta s\Delta t}\right\|_\infty\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{a_3}|P_1(x,r)Q_1(y,s)S_1(z,t)|\Delta t\Delta s\Delta r. \quad (19)$$

Remark 3. Relation (19) is the generalized Ostrowski type inequality in triple integral on general time scales.

The following result is the classical Ostrowski type inequality for triple integrals.

Corollary 2.(continuous case) Let $\mathbb{T}=\boldsymbol{R},\ n=m=q=1;\ k=l=p=0$ and

$$\left\|g'''_{x,y,z}\right\|_\infty = \sup_{(x,y,z)\in(a_1,b_1)\times(a_2,b_2)\times(a_3,b_3)} \left|\frac{\partial^3 g(x,y,z)}{\partial x\partial y\partial z}\right| < \infty.$$

Then (19) takes the form:

$$\Bigg|\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} g(r,s,t)dtdsdr - (b_3-a_3)(b_2-a_2)(b_1-a_1)g(x,y,z)$$
$$+(b_3-a_3)(b_1-a_1)\int_{a_2}^{b_2} g(x,s,z)ds + (b_2-a_2)(b_1-a_1)\int_{a_3}^{b_3} g(x,y,t)dt$$
$$+(b_3-a_3)(b_2-a_2)\int_{a_1}^{b_1} g(r,y,z)dr - (b_3-a_3)\int_{a_1}^{b_1}\int_{a_2}^{b_2} g(r,s,z)dsdr$$
$$-(b_2-a_2)\int_{a_1}^{b_1}\int_{a_3}^{b_3} g(r,y,t)dtdr - (b_1-a_1)\int_{a_2}^{b_2}\int_{a_3}^{b_3} g(x,s,t)dtds\Bigg|$$
$$\le \left\|g'''_{r,s,t}\right\|_\infty \frac{(r-a_1)^2+(b_1-r)^2}{2}\frac{(s-a_2)^2+(b_2-s)^2}{2}\frac{(t-a_3)^2+(b_3-t)^2}{2}. \qquad (20)$$

4. APPLICATIONS FOR GENERALIZED POLYNOMIAL

Example 1. (Discrete case) Let $\mathbb{T}=\mathbb{Z}$. Then, we have

$$h_k(t,s) = \frac{(t-s)^{(k)}}{k!} = (-1)^k\frac{(s-t+k)^{(k)}}{k!}$$

for $s,t \in \mathbb{T}$ and $k \in \mathbb{N}$, where the usual factorial function, (k), is defined as $n^{(k)} = \frac{n!}{k!}$ for $k\in\mathbb{N}$ and $(n)^{(0)} = 1$ for $n\in\mathbb{Z}$. In this case the inequality (6) reduces to the following inequality:

$$\Bigg|\sum_{r=a_1}^{b_1-1}\sum_{s=a_2}^{b_2-1}\sum_{t=a_3}^{b_3-1} g(r+n,s+m,q+t) - \sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\sum_{p=0}^{q-1}\sum_{\alpha=0}^{p}\sum_{\beta=0}^{l}\sum_{\Gamma=0}^{k}(-1)^{k+l+p-\alpha-\beta-\Gamma}$$
$$\times\binom{p}{\alpha}\binom{l}{\beta}\binom{k}{\Gamma}\frac{(b_1-x+k+1)^{(k+1)}+(-1)^k(x-a_1)^{(k+1)}}{(k+1)!}$$
$$\times\frac{(b_2-y+l+1)^{(l+1)}+(-1)^l(y-a_2)^{(l+1)}}{(l+1)!}\frac{(b_3-z+p+1)^{(p+1)}+(-1)^p(z-a_3)^{(p+1)}}{(p+1)!}$$
$$\times g(x+n-k-1+\Gamma,\mathrm{y}+\mathrm{m}+\beta,\mathrm{z}+\mathrm{q}+\alpha) - \sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\sum_{\delta=0}^{q}\sum_{\beta=0}^{l}\sum_{\Gamma=0}^{k}\sum_{t=a_3}^{b_3-1}(-1)^{k+l-\delta-\beta-\Gamma}$$
$$\times\binom{q}{\delta}\binom{l}{\beta}\binom{k}{\Gamma}\frac{(b_1-x+k+1)^{(k+1)}+(-1)^k(x-a_1)^{(k+1)}}{(k+1)!}S_q(z,t)$$
$$\times\frac{(b_2-y+l+1)^{(l+1)}+(-1)^l(y-a_2)^{(l+1)}}{(l+1)!}g(x+n-k-1+\Gamma,\mathrm{y}+\mathrm{m}+\beta,\mathrm{t}+\mathrm{q}+\delta)$$
$$-\sum_{k=0}^{n-1}\sum_{p=0}^{q-1}\sum_{s=a_2}^{b_2-1}\sum_{\alpha=0}^{p}\sum_{\eta=0}^{m}\sum_{\Gamma=0}^{k}(-1)^{k+p-\alpha-\eta-\Gamma}\binom{p}{\alpha}\binom{m}{\eta}\binom{k}{\Gamma}\frac{(b_1-x+k+1)^{(k+1)}+(-1)^k(x-a_1)^{(k+1)}}{(k+1)!}$$

