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Original Article

Sitnikov Problem: It's Extension to Four Body Problem

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Abstract: An analytical expression for the position of the infinitesimal body in the elliptic Sitnikov restricted four body problem is presented. This solution is valid for small bounded oscillations in case of moderate eccentricity of primaries. We have linearized the equation of motion to obtain the Hill's type equation. Using the Courant and Snyder transformation, Hill's equation is transformed into Harmonic oscillator type equation. We have used the Lindstedt - poincare perturbation method and again we have applied the Courant and Snyder transformation to obtain the final result.

Keywords: Sitnikov problem, four body problem, Lindstedt- poincare method, perturbation theory, analytical solution.

1. INTRODUCTION

The Sitnikov problem is a special case of the restricted three body problem where the two primaries of equal masses ($m_1 = m_2 = m=1/2$) are moving in circular or elliptic orbits around the centre of mass under Newtonian force of attraction and the third body of mass m_3 (the mass of the third body is much less than the mass of the primaries) moves along the line which is passing through the centre of mass of the primaries and is perpendicular to the plane of motion of the primaries.

This dynamical model was first described by Pavanini [1].The circular problem was discussed in detail by MacMillan [2] where he showed the integrability of the equation of motion with the aid of elliptic integrals which has been rediscussed by Stumpff [3]. Sitnikov [4] studied the existence of oscillating motion of the three body problem. Sitnikov problem has been studied by many scientists. Perdios et al [5] have studied stability and bifurcation of Sitnikov motion. Liu & Sun [6] have studied the mapping instead of the original differential equation and discovered that there exist a hyperbolic invariant set. Hagel [7] has studied the problem by a new analytical approach. It is valid for bounded small amplitude solution and small eccentricities of the primaries. Belbruno E., Llibre J. and Olle M**.** [8] have studied the family of periodic orbits which bifurcate from the circular Sitnikov motion. Jalali & Pourtakdoust [9] have studied the regular and chaotic solutions of the Sitnikov problem near the 3/2 commensurability. Chasley [10] have studied the global analysis of the generalized Sitnikov problem. Faraque [11] has established the new analytical expression for the position of the infinitesimal body in the elliptic Sitnikov problem. His solution is valid for small bounded oscillation in case of moderate eccentricities of the primaries. Hagel [12] has studied "A high order perturbation analysis of the Sitnikov problem". Perdios [13] has studied the manifold of families of three dimensional periodic orbits and bifurcation in the Sitnikov four body problem. Soulis et al [14] have studied the "Stabality of motion in the Sitnikov problem". Soulis et al [15] has studied the periodic orbits and bifurcation in the Sitnikov four-body problem. Hagel [16] has studied an analytical approach to small amplitude solutions of the extended nearly circular Sitnikov problem. Boutis & Papadakis [17] have studied "The Stability of vertical motion in the N-body circular Sitnikov problem. Kovacs et al [18] studied the relativistic effects in the chaotic Sitnikov problem.

In the present paper we have extended the study of Sitnikov problem to four body problem in elliptic case. We have considered the primaries moving in elliptic orbits of eccentricity

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e. First we have derived the equation of motion and then we have linearized it to obtain the Hill's type of equation .Using the Courant and Snyder transformation Hill's equation is transformed into Harmonic oscillator type equation. Then we have used the Lindstedt -Poincare perturbation method and again we have applied Courant and Snyder transformation to obtain the final result.

2. EQUATION OF MOTION

The system consists of three primaries with equal masses ($m_1 = m_2 = m_3 = m$ =1/3). The fourth body has a mass (m_4) which is much less than the masses of the primaries. All of the primaries are at the vertices of an equilateral triangle [19]. The fourth body is confined to a motion perpendicular to the plane of motion of the three primaries which are equally far away from the barycentre of the system. All the primaries are moving in elliptic orbits around their center of mass O which is taken as origin .The fourth body is moving along the line perpendicular to the plane of motion of the primaries and passing through the centre of mass. In such a system the motion of the fourth body is one dimensional. We have assumed that $r(t)$ is the distance of each of the primaries from the centre of mass, and we further assumed that $z(t)$ is the distance of the fourth particle from the centre of mass.

