Proceedings of the Pakistan Academy of Sciences 49 (1) 25-31 (2012)

ISSN: 0377 - 2969

Pakistan Academy of Sciences

Original Article

A Study on Subordination Results for Certain Subclasses of Analytic Functions defined by Convolution

M.K. Aouf*, A.A. Shamandy, A.O. Mostafa and A.K. Wagdy

Department of Mathematics, Faculty of Science,
Mansoura University, Mansoura 35516, Egypt

Abstract: In this paper, we drive several interesting subordination results of certain classes of analytic functions defined by convolution.

Keywords and phrases: Analytic function, Hadamard product, subordination, factor sequence.
2000 Mathematics Subject Classification: 30C45

1. INTRODUCTION

Let A denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$. Let $\varphi \in A$ be given by

$$\varphi(z) = z + \sum_{k=2}^{\infty} c_k z^k. \tag{1.2}$$

Definition 1. *(Hadamard product or convolution). Given two functions f and φ in the class A, where $f(z)$ is given by (1.1) and $\varphi(z)$ is given by (1.2) the Hadamard product (or convolution) $f * \varphi$ of f and φ is defined (as usual) by*

$$(f * \varphi)(z) = z + \sum_{k=2}^{\infty} a_k c_k z^k = (\varphi * f)(z). \quad (1.3$$

We also denote by K the class of functions $f(z) \in A$ that are convex in $\mathbb{U}$.

Let $M(\beta)$ be the subclass of A consisting of functions $f(z)$ which satisfy the inequality:

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} < \beta \ (z \in \mathbb{U}), \quad (1.4$$

for some $\beta > 1$. Also let $N(\beta)$ denote the subclasse of A consisting of functions $f(z)$ which satisfy the inequality:

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \beta \ (z \in \mathbb{U}), \tag{1.5}$$

for some $\beta > 1$ (see [7], [8], [9] and [10]). For $1 < \beta \leq \frac{4}{3}$, the classes $M(\beta)$ and $N(\beta)$ were investigated earlier by Uralegaddi et al. [14] (see also [12] and [13]).
It follows from (1.4) and (1.5) that

$$f(z) \in N(\beta) \Leftrightarrow zf'(z) \in M(\beta). \tag{1.6}$$

For $0 \leq \lambda < 1, \beta > 1$ and for all $z \in \mathbb{U}$, let $T(g, \lambda, \beta)$ be the subclass of A consisting of functions $f(z)$ of the form (1.1) and functions $g(z)$ given by:

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \ (b_k > 0), \tag{1.7}$$

which satisfying the analytic criterion:

Received, September 2011; Accepted, March 2012
*Corresponding author, M.K. Aouf; Email: mkaouf127@yahoo.com

$$Re\left\{\frac{z(f*g)'(z)}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)}\right\}<\beta. \quad (1.8)$$

We note that:

$$(i)\ T(\tfrac{z}{1-z},0,\beta)=M(\beta)\ and\ T(\tfrac{z}{(1-z)^2},0,\beta)=N(\beta)\ (\beta>1)\ (see\ [7]\ );$$

$$(ii)\ T(g,0,\beta)=M(g,\beta)(\beta>1)(see\ [1]).$$

Also we note that:

$$(i)\ T\left(\tfrac{z}{1-z},\lambda,\beta\right)=T_M(\lambda,\beta)=\left\{f\in A: Re\left\{\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}\right\}<\beta\ (0\le\lambda<1,\beta>1,z\in\mathbb{U})\right\};$$

$$(ii)\ T\left(\tfrac{z}{(1-z)^2},\lambda,\beta\right)=T_N(\lambda,\beta)=\left\{\in A: Re\left\{\frac{f'(z)+zf''(z)}{f'(z)+\lambda zf''(z)}\right\}<\beta\ (0\le\lambda<1,\beta>1,z\in\mathbb{U})\right\};$$

