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Research Article

Concavity Solutions of Second-Order Differential Equations

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Abstract: In this article, we consider varieties of second-order linear differential equations in the unit disk. We show that the solutions of the second-order linear differential equations are concave univalent functions under some conditions.

Keywords: Analytic function, differential equation, concave function , univalent function

AMS Mathematics Subject Classification: 30C45

1. INTRODUCTION

Let A denote the class of functions normalized by

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k, \qquad (z\in \mathrm{D}), \tag{1}$$

which are analytic in the open unit disk $\mathrm{D}=\{z:|z|<1\}$ on the complex plane C. For functions $f\in A$ with $f'(z)\neq 0 \;\; (z\in \mathrm{D})$, we define the Schwarzian derivative of f by

$$S(f,z)=\left(\frac{f''(z)}{f'(z)}\right)'-\frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2, (f\in A;\; f'(z)\neq 0, z\in \mathbf{D}).$$

Let B_k denote the class of bounded functions $q(z)=q_1z+q_2z^2+...$ analytic in the unit disk D, for which $|q(z)|<K$. If $g(z)\in B_k$, then by using the Schwarz lemma [8], the function $q(z)$ defined by $q(z)=z^{-1/2}\int_0^z g(t)t^{-1/2}dt$

is also in B_k. Thus, in terms of derivatives , we have

$$\left|\frac{1}{2}q(z)+zq'(z)\right|<K \Rightarrow |q(z)|<K, (z\in \mathrm{D}). \tag{2}$$

If we let

$$\psi(u,v)=\frac{1}{2}u+v.$$

We can write (2) as

$$|\psi(q(z),zq'(z))|<K \Rightarrow |q(z)|<K. \tag{3}$$

Saitoh [11] and Millar [7] showed that (3) holds true for functions $\psi(u,v)$ in the class H_k given by Definition 1.1. below.

Definition 1.1 (see [7]) Let H_k be the set of complex functions $\psi(u,v)$ satisfying the following conditions:

i. $\psi(u,v)$ is continuous in a domain $\mathrm{D}\subset \mathbf{C}\times\mathbf{C}$;
ii. $(0,0)\in \mathrm{D}$ and $|\psi(0,0)|<K$;
iii. $|\psi(Ke^{i\theta},Te^{i\theta})|\geq K$ when $(Ke^{i\theta},Te^{i\theta})\in \mathrm{D}$, θ is real and $T\geq K$.

Definition 1.2 (see [6]) Let $\psi\in H_k$ with corresponding domain D. We denote by $B_k(\psi)$ those functions $q(z)=q_1z+q_2z^2+...$ which are analytic in D satisfying :

i. $(q(z),zq'(z))\in \mathrm{D}$,

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ii. $|\psi(q(z), zq'(z))| < K \ (z \in \mathbf{D})$.

Many other authors also studied the geometric properties solutions of a class of second-order linear differential equations, for example one can refer to [1, 4, 6, 7, 10, 11, 12].

We now state the following result due to Miller [7].

Theorem 1.3 (Miller [7]**)** Let $p(z)$ be an analytic function in the unit disk D with $|zp(z)| < 1$. Let $v(z), z \in \mathbf{D}$, be the unique solution of

$$v''(z) + p(z)v(z) = 0,$$

with $v(0) = 0$ and $v'(0) = 1$. Then, $\left|\frac{zv'(z)}{v(z)} - 1\right| < 1$ and $v(z)$ is a starlike conformal map of the unit disk D.

Theorem 1.3 is related rather closely to some earlier results of Robertson [10] and Nehari [8], which we recall Theorem 1.4 and Theorem 1.5, respectively, as follows:

Theorem 1.4 (Robertson [10]**)** Let $zp(z)$ be an analytic function in D and $\Re\left\{z^2 p(z)\right\} \le \frac{\pi^2}{4}|z|^2 \ (z \in \mathbf{D})$. Then, the unique solution $v = v(z)$ of the following initial-value problem:

$$v''(z) + p(z)v(z) = 0 \quad (v(0) = 0, v'(0) = 1)$$

is univalent and starlike in D. The constant $\pi^2/4$ is the best possible one.