The equation of motion is

$$\frac{d^2z}{dt^2} = -\frac{z}{(z^2 + r^2(t))^{\frac{3}{2}}}. \tag{1}$$

Since $r(t)$ is given by the solution of the transcendental Keplar's equation, it cannot be written in a finite closed form and is therefore given as an infinite power series in the primaries eccentricity denoted by e. We write

$$r(t) = a\,[\,1 + \sum_{n=1}^{\infty} r_n(t)\,.e^n\,]$$

where

$$r_1(t) = -cost,$$

$$r_2(t) = \frac{1}{2}[\,1 - cos2t\,],$$

$$r_3(t) = \frac{3}{8}[\,cost - cos3t],$$

$$r_4(t) = \frac{1}{3}[cos2t - cos4t], \tag{2}$$

In the present paper we have taken the semi-major axis a equal to unity.

If $e \neq 0$ and $z_{max} \leq r(t)$ for all t we may expand the Equation (1) with respect to z and truncate the so obtain infinite power series after certain order in $z(t)$. Expanding (1) in this, we get

$$\frac{d^2z}{dt^2} + \frac{z}{r(t)^3} - \frac{3z^3}{2\,r(t)^5} = 0. \tag{3}$$

Further more we shall make restriction for sufficiently small value of e, we may expand the coefficients of function of z and z^3 with respect to e. Keeping the terms proportional to the $e^m z^n$ where $m + n \leq 4$, we get

$$\frac{d^2z}{dt^2} + g_1(t)\,z + g_2(t)\,z^3 = 0. \tag{4}$$

where,

$$g_1(t) = 1 + e\,g_{1,1}(t) + e^2\,g_{1,2}(t) + e^3\,g_{1,3}(t) \tag{5}$$

$$g_2(t) = -\frac{3}{2} - \frac{15}{2}\,e\,Cost \tag{6}$$

$$g_{1,1}(t) = 3\,Cost \tag{7}$$

$$g_{1,2}(t) = \frac{3}{2} + \frac{9}{2}\,Cos2\,t \tag{8}$$

$$g_{1,3}(t) = \frac{27}{8}\,Cost + \frac{53}{8}\,Cos3\,t. \tag{9}$$

The most simple limiting case is the one of $e = 0$ and $z << r(t)$. In this case the polynomial equation [3] can be approximated by the simple harmonic oscillator form

$$\frac{d^2z}{dt^2} + z = 0, \tag{10}$$

with the solution

$$z(t) = z_0\,Cost. \tag{11}$$

So for sufficiently small value for the primaries eccentricity and sufficiently small $z(t)$ the number of passages of m_4 through the primaries plane during one primaries revolution it converges.

Taking the eccentricity of primaries constant and turn to the regime of such small values for $z(t)$ that we may truncate the Equation (4) after the linear term in z , we get

$$\frac{d^2 z}{dt^2} + g_1(t)\, z = 0, \tag{12}$$

where $g_1(t)$ is defined in equation (5). We first deal with this equation as a lowest order approximation to the nonlinear equation (4). Since $g_1(t)$ is periodic in t with period 2π the Equation (12) is of *Hill's type*. For this type of equation a closed theory exists and is known as *Floquet theory*. Hence the general solution of the Equation (12) can be written in the form

$$z(t) = a\, \varpi(t)\, Cos[\psi(t) + b] \tag{13}$$

where $\varpi(t)$, the so-called *Floquet Function* is periodic with the period of the coefficient function $g_1(t)$ i.e.