$$(iii)\ T\left(z+\sum_{k=2}^{\infty}\Gamma_k(\alpha_1)z^k,\lambda,\beta\right)=T_{q,s}(\alpha_1,\lambda,\beta)=\left\{\in A: Re\left\{\frac{z(H_{q,s}(\alpha_1,\beta_1)f\ (z))'}{(1-\lambda)H_{q,s}(\alpha_1,\beta_1)f\ (z)+\lambda z(H_{q,s}(\alpha_1,\beta_1)f\ (z))'}\right\}<\beta\right\},$$

where $\Gamma_k(\alpha_1)$ is defined by

$$\Gamma_k(\alpha_1)=\frac{(\alpha_1)_{k-1}\dots(\alpha_q)_{k-1}}{(\beta_1)_{k-1}\dots(\beta_s)_{k-1}(1)_{k-1}} \quad (1.9)$$

$$(\alpha_i>0,i=1,\dots,q;\beta_j>0,j=1,\dots,s;q\le s+1,q,s\in\mathbb{N}_0,\mathbb{N}_0=\mathbb{N}\cup\{0\},\mathbb{N}=\{1,2,..\}),$$

and the operator $H_{q,s}(\alpha_1,\beta_1)$ was introduced and studied by Dziok and Srivastava ([4] and [5]), which is a generalization of many other linear operators considered earlier;

$$(iv)T\left(z+\sum_{k=2}^{\infty}\left[\frac{\ell+1+\mu(k-1)}{\ell+1}\right]^m z^k,\lambda,\beta\right)=T(m,\mu,\ell,\lambda,\beta)=\left\{\in A: Re\left\{\frac{z(I^m(\mu,\ell)f(z))'}{(1-\lambda)I^m(\mu,\ell)f(z)+\lambda z(I^m(\mu,\ell)f(z))'}\right\}<\beta\right\},$$

where $m\in\mathbb{N}_0,\ \mu,\ell\ge 0,\ z\in\mathbb{U}$ and the operator $I^m(\mu,\ell)$ was defined by Cătaş et al. [3], which is a generalization of many other linear operators considered earlier;

$$(v)T\left(z+\sum_{k=2}^{\infty}C_k(b,\mu)z^k,\lambda,\beta\right)=T(\mu,b,\lambda,\beta)=\left\{f\in A: Re\left\{\frac{z(J_b^{\mu}f(z))'}{(1-\lambda)J_b{}^{\mu}f(z)+\lambda z(J_b{}^{\mu}f(z))'}\right\}<\beta\ (0\le\lambda<1,\beta>1,z\in\mathbb{U})\right\},$$

Where $C_k(b,\mu)$ is defined by

$$C_k(b,\mu)=\left(\frac{1+b}{k+b}\right)^{\mu}\quad(\mu\in\mathbb{C},b\in\mathbb{C}\{\mathbb{Z}_0{}^-\},\mathbb{Z}_0{}^-=\mathbb{Z}\backslash\mathbb{N}), \quad (1.10)$$

and the operator J_b^{μ} was introduced by Srivastava and Attiya [11], which is a generalization of many other linear operators considered earlier.

Definition 2. *(Subordination principle). For two functions f and φ, analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $\varphi(z)$ in $\mathbb{U}$, written $f(z)\prec\varphi(z)$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in $\mathbb{U}$ with $w(0)=0$ and $|w(z)|<1$, such that $f(z)=\varphi(w(z))$. Indeed it is known that*

$$f(z)\prec\varphi(z)\Rightarrow f(0)=\varphi(0)\ and\ f(\mathbb{U})\subset\varphi(\mathbb{U}).$$

Furthermore, if the function φ is univalent in $\mathbb{U}$, then we have the following equivalence (see [2] and [6]):

$$f(z)\prec\varphi(z)\Leftrightarrow f(0)=\varphi(0) and\ f(\mathbb{U})\subset\varphi(\mathbb{U}). \quad (1.11)$$

Definition 3. *(Subordinating factor sequence)* [15]. *A sequence $\{d_k\}_{k=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever f of the form* (1.1) *is analytic, univalent* *and convex in $\mathbb{U}$, we have*

$$\sum_{k=2}^{\infty} d_k\, a_k\, z^k \prec f(z)\ (a_1 = 1; z \in \mathbb{U}).$$

2. MAIN RESULTS

Unless otherwise mentioned, we assume throughout this paper that $0 \le \lambda < 1, \beta > 1, z \in \mathbb{U}$ and $g(z)$ is given by (1.7) with $b_{k+1} \ge b_k\ (k \ge 2)$.