Theorem 1.5 (Nehari [8]) If $f(z) \in A$ and it satisfies $|S(f, z)| \le \frac{\pi^2}{2} \ (z \in \mathbf{D})$, then $f(z)$ is univalent.

The next theorems, which are due to Saitoh [11,12] and Owa et al. [9], involve several geometric properties of the solutions of the second-order linear differential equations.

Theorem 1.6 (Saitoh [11]) Let $a(z)$ and $b(z)$ be analytic in D with $\left|z\left(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}[a(z)]^2\right)\right| < \frac{1}{2}$ and $|a(z)| < 1$. Let $v(z)\ (z \in \mathbf{D})$ be the solution of the following second order linear differential equation

$$v''(z) + a(z)\, v'(z) + b(z)\, v(z) = 0,\ v(0) = 0, v'(0) = 1.$$

Then, $v(z)$ is starlike in D.

Theorem 1.7 (Owa et al [9]) Let the function $a(z)$ and $b(z)$ be analytic in D with $\Re\{za(z)\} > -2K$ and $\left|z^2\left(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}[a(z)]^2\right)\right| < K$. Also, let $v(z)$ denote the solution of the initial- value problem equation:

$$v''(z) + a(z)\, v'(z) + b(z)\, v(z) = 0,$$
$$v(0) = 0, v'(0) = 1.$$

Then,

$$1 - K - \frac{1}{2}\Re\{za(z)\} < \Re\left(\frac{zv'(z)}{v(z)}\right) <$$
$$1 + K - \frac{1}{2}\Re\{za(z)\},\ (z \in \mathbf{D}; K > 0).$$

Theorem 1.8 (Saitoh [12]) Let $p_n(z)$ be the non-constant polynomial of degree $n \ge 1$ with $|p_n(z)| < K \quad (z \in \mathbf{D};\ K > 0)$. Let $v(z)$ be the solution of the initial-value problem:

$$v''(z) + p_n(z)v(z) = 0, \quad v(0)=0;\ v'(0) = 1.$$

Then, we have

$$1 - K < \Re\left\{\frac{zv'(z)}{v(z)}\right\} < 1 + K \quad (z \in \mathbf{D}).$$

The following theorem was proved by Abubaker and Darus [1] using the third-order linear differential equation.

Theorem 1.9 (Abubaker & Darus [1]) Let $Q(z) = \sum_{n=0}^{\infty} b_n z^n$ be analytic in D with $\sum_{n=0}^{\infty} |b_n| < K \ (z \in \mathbf{D};\ K > 0)$, and let $v(z)$ denote the solution of the initial-value problem

$$v'''(z) + Q(z)\, v'(z) = 0, \quad z \in \mathbf{D}.$$

Then,

$$1 - K < \Re\left\{1 + \frac{zv''(z)}{v'(z)}\right\} < 1 + K \quad (z \in \mathbf{D};\ K > 0).$$

Next, we state the family of concave functions which is our main focus here.

A function $f:\mathbf{D}\to\mathbb{C}$ is said to belong to the family $C_0(\alpha)$ if f satisfies the following conditions:

i. f is analytic in D with the standard normalization $f(0)=f'(0)-1=0$. In addition it satisfies $f(1)=\infty$.

ii. f maps D conformally onto a set whose complement with respect to C is convex.

iii. The opening angle of $f(\mathbf{D})$ at ∞ is less than or equal $\pi\alpha, \alpha\in(1,2]$.

The class $C_0(\alpha)$ is referred to as the class of concave univalent functions and for a detailed discussion about concave functions, we refer to [2, 3, 5].

We recall the analytic characterization for functions f in $C_0(\alpha), \alpha\in(1,2]: f\in C_0(\alpha)$ if and only if $\Re P_f(z)>0, \quad z\in\mathbf{D},$ where

$$p_f(z)=\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z}-1-z\frac{f''(z)}{f'(z)}\right].$$

Before we establish our main results, we need to indicate to the following theorems to prove our results.

Theorem 1.10 (see [11]) For any $\psi\in H_k, B_k(\psi)\subset B_k, (\psi\in H_k; K>0)$.

