A New Criterion for Meromorphic Multivalent Starlike Functions of Order \( \gamma \) defined by Dziok and Srivastava Operator

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**Abstract:** In this paper we introduce a subclass \( M_{p,q,s}(\alpha; \gamma) \) of meromorphic multivalent starlike functions of order \( \gamma \) defined by Dziok and Srivastava operator. The main object of this paper is to investigate various important properties and characteristics for this class. Further, a property preserving integrals is considered.

**Keywords and phrases:** Meromorphic functions, Hadamard product, generalized hypergeometric function.

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1. INTRODUCTION

Let \( \sum_p \) be the class of functions of the form:

\[
f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_k z^k \quad (p \in \mathbb{N}) \quad \{1,2,\ldots\},
\]

which are analytic and p-valent in the punctured unit disc \( U = \{ z \in \mathbb{C} \text{ and } 0 \leq |z| < 1 \} = U \setminus \{0\} \). For functions \( f(z) \in \sum_p \) given by (1.1) and \( g(z) \in \sum_p \) defined by

\[
g(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} b_k z^k \quad (p \in \mathbb{N}),
\]

then the Hadamard product (or convolution) of \( f(z) \) and \( g(z) \) is given by

\[
(f \ast g)(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_k b_k z^k \quad (g \ast f)(z).
\]

For real numbers \( \alpha_1,\ldots,\alpha_q \) and \( \beta_1,\ldots,\beta_s \)

\( \beta_j \in \mathbb{Z}_0 \setminus \{0,-1,-2,\ldots\}; \quad j = 1,2,\ldots,s \), we now define the generalized hypergeometric function

\[
q F_s(\alpha_1,\ldots,\alpha_q; \beta_1,\ldots,\beta_s; z) \quad \text{by (see, for example, [15, p.19])}
\]

\[
q F_s(\alpha_1,\ldots,\alpha_q; \beta_1,\ldots,\beta_s; z) \quad \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_q)_k}{(\beta_1)_k \cdots (\beta_s)_k} \frac{z^k}{k!}
\]

\( (q \leq s + 1; \quad q,s \in \mathbb{N}_0 \quad \mathbb{N} \cup \{0\}; \quad z \in U) \),

where \( (\theta)_v \) is the Pochhammer symbol defined, in terms of the Gamma function \( \Gamma \), by

\[
(\theta)_v = \Gamma(\theta + v) \Gamma(\theta) \quad \left\{ \begin{array}{ll}
1 & v = 0; \\
\theta(\theta+1)\cdots(\theta+v-1) & (v \in \mathbb{N}; \quad \theta \in \mathbb{C}).
\end{array} \right.
\]

Corresponding to the function

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\( h_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) \), defined by
\[
h_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = z^{-p} F_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z),
\]
we consider a linear operator
\[
H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s): \sum_p \to \sum_p,
\]
which is defined by the following Hadamard product:
\[
H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) f(z) = h_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) * f(z).
\]

We observe that, for a function \( f(z) \) of the form (1.1), we have
\[
H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) f(z) = z^{-p} + \sum_{k=0}^{\infty} \Gamma_{k+p}(\alpha_1)a_k z^k.
\]
where, for convenience
\[
\Gamma_{k+p}(\alpha_1) = (\alpha_1)_{k+p} \cdots (\alpha_q)_{k+p} \frac{1}{(k+p)!}.
\]

If, for convenience, we write
\[
H_{p,q,s}(\alpha_1) = H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s),
\]
then one can easily verify from the definition (1.1) (that see [14])
\[
z(H_{p,q,s}(\alpha_1)f(z)) = \alpha_1 H_{p,q,s}(\alpha_1+1)f(z) - (\alpha_1+\gamma) H_{p,q,s}(\alpha_1)f(z).
\]
The linear operator \( H_{p,q,s}(\alpha_1) \) was investigated recently by Liu and Srivastava [14] and Aouf [3].
Some interesting subclasses of analytic functions associated with the generalized hypergeometric function, were considered recently by (for example) Dziok and Srivastava [6, 7], Gangadharan et al [8], Liu [12].

We note that:
\[ (i) \quad H_{p,2,1}(a,;c) f(z) = L_2(a,c) f(z), \quad f(z) \in \sum_p, \quad a > 0, \; c > 0 \] (see Liu and Srivastava [13]);
\[ (ii) \quad D^{p+1} f(z) = \frac{1}{z^p (1-z)^{n+p}} f(z), \quad n > -p, \; p \in \mathbb{N} \] (see Aouf [1] and Uralegaddi and Somanatha [16]);
\[ (iii) \quad H_{p,2,1}(v,;1) f(z) = \mathcal{F}_{v,p}(f)(z) \] (\( v > 0, \; p \in \mathbb{N} \)) (see Aouf [1], Uralegaddi and Somanatha [16] and Yang [17]).