$$\varpi(t) = \varpi(t + 2\pi). \tag{14}$$

The arbitrary real constant a and b are determined by the initial condition for z and $\mathring{z}$. Puting the value of $z(t)$ from the Equation (13) in the Equation (12) and comparing the coefficient of $\sin\psi$ and $\cos\psi$, we get

$$\frac{d^2\varpi}{dt^2} + g_1(t)\varpi - \frac{1}{\varpi^3} = 0, \tag{15 a}$$

$$\frac{d\psi}{dt} = \frac{1}{\varpi^2}. \tag{15 b}$$

For $z(0) = z_0$ and $\mathring{z}(0) = 0$,
we have

$$a = \frac{z_0}{\varpi_0\ Cos[b]},$$

$$\tan b = \varpi_0\, \dot{\varpi}_0. \tag{16}$$

So the problem of solving the linearized equation (16) is reduced to finding a 2π periodic solution of the Equation (15a) for $\varpi(t)$.

Since we can express $g_1(t)$ as a truncated power series in e, we can apply the same process to approach $\varpi(t)$. We thus have the perturbative series

$$\varpi(t) = \varpi_{(0)}(t) + e\varpi_{(1)}(t) + e^2\varpi_{(2)}(t) + e^3\varpi_{(3)}(t) + \ldots \tag{17}$$

With the help of equation (15a) and (17), we get

$$\frac{d^2\varpi_{(0)}}{dt^2} + \varpi_{(0)} - \frac{1}{\varpi_{(0)}^3} = 0, \tag{18}$$

$$\frac{d^2\varpi_{(1)}}{dt^2} + \varpi_{(1)} + g_{1,1}(t)\,\varpi_{(0)} + \frac{3\varpi_{(1)}}{\varpi_{(0)}^4} = 0, \tag{19}$$

$$\frac{d^2\varpi_{(2)}}{dt^2} + \varpi_{(2)} + g_{1,1}(t)\,\varpi_{(1)} + g_{1,2}(t)\,\varpi_{(0)} + \frac{3\varpi_{(2)}}{\varpi_{(0)}^4} - \frac{6\varpi_{(1)}^2}{\varpi_{(0)}^5} = 0, \tag{20}$$

$$\frac{d^2\varpi_{(3)}}{dt^2} + \varpi_{(3)} + g_{1,1}(t)\,\varpi_{(2)} + g_{1,2}(t)\,\varpi_{(1)} + g_{1,3}(t)\,\varpi_{(0)} + \frac{3\varpi_{(3)}}{\varpi_{(0)}^4} - \frac{12\varpi_{(1)}\varpi_{(2)}}{\varpi_{(0)}^5} + \frac{10\varpi_{(1)}^3}{\varpi_{(0)}^6} = 0. \tag{21}$$

By solving the above system we have to remember that all $\varpi_{(n)}$ should be periodic with period 2π. For $\varpi_{(0)}$ a solution fulfilling this requirement is a simple constant, namely

$$\varpi_{(0)} = 1 = \varpi_{(0,0)}, \tag{22}$$

as can be easily verified by inserting (22) in the Equation (18). To find the solution of the Equation (19), we insert (22) into Equation (19). The Function $g_{(1,1)}(t)$ is taken from (7), we get

$$\frac{d^2\varpi_{(1)}}{dt^2} + 4\varpi_{(1)} + 3Cost = 0. \tag{23}$$

The general solution of this equation is given by

$$\varpi_{(1)} = C_1 Cos2t + C_2 Sin2t - Cost.$$

In order to fulfill the condition of a 2π periodic solution we have to set C_1 and C_2 to zero to obtain

$$\varpi_{(1)}(t) = -Cost = \varpi_{(1),1} Cost.$$

Similarly;

$$\varpi_{(2)}(t) = \frac{3}{4} = \varpi_{(2),0},$$

and

$$\varpi_{(3)}(t) = \varpi_{(3),1}Cost + \varpi_{(3),2}Cos3t. \quad (24)$$

Thus the Equation (17) gives

$$\varpi(t) = \varpi_{(0),0} + e\varpi_{(1),1}Cost + e^2\varpi_{(2),0} + e^3(\varpi_{(3),1}Cost + \varpi_{(3),2}Cos3t), \quad (25)$$

where, coefficients $\varpi_{(m),n}(t)$ (m denotes the order in e and n the number of the associated Fourier component of ϖ)are given by

$$\varpi_{(0),0} = 1, \quad \varpi_{(1),1} = -1 \quad, \varpi_{(2),0} = \frac{3}{4}, \quad \varpi_{(3),1} = \frac{-9}{8}, \quad \varpi_{(3),1} = \frac{3}{8}. \quad (26)$$