$$\times \frac{(b_3 - z + p + 1)^{(p+1)} + (-1)^p (z - a_3)^{(p+1)}}{(p+1)!} g(x + n - k - 1 + \Gamma, s + m + \eta, z + q + \delta)$$

$$- \sum_{k=0}^{n-1} \sum_{s=a_2}^{b_2-1} \sum_{t=a_3}^{b_3-1} \sum_{\delta=0}^{q} \sum_{\eta=0}^{m} \sum_{\Gamma=0}^{k} (-1)^{k-\delta-\eta-\Gamma} \frac{(b_1 - x + k + 1)^{(k+1)} + (-1)^k (x - a_1)^{(k+1)}}{(k+1)!}$$

$$\times g(x + n - k - 1 + \Gamma, s + m + \eta, t + q + \delta) Q_m(y,s) S_q(z,t)$$

$$- \sum_{l=0}^{m-1} \sum_{p=0}^{q-1} \sum_{r=a_1}^{b_1-1} \sum_{\psi=0}^{n} \sum_{\beta=0}^{l} \sum_{\alpha=0}^{p} (-1)^{l+p-\psi-\beta-\alpha} P_n(x,r) \frac{(b_2 - y + l + 1)^{(l+1)} + (-1)^l (y - a_2)^{(l+1)}}{(l+1)!}$$

$$\times \frac{(b_3 - z + p + 1)^{(p+1)} + (-1)^p (z - a_3)^{(p+1)}}{(p+1)!} g(r + \psi, y + m + \beta, z + q + \alpha)$$

$$- \sum_{l=0}^{m-1} \sum_{r=a_1}^{b_1-1} \sum_{t=a_3}^{b_3-1} \sum_{\psi=0}^{n} \sum_{\beta=0}^{l} \sum_{\delta=0}^{q} (-1)^{l-\psi-\beta-\delta} P_n(x,r) S_q(z,t) \frac{(b_2 - y + l + 1)^{(l+1)} + (-1)^l (y - a_2)^{(l+1)}}{(l+1)!}$$

$$\times g(r + \psi, y + m + \beta, t + q + \delta) - \sum_{p=0}^{q-1} \sum_{r=a_1}^{b_1-1} \sum_{s=a_2}^{b_2-1} \sum_{\alpha=0}^{p} \sum_{\eta=0}^{m} \sum_{\psi=0}^{n} (-1)^{p-\psi-\eta-\alpha} P_n(x,r) Q_m(y,s)$$

$$\times \frac{(b_3 - z + p + 1)^{(p+1)} + (-1)^p (z - a_3)^{(p+1)}}{(p+1)!} g(r + \psi, s + m + \eta, z + q + \alpha) \Bigg|$$

$$\le M \sum_{r=a_1}^{b_1-1} \sum_{s=a_2}^{b_2-1} \sum_{t=a_3}^{b_3-1} \left| P_n(x,r) Q_m(y,s) S_q(z,t) \right|.$$