Making use of the operator \( H_{p,q,s}(\alpha_1) f(z) \), we say that a function \( f(z) \in \sum_p \) is in the class
\[ M_{p,q,s}(\alpha_1; \gamma) \] if it satisfies the following inequality:
\[
\Re \left\{ z \left( H_{p,q,s}(\alpha_1)f(z) \right)' \right\} < -\gamma
\]
\[ (\alpha_1, \ldots, \alpha_q \in \mathbb{R} \; \text{and} \; \beta_1, \ldots, \beta_s \in \mathbb{R} \setminus \mathbb{Z}; \quad q \in \mathbb{N}; \; q < s+1; \; 0 < \gamma < p; \; z \in \mathbb{U}).
\]

We note the following interesting relationship with some of the special function classes which were investigated recently:
\[ (i) \quad M_{p,2,1}(n+1; 0) = M_n \left( n \in \mathbb{N}_0 \right) \] (see Aouf [2]);
\[ (ii) \quad M_{1,2,1}(n+1; \alpha) = M_n(\alpha) \left( n \in \mathbb{N}_0; \; 0 \leq \alpha < 1 \right) \] (see Aouf and Hossen [4]).

Also, we note that:
\[ (i) \quad M_{p,2,1}(n+p; \gamma) = M_p(n; \gamma \gamma - n) \] (\( n > -p \) )
\[
= \Re \left\{ \frac{D^{p+1} f(z)}{D^{p+1} f(z) (p+1)} \right\} < \frac{p+1}{n+1};
\]
\[ (ii) \quad M_{p,2,1}(a,c; \gamma) = M_p(a,c; \gamma) \left( a,c > 0 \right) \]
Meromorphic Multivalent Starlike Functions of Order $\gamma$

In this paper along with other things we shall show that a function $f(z) \in \Sigma_p$, which satisfies the condition (1.12) is meromorphic multivalent starlike in $U^*$. More precisely it is proved that for the classes $M_{p,q,s}(\alpha_i;\gamma)$ of functions in $\Sigma_p$,

$$M_{p,q,s}(\alpha_i + 1;\gamma) \subseteq M_{p,q,s}(\alpha_i;\gamma)$$

holds. If $q = 2$, $s = 1$, $\alpha_1 = \beta_1 = 1$, and $\alpha_2 = 1$, then $M_{p,2,1}(1;\gamma) = \Sigma^*_p(\gamma)$ is the class of meromorphic multivalent starlike functions of order $\gamma$ ($0 \leq \gamma < p$). The starlikeness of members of $M_{p,q,s}(\alpha_i;\gamma)$ is a consequence of (1.16). Further for $\mu > 0$, let

$$F(z) = \frac{\mu}{z^{\mu + p}} \int_0^z t^{\mu + p - 1} f(t) dt,$$

it is shown that $F(z) \in M_{p,q,s}(\alpha_i;\gamma)$ whenever $f(z) \in M_{p,q,s}(\alpha_i;\gamma)$. Also it shown that if $f(z) \in M_{p,q,s}(\alpha_i;\gamma)$ then

$$F(z) = \frac{n + 1}{z^{n + p + 1}} \int_0^z t^{n + p} f(t) dt$$

belongs to $M_{p,q,s}(\alpha_i + 1;\gamma)$ for $F(z) \neq 0$ in $U^*$. Some known results Bajpaj [5], Goel and Sohi [10], Ganigi and Uralegaddi [9], Aouf and Hossen [4] and Aouf [2] are extended.

2. PROPERTIES OF THE CLASS $M_{p,q,s}(\alpha_i;\gamma)$

Unless otherwise mentioned, we assume throughout this paper that:

$\alpha_1, \ldots, \alpha_q \in \mathbb{R}$ and

$\beta_1, \ldots, \beta_s \in \mathbb{R} \setminus \mathbb{Z}_0$, $p \in \mathbb{N}$, $q, s \in \mathbb{N}_0$, $q \leq s + 1$, $\alpha_i > 0$, $0 \leq \gamma < p$, $z \in U$.

In proving our main results, we shall need the following lemma due to Jack [11].