Now from the Equation (15)

$$\psi(t) = \int_0^t \frac{dt}{\varpi^2}.$$

Thus we have,

$$\psi(t) = \left[1 + 3e^2\right]t + \left[2e + \frac{3}{4}e^3\right]Sint + 3e^2Sin2t - \frac{2}{3}e^3Sin3t. \quad (27)$$

From the Equations (13), (25) and (27), we finally find the analytic solution

$$z(t) = \frac{z_0\varpi(t)}{\varpi(0)}Cos\psi(t). \quad (28)$$

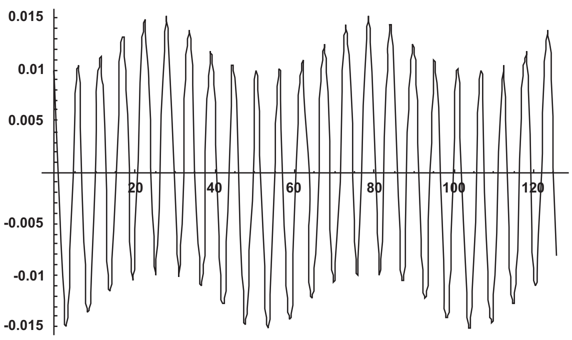


Fig. 1(a). Sitnikov four body motion for e = 0.02, z = 0.01.

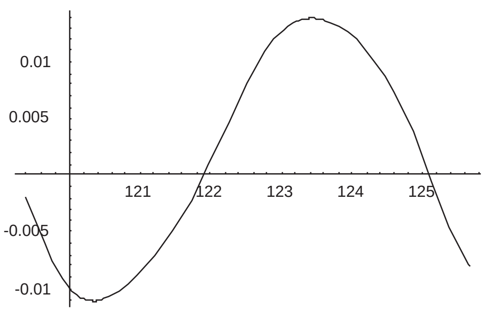


Fig. 1(b). Sitnikov four body motion for e = 0.02, z = 0.01

Here Figure-1(a) shows the graph of the solution z[t] over the time interval $0 < t \le 20\pi$. $z(0) = 0.01$ and $e = 0.02$. We observe that the solution has a periodic envelope, and the Figure- 1(b) shows the same graph for the short time interval $119 < t \le 128$.

3. TRANSFORMATION OF HILL'S EQUATION TO A HARMONIC OSCILLATOR

When $g_2(t) = 0$, the Equation (4) reduces to Hill's Equation

$$\ddot{z}(t) + g_1(t)z = 0. \quad (29)$$

For this Equation we can write the solution as

$$z(t) = a\varpi(t)Cos\psi(t). \quad (30)$$

Now we transform the dependent variables as

$$y = \frac{z}{\varpi}. \quad (31)$$

And let ψ be the new independent variable then obviously $y(\psi)$ satisfies the Harmonic oscillator equation

$$\frac{d^2y}{d\psi^2} + y = 0. \quad (32)$$

Now we apply this transformation to the nonlinear equation (4). From (31), we get

$$\frac{d^2z}{dt^2} = \varpi\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\frac{d\varpi}{dt} + y\frac{d^2\varpi}{dt^2}. \quad (33)$$

Inserting this into the Equation (4), we get

$$\varpi\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\frac{d\varpi}{dt} + y\frac{d^2\varpi}{dt^2} + g_1(t)y\varpi + g_2(t)y^3\varpi^3 = 0. \quad (34)$$