Where, M =maximum value of the absolute value of

$$\sum_{\psi=0}^{n} \sum_{\eta=0}^{l} \sum_{\delta=0}^{q} (-1)^{n+m+q-\psi-\eta-\delta} g(r + \eta, s + m + \eta, t + q + \delta),$$

over$[a_1, b_1 - 1]_{\mathbb{Z}} \times [a_2, b_2 - 1]_{\mathbb{Z}} \times [a_3, b_3 - 1]_{\mathbb{Z}}$

Example 2. (Quantum calculus case) Let $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$.Then in this case Ostrowski inequality (19) takes the following form:

$$\Bigg| \sum_{r=0}^{log_q^{(b_1/(q\alpha_1))}} \sum_{s=0}^{log_q^{(b_2/(q\alpha_2))}} \sum_{t=0}^{log_q^{(b_3/(q\alpha_3))}} g(a_1 q^{r+1}, a_2 q^{s+1}, a_3 q^{t+1}) - (b_3 - a_3)(b_2 - a_2)$$

$$\times (b_1 - a_1) g(x, qy, qz) + (b_2 - a_2)(b_1 - a_1) \sum_{t=0}^{log_q^{(b_3/(q\alpha_3))}} S_1(z, a_3 q^t)$$

$$\times \frac{g(x, qy, q^2 t) - g(x, qy, qt)}{q(q-1)t} + (b_3 - a_3)(b_1 - a_1) \sum_{s=0}^{log_q^{(b_2/(q\alpha_2))}} Q_1(y, a_2 q^s)$$

$$\times \frac{g(x,q^2s,qz)-g(x,qs,qz)}{q(q-1)s} + (b_3-a_3)(b_2-a_2) \sum_{r=0}^{log_q^{(b_1/(q\alpha_1))}} P_1(x,a_1q^r)$$

$$\times \frac{g(qr,qy,qz)-g(r,qy,qz)}{(q-1)r} - (b_1-a_1) \sum_{s=0}^{log_q^{(b_2/(q\alpha_2))}} \sum_{t=0}^{log_q^{(b_3/(q\alpha_3))}} Q_1(y,a_2q^s)\, S_1(z,a_3q^t)$$

$$\times \frac{g(x,sq^2,tq^2)-g(x,sq^2,qt)-g(x,sq,tq^2)+g(x,sq,tq)}{st(q^2-q)^2} - (b_2-a_2)$$

$$\times \sum_{r=0}^{log_q^{(b_1/(q\alpha_1))}} \sum_{t=0}^{log_q^{(b_3/(q\alpha_3))}} \frac{g(qr,qy,tq^2)-g(r,qy,tq^2)-g(qr,qy,qt)+g(r,qy,tq)}{rqt(q-1)^2}$$

$$\times P_1(x,a_1q^r)\ S_1(z,a_3q^t) - (b_3-a_3) \sum_{r=0}^{log_q^{(b_1/(q\alpha_1))}} \sum_{s=0}^{log_q^{(b_2/(q\alpha_2))}} P_1(x,a_1q^r)\, Q_1(y,a_2q^s)$$

$$\left. \times \frac{g(qr,sq^2,qz)-g(qr,qs,qz)-g(r,sq^2,qz)+g(r,qs,qz)}{rqs(q-1)^2} \right|$$

$$\leq M \sum_{r=0}^{log_q^{(b_1/(q\alpha_1))}} \sum_{s=0}^{log_q^{(b_2/(q\alpha_2))}} \sum_{t=0}^{log_q^{(b_3/(q\alpha_3))}} |P_1(x,a_1q^r)Q_1(y,a_2q^s)S_1(z,a_3q^t)|.$$

Where, M is the maximum value of the absolute value of

$$\frac{g(qr,qs,qt)+g(r,s,qt)+g(r,qs,t)+g(qr,s,t)}{rst(q-1)^3} - \frac{g(r,qs,qt)+g(qr,s,qt)+g(qr,qs,t)+g(r,s,t)}{rst(q-1)^3},$$

over $[a_1,b_1/q]_{q^{\mathbb{N}_0}} \times [a_2,b_2/q]_{q^{\mathbb{N}_0}} \times [a_3,b_3/q]_{q^{\mathbb{N}_0}}$.

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