**Lemma.** (Jack [11]) Suppose $w(z)$ be a nonconstant analytic function in $U$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value at a point $z_0 \in U$ on the circle $|z| = r < 1$, then $z_0 w'(z_0) = \zeta w(z_0)$, where $\zeta \geq 1$ is some real number.

**Theorem 1.** $M_{p,q,s}(\alpha_i + 1;\gamma) \subseteq M_{p,q,s}(\alpha_i;\gamma)$

**Proof.** Let $f(z) \in M_{p,q,s}(\alpha_i + 1;\gamma)$. Then

$$\Re\left\{\frac{H_{p,q,s}(\alpha_i + 2)f(z)}{H_{p,q,s}(\alpha_i + 1)f(z)} - (p + 1)\right\} < -\frac{p\alpha_1 + \gamma}{\alpha_1}.$$  \hspace{1cm} (2.1)

We have to show that (2.1) implies the inequality

$$\Re\left\{\frac{H_{p,q,s}(\alpha_i + 1)f(z)}{H_{p,q,s}(\alpha_i)f(z)} - (p + 1)\right\} < -\frac{p(\alpha_i - 1) + \gamma}{\alpha_i}.$$  \hspace{1cm} (2.2)

Define $w(z)$ in $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ by

$$\frac{H_{p,q,s}(\alpha_i + 1)f(z)}{H_{p,q,s}(\alpha_i)f(z)} - (p + 1)$$

which may be written as

$$\frac{H_{p,q,s}(\alpha_i + 2)f(z)}{H_{p,q,s}(\alpha_i + 1)f(z)} - (p + 1) = \frac{\alpha_i + (\alpha_i + 2p - 2\gamma)w(z)}{\alpha_i(1 + w(z))}.$$  \hspace{1cm} (2.3)

Clearly $w$ is regular and $w(0) = 0$. Equation (2.3) may be written as

$$\frac{H_{p,q,s}(\alpha_i + 2)f(z)}{H_{p,q,s}(\alpha_i + 1)f(z)} - (p + 1)$$

Differentiating (2.4) logarithmically and using the identity (1.11), we obtain

$$\frac{H_{p,q,s}(\alpha_i + 2)f(z)}{H_{p,q,s}(\alpha_i + 1)f(z)} - (p + 1) = \frac{\alpha_i + (\alpha_i + 2p - 2\gamma)w(z)}{\alpha_i(1 + w(z))}.$$  \hspace{1cm} (2.4)

that is

$$\frac{H_{p,q,s}(\alpha_i + 2)f(z)}{H_{p,q,s}(\alpha_i + 1)f(z)} - (p + 1) = \frac{\alpha_i + (\alpha_i + 2p - 2\gamma)w(z)}{\alpha_i(1 + w(z))}.$$  \hspace{1cm} (2.5)
We claim that \( |w(z)| < 1 \) in \( U \). For otherwise (by Jack’s Lemma) there exists \( z_0 \in U \) such that
\[
z_0 w'(z_0) = \zeta w(z_0) \quad (2.7)
\]
where \( |w(z_0)| = 1 \) and \( \zeta \geq 1 \). From (2.6) and (2.7), we obtain
\[
\begin{align*}
H_{p,q,s}(\alpha_1 + 2)f(z_0) / \frac{H_{p,q,s}(\alpha_1 + 1)f(z_0)}{H_{p,q,s}(\alpha_1 + 1)f(z_0)} &= (p + 1) + \frac{p\alpha_1 + \gamma}{\alpha_1 + 1}h
\end{align*}
\]
\[
= \frac{p - \gamma}{\alpha_1 + 1} + \frac{1 - w(z_0)}{2w(z_0)} + \frac{2w(z_0)}{1 + w(z_0)}(\alpha_1 + (\alpha_1 + 2p - 2\gamma)w(z_0))
\]
Thus
\[
\begin{align*}
\text{Re} \left\{ \frac{H_{p,q,s}(\alpha_1 + 2)f(z_0) / \frac{H_{p,q,s}(\alpha_1 + 1)f(z_0)}{H_{p,q,s}(\alpha_1 + 1)f(z_0)}}{H_{p,q,s}(\alpha_1 + 1)f(z_0)}\right\} &= (p + 1) + \frac{p\alpha_1 + \gamma}{\alpha_1 + 1}h
\end{align*}
\]
\[
\geq \frac{p - \gamma}{2(\alpha_1 + 1)(\alpha_1 + p - \gamma)} > 0
\]
which contradicts (2.1). Hence \( |w(z)| < 1 \) in \( U \) and from (2.3) it follows that \( f(z) \in M_{p,q,s}(\alpha_1; \gamma) \).