We express the time derivatives of y by the derivatives of y with respect to ψ with the relation

$$\frac{d\psi}{dt} = \frac{1}{\varpi^2}, \quad (35)$$

we get

$$\frac{dy}{dt} = \frac{dy}{d\psi} \cdot \frac{d\psi}{dt} = \left(\frac{1}{\varpi^2}\right)\frac{dy}{d\psi},$$

$$\frac{d^2y}{dt^2} = \left(\frac{\frac{d^2y}{d\psi^2} - 2\frac{dy}{d\psi} \cdot \frac{d\varpi}{dt} \cdot \varpi}{\varpi^4}\right). \quad (36)$$

After putting these in (34), the new relation becomes

$$\frac{d^2y}{d\psi^2} + y\varpi^3\left[\frac{d^2\varpi}{dt^2} + g_1(t)\varpi\right] + g_2(t)y^3\varpi^6 = 0. \quad (37)$$

Applying the Equation (15 a) in the Equation (37), we get

$$\frac{d^2y}{d\psi^2} + y + g_2(t(\psi))y^3\varpi^6(t(\psi)) = 0, \quad (38)$$

which is an equation of the form of a Harmonic Oscillator with a cubic perturbation .The explicit representation of t as function of ψ to the first order in e is

$$t = \psi - 2\, eSin\, \psi. \quad (39)$$

From (25), we have

$$\varpi^6(t) = 1 - 6eCos\, t. \quad (40)$$

Substituting the equation (6) and (40) into (38), we get

$$\frac{d^2y}{d\psi^2} + y = (\alpha' + \beta' eCos\psi)\, y^3, \quad (41)$$

where,

$$\alpha' = \frac{3}{2},\ \beta' = -\frac{3}{2}. \quad (42)$$

Once we get the solution of this equation we can determine z .The solution of the full equation can therefore be generated by further application of a perturbation determined by small perimeter , $\varepsilon = \beta' e$ .We assume from the outset that the solution depends only on the time parameter ψ as appears in the Equation (41). In corporating $\varepsilon = \beta' e$ the Equation (41) becomes

$$\frac{d^2y}{d\psi^2} + y = \alpha' y^3 + \varepsilon Cos\psi\, y^3. \quad (43)$$

Now, we express the solution as a perturbative series

$$y(\psi, \varepsilon) = y_0(\psi) + \varepsilon y_1(\psi) + \varepsilon^2 y_2(\psi) + \ldots \quad (44)$$

Differentiating twice and putting the result into the above equation and equating the coefficient of like powers of ε, we get

$$\frac{d^2y_0}{d\psi^2} + y = \alpha' y_0^3, \quad (45)$$

$$\frac{d^2y_1}{d\psi^2} + y_1 = 3\alpha' y_0^2 y_1 + Cos\psi\, y_0^3. \quad (46)$$

The Equation (45) is of the form of duffing oscillator equation and the solution of this Equation can be found by perturbation method .The non linear term on the right hand side of the Equation (45) can be treated as a perturbation to the linear problem by introducing a formal perturbation parameter ε. This is necessary because α' is much greater than 1, and can not be treated as perturbation parameter. So a formal parameter ε is used to develop the solution and in the end, it is set to unity.

Let the solution of the Equation (45) as a power series in the parameter ε be

$$y_0 = y_{0,0}(\psi) + \varepsilon y_{0,1}(\psi) + \varepsilon^2 y_{0,2}(\psi) + \ldots$$

By introducing the above mentioned parameter the Equation (45) becomes

$$\frac{d^2y_0}{d\psi^2} + y = \varepsilon\alpha' y_0^3.$$

It is easy to show that the solution of higher order equation for $y_{0,1}$, $y_{0,2}$ etc will contain secular term proportional to ψ^m, m being the order. Now we shall apply *Lindstedt-Poincare method* to get the solution .In order to implement this , we introduce a new independent variable $\tau : \tau = \mu\psi$,

where $\mu = 1 + \varepsilon\mu_1 + \varepsilon^2\mu_2 + \ldots$, and expand the solution in a power series of $\varepsilon : y_0 = y_{0,0}(\tau) + \varepsilon y_{0,1}(\tau) + \varepsilon^2 y_{0,2}(\tau) + \ldots$