To prove our main result we need the following lemmas.

Lemma 1. [15]. *The sequence $\{d_k\}_{k=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$Re\left\{1 + 2\sum_{k=1}^{\infty} d_k\, z^k\right\} > 0. \qquad (2.1)$$

Now, we prove the following lemma which gives a sufficient condition for functions belonging to the class $T(g,\lambda,\beta)$:

Lemma 2. A function $f(z)$ of the form (1.1) is said to be in the class $T(g,\lambda,\beta)$ if

$$\sum_{k=2}^{\infty}\left\{\begin{matrix}(1-\lambda)(k-1)\\ +\left|\begin{matrix}k-(2\beta-1)\\ [1+\lambda(k-1)]\end{matrix}\right|\end{matrix}\right\} b_k\,|a_k| \le 2(\beta-1). \qquad (2.2)$$

Proof. Assume that the inequality (2.2) holds true. Then it suffices to show that

$$\left|\frac{\dfrac{z(f*g)'(z)}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)} - 1}{\dfrac{z(f*g)'(z)}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)} - (2\beta-1)}\right| < 1.$$

We have

$$\left|\frac{\dfrac{z(f*g)'(z)}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)} - 1}{\dfrac{z(f*g)'(z)}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)} - (2\beta-1)}\right|$$

$$\le \frac{\sum_{k=2}^{\infty}(1-\lambda)(k-1)\, b_k\,|a_k|\,|z|^{k-1}}{2(\beta-1) - \sum_{k=2}^{\infty}|k-(2\beta-1)[1+\lambda(k-1)]|\, b_k\,|a_k|\,|z|^{k-1}}$$

$$< \frac{\sum_{k=2}^{\infty}(1-\lambda)(k-1)\, b_k\,|a_k|}{2(\beta-1) - \sum_{k=2}^{\infty}|k-(2\beta-1)[1+\lambda(k-1)]|\, b_k\,|a_k|} < 1.$$

This completes the proof of Lemma 2.

Corollary 1. *Let the function $f(z)$ defined by (1.1) be in the class $T(g;\lambda,\beta)$, then*

$$|a_k| \le \frac{2(\beta-1)}{\{(1-\lambda)(k-1) + |k-(2\beta-1)[1+\lambda(k-1)]|\}b_k}. \qquad (2.3)$$

The result is sharp for the function

$$f(z) = z + \frac{2(\beta-1)}{\{(1-\lambda)(k-1) + |k-(2\beta-1)[1+\lambda(k-1)]|\}b_k}. \qquad (2.4)$$

Let $T^*(g;\lambda,\beta)$ denote the subclass of functions $f(z) \in A$ whose coefficients satisfy the condition (2.2). We note that $T^*(g;\lambda,\beta) \subseteq T(g,\lambda,\beta)$.

Thereom 1. *Let $f(z) \in T^*(g;\lambda,\beta)$. Then*

$$\frac{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2}{2\{2(\beta-1)+[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2\}}(f*h)(z) \prec h(z), \qquad (2.5)$$

for every function $h \in K$, and

$$Re\{f(z)\} > -\frac{\{2(\beta-1)+[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2\}}{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2}. \qquad (2.6)$$

The constant factor $\frac{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2}{2\{2(\beta-1)+[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2\}}$ in the subordination result (2.5) is the best estimate.