Putting \( q = 2, s = 1, \alpha_1 = n + p (n > p) \) and \( \alpha_2 = \beta_1 = p (p \in \mathbb{N}) \) in Theorem 1, we obtain the following corollary.

**Corollary 1.** \( M_p(n + 1; \gamma) \subset M_p(n; \gamma) \).

Putting \( q = 2, s = 1, \alpha_1 = \alpha > 0, \alpha_2 = 1 \) and \( \beta_1 = \delta > 0 \) in Theorem 1, we obtain the following corollary.

**Corollary 2.** \( M_p(a + 1, c; \gamma) \subset M_p(a, c; \gamma) \).

**Theorem 2.** Let \( f(z) \in \sum_p \) satisfy the condition
\[
\begin{align*}
\frac{H_{p,q,s}(\alpha_1 + 1)f(z)}{H_{p,q,s}(\alpha_1) f(z)} &> (p + 1) \\
&< \frac{(p - \gamma) - 2 (p\alpha_1 - p + \gamma)(c + p - \gamma)}{2\alpha_1(c + p - \gamma)}
\end{align*}
\]
Then
\[
F(z) = \frac{\mu}{z^{\alpha_1 + 2}} \int_0^z t^{\alpha_1 + 1} f(t) dt \quad (\mu > 0)
\]
belong to \( M_{p,q,s}(\alpha_1; \gamma) \).

**Proof.** From the definition of \( F(z) \), we have
\[
\begin{align*}
\text{Re} \left\{ \frac{H_{p,q,s}(\alpha_1 + 1)f(z)}{H_{p,q,s}(\alpha_1) f(z)}\right\} &= (p + 1) + \frac{p\alpha_1 + \gamma}{\alpha_1 + 1}h
\end{align*}
\]
using (2.11) and the identity (1.11), the condition (2.9) may be written as
\[
\begin{align*}
\text{Re} \left\{ \frac{H_{p,q,s}(\alpha_1 + 1)f(z)}{H_{p,q,s}(\alpha_1) f(z)}\right\} &= (p + 1) + \frac{p\alpha_1 + \gamma}{\alpha_1 + 1}h
\end{align*}
\]
\[
\geq \frac{p - \gamma}{2(\alpha_1 + 1)(\alpha_1 + p - \gamma)} > 0
\]
which contradicts (2.1). Hence \( |w(z)| < 1 \) in \( U \) and from (2.3) it follows that \( f(z) \in M_{p,q,s}(\alpha_1; \gamma) \).

Define \( w(z) \) in \( U \) by
\[
\begin{align*}
\text{Re} \left\{ \frac{H_{p,q,s}(\alpha_1 + 1)f(z)}{H_{p,q,s}(\alpha_1) f(z)}\right\} &= (p + 1) + \frac{p\alpha_1 + \gamma}{\alpha_1 + 1}h
\end{align*}
\]
\[
\geq \frac{p - \gamma}{2(\alpha_1 + 1)(\alpha_1 + p - \gamma)} > 0
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\begin{align*}
\frac{H_{p,q,s}(\alpha_1 + 1)f(z)}{H_{p,q,s}(\alpha_1) f(z)} &= (p + 1) + \frac{p\alpha_1 + \gamma}{\alpha_1 + 1}h
\end{align*}
\]
\[
\geq \frac{p - \gamma}{2(\alpha_1 + 1)(\alpha_1 + p - \gamma)} > 0
\]
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\frac{H_{p,q,s}(\alpha_1 + 1)f(z)}{H_{p,q,s}(\alpha_1) f(z)} &= (p + 1) + \frac{p\alpha_1 + \gamma}{\alpha_1 + 1}h
\end{align*}
\]
\[
\geq \frac{p - \gamma}{2(\alpha_1 + 1)(\alpha_1 + p - \gamma)} > 0
\]
which contradicts (2.1). Hence \( |w(z)| < 1 \) in \( U \) and from (2.3) it follows that \( f(z) \in M_{p,q,s}(\alpha_1; \gamma) \).
Differentiating (2.14) logarithmically and using the identity (1.11), we obtain
\[
\begin{aligned}
\frac{(\alpha_1+1)H_{p,q,s}(\alpha_1+2)F(z)}{H_{p,q,s}(\alpha_1+1)F(z)} - & \frac{\alpha_1H_{p,q,s}(\alpha_1+1)F(z)}{H_{p,q,s}(\alpha_1)F(z)} - 1 = \frac{2(p-\gamma)zw'(z)}{[1+w(z)][\alpha_1 + (\alpha_1+2p-2\gamma)w(z)]} \\
\end{aligned}
\tag{2.15}
\]