Differentiating twice and putting in the above equation and equating the like power of ε, we get the following equations

$$\frac{d^2 y_{0,0}}{d\tau^2} + y_{0,0}(\tau) = 0, \tag{47}$$

$$\frac{d^2 y_{0,1}}{d\tau^2} + y_{0,1}(\tau) = -2\mu_1 \frac{d^2 y_{0,0}}{d\tau^2} + \alpha' y_{0,0}^3(\tau). \tag{48}$$

The solution of the Equation (45) can be found by using the same strategy. We get the solution

$$y_0(\psi) = \left[A + \frac{3A^3}{64}\right] Cos(\mu\psi) - \frac{3A^3}{64} Cos(3\mu\psi). \tag{49}$$

For sufficiently small value 'A', we can approximate $y_0(\psi)$ by

$$y_0(\psi) = ACos(\mu\psi). \tag{50}$$

And inserting this into the Equation (46), we get

$$\frac{d^2 y_1}{d\psi^2} + \left[\lambda - \delta Cos(2\mu\psi)\right] y_1 = Cos\psi.A^3 Cos^3(\mu\psi), \tag{51}$$

Where

$$\lambda = 1 - \frac{9\alpha' A^2}{2}, \tag{52}$$

$$\delta = \frac{9\alpha' A^2}{2}. \tag{53}$$

For sufficiently small value of A, we have the approximate equation as

$$\frac{d^2 y_1}{d\psi^2} + f_1(\psi) y_1 = 0, \tag{54}$$

where

$$f_1(\psi) = \left[\lambda - f_{1;1}(\psi)\right], \tag{55}$$

$$f_{1;1}(\psi) = \delta Cos(2\mu\psi). \tag{56}$$

The Equation (54) is a *Hill's type* equation. The general solution of the Equation (54) can be written as;

$$y_1(\psi) = \rho(\psi)\, Cos\xi. \tag{57}$$

We can find the result upto the first order by applying the same process as in the Equation (12), we get

$$\rho(\psi) = \frac{1}{\sqrt[4]{\lambda}} + \frac{\delta Cos(2\mu\psi)}{4\sqrt[4]{\lambda}\,(\lambda - \mu^2)},$$

$$\approx \frac{1}{\sqrt[4]{\lambda}}\left(1 - \frac{3Cos(2\mu\psi)}{10}\right) \tag{58}$$

and

$$\xi(\psi) = \sqrt{\lambda}\psi + \frac{3\sqrt{\lambda}}{10\mu}.Sin(2\mu\psi). \tag{59}$$

Now, we shall again apply the *Courant and Snyder* transformation defined as

$$\zeta = \frac{y_1}{\rho}, \tag{60}$$

to the Equation (54) and apply the similar procedure as in the derivation of the Equation (29) and find

$$\frac{d^2\zeta}{d\xi^2} + \zeta = \rho^3.Cos\psi.A^3 Cos^3(\mu\psi). \tag{61}$$

with

$$\psi = \frac{\xi}{\sqrt{\lambda}} - \frac{3}{10\mu}.Sin\left(\frac{2\mu\xi}{\sqrt{\lambda}}\right). \tag{62}$$

We define a parameter χ by

$$\chi = \frac{\xi}{\sqrt{\lambda}}, \tag{63}$$

the use of which in the Equation (61) results the following equation

$$\frac{d^2\zeta}{d\chi^2} + \lambda.\zeta = \lambda.\rho^3.Cos\chi.A^3 Cos^3(\mu\chi). \tag{64}$$

Now, we shall put the value of ρ^3 upto the first order of δ i.e.,

$$\rho^3 = \frac{1}{(\lambda)^{\frac{3}{4}}}\left[1 - \frac{9}{10} Cos\,(2\chi\mu)\right]. \tag{65}$$