Theorem 1.10 leads us to immediately to the following result, which was also given by Saitoh [11].

Theorem 1.11 (see [11]) Let $\psi\in H_k$ and $b(z)$ be an analytic function in D with $|b(z)|<K$. If the differential equation

$$\psi(q(z), zq'(z))=b(z), q(0)=0, q'(0)=1$$

has a solution $q(z)$ analytic in D, then $|q(z)|<K$.

The objective of the present paper is to investigate the concavity of solutions of the second-order linear differential equations.

2. MAIN RESULTS

We derive the following results by employing Theorem 1.11. First, we concentrate on the concavity of the solution of the following initial-value problem:

$$q''(z)+a(z)q'(z)+b(z)q(z)=0. \tag{4}$$

Theorem 2.1 Let $a(z), b(z)$ be analytic functions in D such that

$$\left|z^2 b(z)\right|<K, \quad (z\in \mathrm{D}; K>0). \tag{5}$$

Let $q(z), z\in\mathbf{D}$ be the solution of the initial value problem (4) in D. Then,

$$\frac{2}{\alpha-1}\left(\frac{\alpha+1}{2}-k\right)<$$
$$\Re\left\{\frac{2}{\alpha-1}\left(\frac{\alpha+1}{2}\frac{1+z}{1-z}-\frac{zq'(z)}{q(z)}\right)\right\}<$$
$$\frac{20}{\alpha-1}(\alpha+K+1), \tag{6}$$

where $\alpha\in(1,2]$.

Proof. We recall $f\in C_0(\alpha)$ if and only if $\Re P_f(z)>0$ in D, where

$$p_f(z)=\frac{2}{\alpha-1}\left(\frac{\alpha+1}{2}\frac{1+z}{1-z}-\frac{zg'(z)}{g(z)}\right)$$

with $g(z)=zf'(z)$. We note that p is analytic in D with $p(0)=1$.

If we set

$$r(z)=-\frac{zq'(z)}{q(z)} \qquad (z\in\mathbf{D}) \tag{7}$$

then, $r(z)$ is analytic in D, $r(0)=0$ and (4) becomes

$$\left(r(z)\right)^2+(1-za(z))r(z)-zr'(z)=-z^2b(z). \tag{8}$$

Thus (8) can be rewritten as

$$\psi(r(z), zr'(z))=-z^2b(z),$$

where $\psi(s,t)=s^2+(1-za(z))s-t$.

Since

i. $\psi(s,t)$ is continuous in a domain $\mathbf{D}\subset\mathbf{C}\times\mathbf{C}$;

ii. $(0,0)\in \mathbf{D}$ and $|\psi(0,0)| = 0 < K;$

iii. For $(Ke^{i\theta}, Te^{i\theta})\in \mathbf{D}$, θ is real and $T \geq K, \left|\psi(Ke^{i\theta}, Te^{i\theta})\right| = \left|K^2 e^{i\theta} + K - T\right| > T \geq K.$

We conclude that $\psi(s,t)\in H_k$.

From the hypothesis (5) and by employing Theorem 1.11 , we obtain that

$$|r(z)| < K, \quad K > 0.$$

Combine this with (7) we have

$$\left|-\frac{zq''(z)}{q'(z)}\right| < K, \quad K > 0.$$

This leads to the following relations

$$\Re\left\{\frac{2}{\alpha-1}\left(\frac{\alpha+1}{2}\frac{1+z}{1-z} - K\right)\right\} <$$
$$\Re\left\{\frac{2}{\alpha-1}\left(\frac{\alpha+1}{2}\frac{1+z}{1-z} - \frac{zq'(z)}{q(z)}\right)\right\} <$$
$$\Re\left\{\frac{2}{\alpha-1}\left(\frac{\alpha+1}{2}\frac{1+z}{1-z} + K\right)\right\}.$$

We find that

$$\frac{2}{\alpha-1}\left(\frac{\alpha+1}{2} - K\right) < \Re\left\{\frac{2}{\alpha-1}\left(\frac{\alpha+1}{2}\frac{1+z}{1-z} - K\right)\right\}$$

and

$$\Re\left\{\frac{2}{\alpha-1}\left(\frac{\alpha+1}{2}\frac{1+z}{1-z} + K\right)\right\} < \frac{2}{\alpha-1}\left(20\frac{\alpha+1}{2} + K\right).$$

We can simplify the last expressions and obtain (6). This completes the proof of the theorem.