The above equation may be written as
\[
\begin{aligned}
\frac{(\alpha_1+1)H_{p,q,s}(\alpha_1+2)F(z)}{H_{p,q,s}(\alpha_1+1)F(z)} - (\alpha_1+1-\mu) &= \frac{\alpha_1H_{p,q,s}(\alpha_1+1)F(z)}{H_{p,q,s}(\alpha_1)F(z)} - (p+1) \\
&= \frac{H_{p,q,s}(\alpha_1+1)F(z)}{H_{p,q,s}(\alpha_1)F(z)} - (p+1) \\
&+ \left[ \frac{2(p-\gamma)zw'(z)}{[1+w(z)][\alpha_1 + (\alpha_1+2p-2\gamma)w(z)]} \right] \left[ \alpha_1 - (\alpha_1-\mu) \frac{H_{p,q,s}(\alpha_1)F(z)}{H_{p,q,s}(\alpha_1+1)F(z)} \right] \\
&= \frac{1}{\alpha_1 - (\alpha_1-\mu) \frac{H_{p,q,s}(\alpha_1)F(z)}{H_{p,q,s}(\alpha_1+1)F(z)}} \\
\end{aligned}
\tag{2.16}
\]

which, by using (2.13) and (2.14), reduces to
\[
\begin{aligned}
\frac{(\alpha_1+1)H_{p,q,s}(\alpha_1+2)F(z)}{H_{p,q,s}(\alpha_1+1)F(z)} - (\alpha_1+1-\mu) &= \frac{\alpha_1H_{p,q,s}(\alpha_1+1)F(z)}{H_{p,q,s}(\alpha_1)F(z)} - (p+1) \\
&= \frac{p(\alpha_1-1)+\gamma}{\alpha_1} + \frac{p-\gamma}{\alpha_1} \frac{1-w(z)}{1+w(z)} \\
&+ \left[ \frac{2(p-\gamma)zw'(z)}{[1+w(z)][\mu + (\mu+2p-2\gamma)w(z)]} \right] \left[ \alpha_1 - (\alpha_1-\mu) \frac{H_{p,q,s}(\alpha_1)F(z)}{H_{p,q,s}(\alpha_1+1)F(z)} \right] \\
&= \frac{1}{\alpha_1 - (\alpha_1-\mu) \frac{H_{p,q,s}(\alpha_1)F(z)}{H_{p,q,s}(\alpha_1+1)F(z)}} \\
\end{aligned}
\tag{2.17}
\]

We claim that \(|w(z)| < 1\) in \(U\). For otherwise (by Jack's Lemma) there exists \(z_0 \in U\) such that
\[
z_0 w'(z_0) = \zeta w(z_0) \tag{2.18}
\]
where \(|w(z_0)| = 1\) and \(\zeta \geq 1\). From (2.17) and
\[
\begin{aligned}
\frac{(\alpha_1+1)H_{p,q,s}(\alpha_1+2)F(z_0)}{H_{p,q,s}(\alpha_1+1)F(z_0)} - (\alpha_1+1-\mu) &= \frac{\alpha_1H_{p,q,s}(\alpha_1+1)F(z_0)}{H_{p,q,s}(\alpha_1)F(z_0)} - (p+1) \\
&= \frac{p(\alpha_1-1)+\gamma}{\alpha_1} + \frac{p-\gamma}{\alpha_1} \frac{1-w(z)}{1+w(z)} \\
&+ \left[ \frac{2(p-\gamma)zw'(z)}{[1+w(z)][\mu + (\mu+2p-2\gamma)w(z)]} \right] \left[ \alpha_1 - (\alpha_1-\mu) \frac{H_{p,q,s}(\alpha_1)F(z_0)}{H_{p,q,s}(\alpha_1+1)F(z_0)} \right] \\
&= \frac{1}{\alpha_1 - (\alpha_1-\mu) \frac{H_{p,q,s}(\alpha_1)F(z_0)}{H_{p,q,s}(\alpha_1+1)F(z_0)}} \\
\end{aligned}
\]

which contradicts (2.9). Hence \(|w(z)| < 1\) in \(U\) and from (2.13) it follows that \(F(z) \in M_{p,q,s}^\ast(\alpha_1; \gamma)\).