Then we get,

$$\frac{d^2\zeta}{d\chi^2} + \lambda.\zeta = \sqrt[4]{\lambda}.A^3 \Bigg[-\frac{9}{160} Cos\,(1 + 5\mu)\,\chi - \frac{7}{160} Cos\,(1 + 3\mu)\,\chi + \frac{3}{20} Cos\,(1 + \mu)\,\chi - \frac{9}{160} Cos\,(1 - 5\mu)\,\chi - \frac{7}{160} Cos\,(1 - 3\mu)\,\chi + \frac{3}{20} Cos\,(1 - \mu)\,\chi\Bigg]. \tag{66}$$

Solution of the Equation (66) can be found using algebraic methods with the help of *Mathematica*. We obtain

$$\zeta = -\frac{9.\sqrt[4]{\lambda}.A^3}{160\left\{\lambda-(1+5\mu)^2\right\}}\,Cos\,(1+5\mu)\,\chi\, -$$

$$\frac{7.\sqrt[4]{\lambda}.A^3}{160\left\{\lambda-(1+3\mu)^2\right\}}\,Cos\,(1+3\mu)\,\chi$$

$$+\frac{3.\sqrt[4]{\lambda}.A^3}{20\left\{\lambda-(1+\mu)^2\right\}}\,Cos\,(1+\mu)\,\chi\, -$$

$$\frac{9.\sqrt[4]{\lambda}.A^3}{160\left\{\lambda-(1-5\mu)^2\right\}}\,Cos\,(1-5\mu)\,\chi$$

$$-\frac{7.\sqrt[4]{\lambda}.A^3}{170\left\{\lambda-(1-3\mu)^2\right\}}\,Cos\,(1-3\mu)\,\chi\, +$$

$$\frac{3.\sqrt[4]{\lambda}.A^3}{20\left\{\lambda-(1-\mu)^2\right\}}\,Cos\,(1-\mu)\,\chi\, + C_1, \tag{67}$$

where C_1 is a constant to be determined by the initial conditions .We choose C_1 in such a way that $\chi = 0,\ \zeta = 0$. Hence,

$$C_1 = \left[\frac{9}{160}\left\{\frac{\sqrt[4]{\lambda}.A^3}{\left\{\lambda-(1+5\mu)^2\right\}}+\frac{\sqrt[4]{\lambda}.A^3}{\left\{\lambda-(1-5\mu)^2\right\}}\right\}+\right.$$

$$\frac{7}{160}\left\{\frac{\sqrt[4]{\lambda}.A^3}{\left\{\lambda-(1+3\mu)^2\right\}}+\frac{\sqrt[4]{\lambda}.A^3}{\left\{\lambda-(1-3\mu)^2\right\}}\right\}$$

$$\left.-\frac{3}{20}\left\{\frac{\sqrt[4]{\lambda}.A^3}{\left\{\lambda-(1+\mu)^2\right\}}+\frac{\sqrt[4]{\lambda}.A^3}{\left\{\lambda-(1-\mu)^2\right\}}\right\}\right]. \tag{68}$$

Our next aim is to find out $y_1(\psi)$ using (60). To proceed, we note that to the first order $\chi = \psi - \frac{3}{10\mu}\,Sin\,(2\mu\psi)$. Since we have $\psi = t + 2\,eSin\,t$, $y_1(t) = \rho(t)\xi(t)$ and

$$\rho(\psi) \approx \frac{1}{\sqrt[4]{\lambda}}\left(1-\frac{3Cos(2\mu\psi)}{10}\right),$$

we get

$$\rho(t) \approx \frac{1}{\sqrt[4]{\lambda}}\left(1-\frac{3Cos(2\mu t+4\mu eSin\,t))}{10}\right) \tag{69}$$

and then from (67), we get

$$\zeta(t) = -\frac{9\sqrt[4]{\lambda}A^3}{160\left\{\lambda-(1+5\mu)^2\right\}}\,Cos\Big[(1+5\mu)\,t + 2e\,Sin\,t +$$