Proof. Let $f(z) \in T^*(g;\lambda,\beta)$ and suppose that $h(z) = z + \sum_{k=2}^{\infty} h_k\, z^k \in K$, then

$$\frac{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2}{2\{2(\beta-1)+[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2\}}(f*h)(z)$$

$$= \frac{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2}{2\{2(\beta-1)+[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2\}}\left(z + \sum_{k=2}^{\infty} h_k\, a_k\, z^k\right). \qquad (2.7)$$

Thus, by using Definition 3, the subordination result holds true if

$$\left\{\frac{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2}{2\{2(\beta-1)+[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2\}}a_k\right\}_{k=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1, this is equivalent to the following inequality:

$$Re\left\{1+\sum_{k=1}^{\infty}\frac{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2}{\{2(\beta-1)+[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2\}}a_k z^k\right\}>0. \quad (2.8)$$

Now, since

$$\Psi(k)=\{(1-\lambda)(k-1)+|k-(2\beta-1)[1+\lambda(k-1)]|\}b_k$$

is an increasing function of k $(k \geq 2)$, we have

$$Re\left\{1+\frac{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2}{\{2(\beta-1)+[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2\}}\sum_{k=1}^{\infty}a_k z^k\right\}$$

$$=Re\left\{1+\frac{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2}{\{2(\beta-1)+[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2\}}z+\frac{\sum_{k=2}^{\infty}[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2 a_k z^k}{\{2(\beta-1)+[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2\}}\right\}$$

$$\geq 1-\frac{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2}{\{2(\beta-1)+[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2\}}r-\frac{1}{\{2(\beta-1)+[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2\}}\sum_{k=2}^{\infty}[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2|a_k|r^k$$

$$\geq 1-\frac{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2}{\{2(\beta-1)+[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2\}}r-\frac{1}{\{2(\beta-1)+[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2\}}\cdot\sum_{k=2}^{\infty}\{(1-\lambda)(k-1)+|k-(2\beta-1)[1+\lambda(k-1)]|\}b_k|a_k|r^k$$

$$\geq 1-\frac{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2}{\{2(\beta-1)+[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2\}}r-\frac{2(\beta-1)}{\{2(\beta-1)+[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2\}}r$$

$$\geq 1-\frac{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2}{\{2(\beta-1)+[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2\}}-\frac{2(\beta-1)}{\{2(\beta-1)+[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2\}}>0 \ (|z|=r<1),$$

where we have also made use of assertion (2.2) of Lemma 2. Thus (2.8) holds true in $\mathbb{U}$. This proves the inequality (2.5). The inequality (2.6) follows from (2.5) by taking the convex function

$$h(z)=\frac{z}{1-z}=z+\sum_{k=2}^{\infty}z^k \in K. \quad (2.9)$$

To prove the sharpness of the constant

$$\frac{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2}{2\{2(\beta-1)+[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2\}},$$

we consider the function $f_0(z) \in T^*(g;\lambda,\beta)$ given by

$$f_0(z)=z-\frac{2(\beta-1)}{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2}z^2.$$

Thus from (2.5), we have

$$\frac{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2}{2\{2(\beta-1)+[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2\}}f_0(z)\prec\frac{z}{1-z}$$

It is easily verified that

$$\min_{|z|\leq r}\left\{Re\left(\frac{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2}{2\{2(\beta-1)+[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2\}}f_0(z)\right)\right\}=-\frac{1}{2}.$$

This show that the constant $\frac{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2}{2\{2(\beta-1)+[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]b_2\}}$ is the best possible. This completes the proof of Theorem 1.

Remark. (i) Taking $g(z)=\frac{z}{1-z}$ and $\lambda=0$ in Lemma 2 and Theorem 1, we obtain the result obtained by Srivastava and Attiya [10, Corollary 2] and Nishiwaki and Owa [7, Theorem 2.1];

(ii) Taking $g(z)=\frac{z}{(1-z)^2}$ and $\lambda=0$ in Lemma 2 and Theorem 1, we obtain the result obtained by Srivastava and Attiya [10, Corollary 4] and Nishiwaki and Owa [7, Corollary 2.2].

Also, we establish subordination results for the associated subclasses, $M^*(g,\beta)$, $T_M^*(\lambda,\beta)$, $T_N^*(\lambda,\beta)$, $T_{q,s}^*(\alpha_1,\lambda,\beta)$, $T^*(m,\mu,\ell,\lambda,\beta)$ and $T^*(\mu,b,\lambda,\beta)$, whose coefficients satisfy the condition (2.2) in the special cases as mentioned in the introduction.