Putting \(q = 2, \ s = 1, \alpha_1 = n + 1 (n > -1)\) and \(\alpha_2 = \beta_1 = 1\) in Theorem 2, we obtain the following corollary.

Corollary 3. Let \(f(z) \in \Sigma_p\) satisfy the condition
\[
\Re \left\{ \frac{D^{s+p}f(z)}{D^{s+p-1}f(z)} - (p+1) \right\} < \frac{(p-\gamma) - 2(pn+\gamma)(\mu + p - \gamma)}{2(n+1)\mu + p - \gamma},
\tag{2.20}
\]
then \(F(z)\) is given by (2.10) belongs to \(M_p^\ast(n; \gamma)\).

Putting \(q = 2, \ s = 1, \alpha_1 = a > 0, \alpha_2 = 1\) and \(\beta_1 = c > 0\) in Theorem 2, we obtain the following corollary.

Corollary 4. Let \(f(z) \in \Sigma_p\) satisfy the condition
\[
\Re \left\{ \frac{L_p(a+1,c)f(z)}{L_p(a,c)f(z)} - (p+1) \right\} < \frac{(p-\gamma) - 2(pn+\gamma)(\mu + p - \gamma)}{2(n+1)\mu + p - \gamma},
\tag{2.20}
\]
\[
\left( \frac{p - \gamma} {2a} \right) - 2 \left( \frac{pa - p + \gamma}{\mu + p - \gamma} \right) \mu + p - \gamma
\]

(2.21)

then \( F(z) \) is given by (2.10) belongs to \( M_p(a,c;\gamma) \).

Theorem 3. If \( f(z) \in M_{p,q,s}(\alpha;\gamma) \), then

\[
F(z) = \frac{n + 1}{z^{n+p+1}} \int_0^z f(t)dt
\]

(2.22)

belongs to \( M_{p,q,s}(\alpha + 1;\gamma) \) for \( F(z) \neq 0 \) in \( U^* \).

Proof. We have

\[
\mu H_{p,q,s}(\alpha) f(z) = \alpha H_{p,q,s}(\alpha + 1) F(z)
\]

(2.23)

\[
-(\alpha - \mu) H_{p,q,s}(\alpha) F(z)
\]

and

\[
\mu H_{p,q,s}(\alpha + 1) f(z) = (\alpha + 1) H_{p,q,s}(\alpha + 2) F(z)
\]

(2.24)

\[
-(\alpha + 1 - \mu) H_{p,q,s}(\alpha + 1) F(z).
\]

Taking \( \mu = \alpha \) in the above relations, we obtain

\[
\frac{(\alpha + 1) H_{p,q,s}(\alpha + 2) F(z) - H_{p,q,s}(\alpha + 1) F(z)}{\alpha H_{p,q,s}(\alpha + 1) F(z)} = \frac{H_{p,q,s}(\alpha + 1) f(z)}{H_{p,q,s}(\alpha) f(z)}
\]

(2.25)

which reduces to

\[
\frac{(\alpha + 1) H_{p,q,s}(\alpha + 2) F(z)}{\alpha H_{p,q,s}(\alpha + 1) F(z)} = \frac{1}{\alpha} - \frac{1}{\alpha}
\]

(2.26)

Thus

\[
\text{Re} \left\{ \frac{(\alpha + 1) H_{p,q,s}(\alpha + 2) F(z)}{\alpha H_{p,q,s}(\alpha + 1) F(z)} - \frac{1}{\alpha} - (p + 1) \right\} = \frac{1}{\alpha}
\]

Then \( F(z) \in M_{p,q,s}(\alpha + 1;\gamma) \). This complete the proof of Theorem 3.

Remarks:

(i) Taking \( q = 2, s = 1, \alpha = n + 1 (n > -1) \), \( \alpha = 1, \beta = 1 \) and \( \gamma = 0 \), in all our results, we obtain the results obtained by Aouf [2];

(ii) Taking \( q = 2, s = 1, \alpha = n + 1 (n > -1) \) and \( \alpha = 1, \beta = p = 1 \) in all our results, we obtain the results obtained by Aouf and Hossen [4];

(iii) Taking \( q = 2, s = 1, \alpha = n + 1 (n > -1) \), \( \alpha = 1, \beta = p = 1 \) and \( \gamma = 0 \), in all our results, we obtain the results obtained by Ganigi and Uralegaddi [9].

3. REFERENCES