If we take $K < \frac{\alpha+1}{2}$ in Theorem 2.1, then we deduce the following corollary.

Corollary 2.2 Let $a(z), b(z)$ be analytic functiosn in D such that $\left|z^2 b(z)\right| < \frac{\alpha+1}{2}, (z\in \mathbf{D}; \alpha\in (1,2]).$ Let $q(z)$ be the solution of the initial –value problem (4). Then, $q(z)\in C_0(\alpha)$.

Example 2.3 Let $a(z) = 0$ and $b(z) = 1$ in Corollary 2.2 . Then, for $z \to 1$ and $\alpha = 2$, the solution of the following initial-value problem :

$$q''(z) + q(z) = 0, \quad q(0) = 0, q'(0) = 1$$

is given by

$q(z) = \sin z \in C_0(2).$

We next show that the following differential equation

$$q''(z) + M(z)q(z) = 0 \tag{9}$$

has a solution $q(z)$, which is concave univalent in D.

Theorem 2.4 Let $M(z)$ be analytic functions in D such that

$$\left|z^2 M(z)\right| < K \quad (z\in \mathbf{D}, K > 0). \tag{10}$$

Let $q(z), z \in D$ be the solution of the initial value problem (9) . Then,

$$\frac{2}{\alpha-1}\left(\frac{\alpha+1}{2} - K\right) <$$
$$\Re\left\{\frac{2}{\alpha-1}\left(\frac{\alpha+1}{2}\frac{1+z}{1-z} - \frac{zq'(z)}{q(z)}\right)\right\} <$$
$$\frac{20}{\alpha-1}(\alpha + K + 1), \tag{11}$$

where $\alpha\in (1,2]$.

Proof. If we put

$$r(z) = -\frac{zq'(z)}{q(z)} \qquad (z\in \mathbf{D}), \tag{12}$$

we see that $r(z)$ is analytic in D, $r(0) = 0$ and (9) becomes

$$\left(r(z)\right)^2 + r(z) - zr'(z) = -z^2 M(z). \tag{13}$$

We can write this equality as

$$\psi(r(z), zr'(z)) = -z^2 M(z),$$

where $\psi(s,t) = s^2 + s - t$.

It is easy to check that the conditions of Definition 1.1 are satisfied.

Therefore from (12) and in order to apply Theorem 1.11, we obtain

$$|r(z)| < K, \quad K > 0,$$

which implies that

$$\left|-\frac{zq'(z)}{q(z)}\right| < K, \quad K > 0.$$

Hence we conclude that

$$\frac{2}{\alpha-1}\left(\frac{\alpha+1}{2}-K\right)<\Re\left\{\frac{2}{\alpha\ \ 1}\left(\frac{\alpha+1}{2}\frac{1+z}{1\ \ z}-\frac{zq'(z)}{q(z)}\right)\right\}<$$

$$\frac{20}{\alpha-1}(\alpha+K+1)$$

$(z\in \mathrm{D};\ \mathrm{K}>0;\ \alpha\in(1,2]).$

Thus, the proof is complete.

Next we obtain the Corollary by following substituting $K<\frac{\alpha+1}{2}$ in Theorem 2.4.

Corollary 2.5 Let $M(z)$ be analytic functions in D such that $\left|z^2M(z)\right|<\frac{\alpha+1}{2}$, $(z\in \mathbf{D};\ \alpha\in(1,2])$. Let $q(z)$ be the solution of the initial –value problem (9) . Then, $q(z)\in C_0(\alpha)$.

3. CONCLUSIONS

The varieties of second-order linear differential equations in the unit disk are discussed. Moreover, we showed that the solutions of the second-order linear differential equations are concave univalent functions under some conditions.

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