By taking $\lambda=0$ in Lemma 2 and Theorem 1, we obtain the following corollary:

Corollary 2. *Let the function $f(z)$ defined by (1.1) be in the class $M^*(g,\beta)$ and satisfy the condition*

$$\sum_{k=2}^{\infty}\{k-1+|k-(2\beta-1)|\}b_k|a_k|\leq 2(\beta-1). \quad (2.11)$$

Then for every function $h \in K$, we have:

$$\frac{[1+|3-2\beta|]b_2}{2\{2(\beta-1)+(1+|3-2\beta|)b_2\}}(f*h)(z)\prec h(z) \tag{2.12}$$

and

$$Re\{f(z)\}>-\frac{\{2(\beta-1)+(1+|3-2\beta|)b_2\}}{[1+|3-2\beta|]b_2}. \tag{2.13}$$

The constant factor $\frac{[1+|3-2\beta|]b_2}{2\{2(\beta-1)+(1+|3-2\beta|)b_2\}}$ *in the subordination result (2.12) can not be replaced by a larger one and the function*

$$f_0(z)=z-\frac{2(\beta-1)}{[1+|3-2\beta|]b_2}z^2 \tag{2.14}$$

gives the sharpness.

By taking $g(z)=\frac{z}{1-z}$ in Lemma 2 and Theorem 1, we obtain the following corollary:

Corollary 3. *Let the function* $f(z)$ *defined by (1.1) be in the class* $T_M^*(\lambda,\beta)$ *and satisfy the condition*

$$\sum_{k=2}^{\infty}\left\{\begin{matrix}(1-\lambda)(k-1)+\\ |k-(2\beta-1)[1+\lambda(k-1)]|\end{matrix}\right\}|a_k|\le 2(\beta-1). \tag{2.15}$$

Then for every function $h\in K$*, we have:*

$$\frac{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]}{2[2\beta-1-\lambda+|3-2\beta-\lambda(2\beta-1)|]}(f*h)(z)\prec h(z) \tag{2.16}$$

and

$$Re\{f(z)\}>-\frac{[2\beta-1-\lambda+|3-2\beta-\lambda(2\beta-1)|]}{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]}. \tag{2.17}$$

The constant factor $\frac{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]}{2[2\beta-1-\lambda+|3-2\beta-\lambda(2\beta-1)|]}$ *in the subordination result (2.16) can not be replaced by a larger one and the function*

$$f_0(z)=z-\frac{2(\beta-1)}{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]}z^2 \tag{2.18}$$

gives the sharpness.

By taking $g(z)=\frac{z}{(1-z)^2}$ in Lemma 2 and Theorem 1, we obtain the following corollary:

Corollary 4. *Let the function* $f(z)$ *defined by (1.1) be in the class* $T_N^*(\lambda,\beta)$ *and satisfy the condition*

$$\sum_{k=2}^{\infty}k\left\{\begin{matrix}(1-\lambda)(k-1)+\\ |k-(2\beta-1)[1+\lambda(k-1)]|\end{matrix}\right\}|a_k|\le 2(\beta-1). \tag{2.19}$$

Then for every function $h\in K$*, we have:*

$$\frac{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]}{2[\beta-\lambda+|3-2\beta-\lambda(2\beta-1)|]}(f*h)(z)\prec h(z) \tag{2.20}$$

and

$$Re\{f(z)\}>-\frac{[\beta-\lambda+|3-2\beta-\lambda(2\beta-1)|]}{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]}. \tag{2.21}$$

The constant factor $\frac{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]}{2[\beta-\lambda+|3-2\beta-\lambda(2\beta-1)|]}$ *in the subordination result (2.20) can not be replaced by a larger one and the function*

$$f_0(z)=z-\frac{\beta-1}{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]}z^2 \tag{2.22}$$

gives the sharpness.

By taking $b_k=\Gamma_k(\alpha_1)$, where $\Gamma_k(\alpha_1)$ defined by (1.9), in Lemma 2 and Theorem 1, we obtain the following corollary:

Corollary 5. *Let the function* $f(z)$ *defined by (1.1) be in the class* $T_{q,s}^*(\alpha_1,\lambda,\beta)$ *and satisfy the condition*

$$\sum_{k=2}^{\infty}\left\{\begin{matrix}(1-\lambda)(k-1)\\ +\left|\begin{matrix}k-(2\beta-1)\\ [1+\lambda(k-1)]\end{matrix}\right|\end{matrix}\right\}\Gamma_k(\alpha_1)|a_k|\le 2(\beta-1). \tag{2.23}$$

Then for every function $h\in K$*, we have:*

$$\frac{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]\Gamma_2(\alpha_1)}{2\left\{\begin{matrix}2(\beta-1)\\ +[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]\Gamma_2(\alpha_1)\end{matrix}\right\}}(f*h)(z)\prec h(z) \tag{2.24}$$

and

$$Re\{f(z)\} > -\frac{\left\{2(\beta-1) + \left[1-\lambda+\left|3-2\beta-\lambda(2\beta-1)\right|\right]\Gamma_2(\alpha_1)\right\}}{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]\Gamma_2(\alpha_1)}. \tag{2.25}$$

The constant factor $\frac{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]\Gamma_2(\alpha_1)}{2\{2(\beta-1)+[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]\Gamma_2(\alpha_1)\}}$ *in the subordination result (2.24) can not be replaced by a larger one and the function*

$$f_0(z) = z - \frac{2(\beta-1)}{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]\Gamma_2(\alpha_1)} z^2 \tag{2.26}$$

gives the sharpness.

By taking $b_k = \left[\frac{\ell+1+\mu(k-1)}{\ell+1}\right]^m$ $(m \in \mathbb{N}_0, \mu, \ell \geq 0)$ in Lemma 2 and Theorem 1, we obtain the following corollary:

Corollary 6. *Let the function* $f(z)$ *defined by (1.1) be in the class* $T^*(m, \mu, \ell, \lambda, \beta)$ *and satisfy the condition*

$$\sum_{k=2}^{\infty} \left\{ (1-\lambda)(k-1) + \left| \frac{k-(2\beta-1)}{[1+\lambda(k-1)]} \right| \right\} \left[\frac{\ell+1+\mu(k-1)}{\ell+1}\right]^m |a_k| \leq 2(\beta-1). \tag{2.27}$$

Then for every function $h \in K$, *we have:*

$$\frac{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|](\ell+1+\mu)^m}{2\left\{2(\ell+1)^m(\beta-1) + \left[1-\lambda+\left|3-2\beta-\lambda(2\beta-1)\right|\right](\ell+1+\mu)^m\right\}} (f*h)(z) \prec h(z) \tag{2.28}$$

and

$$Re\{f(z)\} > -\frac{\left\{2(\ell+1)^m(\beta-1) + \left[1-\lambda+\left|3-2\beta-\lambda(2\beta-1)\right|\right](\ell+1+\mu)^m\right\}}{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|](\ell+1+\mu)^m}. \tag{2.29}$$

The constant factor $\frac{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|](\ell+1+\mu)^m}{2\{2(\ell+1)^m(\beta-1)+[1-\lambda+|3-2\beta-\lambda(2\beta-1)|](\ell+1+\mu)^m\}}$ *in the subordination result (2.28) can not be replaced by a larger one and the function*

$$f_0(z) = z - \frac{2(\beta-1)(\ell+1)^m}{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|](\ell+1+\mu)^m} z^2 \tag{2.30}$$

gives the sharpness.

By taking $b_k = C_k(b, \mu)$, *where* $C_k(b, \mu)$ defined by (1.10), in Lemma 2 and Theorem 1, we obtain the following corollary:

Corollary 7. *Let the function* $f(z)$ *defined by (1.1) be in the class* $T^*(\mu, b, \lambda, \beta)$ *and satisfy the condition*

$$\sum_{k=2}^{\infty} \left\{ (1-\lambda)(k-1) + \left| \frac{k-(2\beta-1)}{[1+\lambda(k-1)]} \right| \right\} C_k(b, \mu) |a_k| \leq 2(\beta-1). \tag{2.31}$$

Then for every function $h \in K$, *we have:*

$$\frac{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]C_2(b,\mu)}{2\left\{2(\beta-1) + \left[1-\lambda+\left|3-2\beta-\lambda(2\beta-1)\right|\right]C_2(b,\mu)\right\}} (f*h)(z) \prec h(z) \tag{2.32}$$

and

$$Re\{f(z)\} > -\frac{\left\{2(\beta-1) + \left[1-\lambda+\left|3-2\beta-\lambda(2\beta-1)\right|\right]C_2(b,\mu)\right\}}{\left[1-\lambda+\left|3-2\beta-\lambda(2\beta-1)\right|\right]C_2(b,\mu)}. \tag{2.33}$$

The constant factor $\frac{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]C_2(b,\mu)}{2\{2(\beta-1)+[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]C_2(b,\mu)\}}$ *in the subordination result (2.32) can not be replaced by a larger one and the function*

$$f_0(z) = z - \frac{2(\beta-1)}{[1-\lambda+|3-2\beta-\lambda(2\beta-1)|]C_2(b,\mu)} z^2 \tag{2.34}$$

gives the sharpness.

3. ACKNOWLEDGEMENTS

The authors thank the anonymous referees of the paper for their helpful suggestions.

4. REFERENCES

1. Aouf, M.K., A.A.. Shamandy, A.O. Mostafa & E.A. Adwan. Subordination results for certain class of analytic functions defined by convolution.

Rend. del Circolo Math. di Palermo (in press).
2. Bulboaca, T. *Differential Subordinations and Superordinations, Recent Results*. House of Scientific Book Publ., Cluj-Napoca (2005).
3. A Cătaş, G.I. Oros & G. Oros. Differential subordinations associated with multiplier transformations. *Abstract Appl. Anal.* ID845724: 1-11 (2008).
4. Dziok, J. & H.M. Srivastava. Classes of analytic functions with the generalized hypergeometric function. *Appl. Math. Comput.* 103: 1-13 (1999).
5. Dziok, J. & , H.M. Srivastava. Certain subclasses of analytic functions associated with the generalized hypergeometric function. *Integral Transform. Spec. Funct*. 14: 7-18 (2003).
6. Miller, S.S. & P.T. Mocanu. *Differential Subordinations Theory and Applications.* In: Series on Monographs and Textbooks in Pure and Applied Mathematics 255. Marcel Dekker, New York (2000).
7. Nishiwaki, J. & S Owa.. Coefficient inequalities for certain analytic functions, *Internat. J. Math. Math. Sci.* 29 (5): 285-290 (2002).
8. Owa, S. & J. Nishiwaki. Coeffocient estimates for certain classes of analytic functions. *J. Inequal. Pure Appl. Math.* 3 (5), Art. 72: 1-12 (2002).
9. Owa, S. & H.M. Srivastava. Some generalized convolution properties associated with certain subclasses of analytic functions. *J. Inequal. Pure Appl. Math.* 3 (3), Art. 42: 1-27 (2002).
10. Srivastava, H.M. & A.A Attiya. Some subordination results associated with certain subclasses of analytic functions. *J. Inequal. Pure Appl. Math.* 5 (4), Art. 82: 1-14 (2004).
11. Srivastava, H.M. & A.A Attiya.. An integral operator associated with the Hurwitz-Lerch Zeta function and differential subordination. *Integral Transform. Spec. Funct.* 18: 207-216 (2007).
12. Srivastava, H.M. & S. Owa. *Current Topics in Analytic Function Theory.* World Scientific Publishing Company, Singapore (1992).
13. Uralegaddi, BA. & A.R. Desa. Convolutions of univalent functions with positive coefficients. *Tamkang J. Math.* 29: 279-285 (1998).
14. Uralegaddi, B.A., M.D. Ganigi & S.M Sarangi. Univalent functions with positive coefficients. *Tamkang J. Math.* 25: 225-230 (1994).
15. Wilf, S. Subordinating factor sequence for convex maps of the unit circle. *Proc. Amer. Math. Soc.* 12: 689-693 (1961).