$$\left.10\mu eSin\,t - \frac{3\,(1+5\mu)}{10\mu}\,Sin\,\big(2\mu t + 4e\mu Sin\,t\big)\right]$$

$$-\frac{7\sqrt[4]{\lambda}A^3}{160\left\{\lambda-(1+3\mu)^2\right\}}\,Cos\Big[(1+3\mu)\,t + 2e\,Sin\,t +$$

$$\left.6\mu eSin\,t - \frac{3\,(1+3\mu)}{10\mu}\,Sin\,\big(2\mu t + 4e\mu Sin\,t\big)\right]$$

$$+\frac{3\sqrt[4]{\lambda}A^3}{20\left\{\lambda-(1+\mu)^2\right\}}\,Cos\Big[(1+\mu)\,t + 2e\,Sin\,t +$$

$$\left.2\mu eSin\,t - \frac{3\,(1+\mu)}{10\mu}\,Sin\,\big(2\mu t + 4e\mu Sin\,t\big)\right]$$

$$-\frac{9\sqrt[4]{\lambda}A^3}{160\left\{\lambda-(1-5\mu)^2\right\}}\,Cos\Big[(1-5\mu)\,t + 2e\,Sin\,t -$$

$$\left.10\mu eSin\,t - \frac{3\,(1-5\mu)}{10\mu}\,Sin\,\big(2\mu t + 4e\mu Sin\,t\big)\right]$$

$$-\frac{7\sqrt[4]{\lambda}A^3}{160\left\{\lambda-(1-3\mu)^2\right\}}\,Cos\Big[(1-3\mu)\,t + 2e\,Sin\,t -$$

$$\left.6\mu eSin\,t - \frac{3\,(1-3\mu)}{10\mu}\,Sin\,\big(2\mu t + 4e\mu Sin\,t\big)\right]$$

$$+\frac{3\sqrt[4]{\lambda}A^3}{2\left\{\lambda-(1-\mu)^2\right\}}\,Cos\Big[(1-\mu)\,t + 2e\,Sin\,t -$$

$$\left.-\,2\mu eSin\,t - \frac{3\,(1-\mu)}{20\mu}\,Sin\,\big(2\mu t + 4e\mu Sin\,t\big)\right] + C_1, \tag{70}$$

where C_1 is given by the Equation (68). And from the Equation (49), we get

$$y_0(t) = \left[A + \frac{3A^3}{64}\right]Cos(\mu t + 2\mu e\ Sint) -$$

$$\frac{3A^3}{64}\,Cos(3\mu t + 6\mu e\ Sint)\,. \tag{71}$$

Finally we obtain z by the Equation (31)

$$z = y\varpi = (1 - eCost)\,(y_0 + \beta_1 e y_1)\,. \tag{72}$$

where y_0 and y_1 are given by the Equations (71), (69) and (70). Next we determine the constant A. A little algebra provides that A is related to the initial value of z, which is z_0 by the following:

$$A = z_0. \tag{73}$$

Finally, we have to normalize the expression (72) to arrive at a solution which results in z_0 at $t = 0$. Carrying out this process results in the final expression for z

$$z = \frac{z_0\left(1 - e\,Cost\right)}{(1 - e)}\left[\left(1 - \frac{3}{32}z_0^2\right)\right.$$

$$Cos\left(\mu t + 2\mu\, e\, Sint\right) - \frac{3}{64}z_0^2 Cos\left(3\mu t + 6\mu\, e\, Sint\right)$$

$$+\beta_1 e\left(1 + \frac{1}{2}Cos\left(2\mu t + 4\mu e Sint\right)\right) \times$$

$$\left(-\frac{9z_0^2}{160\left\{\lambda - (1 + 5\mu)^2\right\}}Cos\xi_1\right.$$

$$-\frac{7z_0^2}{160\left\{\lambda - (1 + 3\mu)^2\right\}}Cos\xi_2 +$$

$$\frac{3z_0^2}{20\left\{\lambda - (1 + \mu)^2\right\}}Cos\xi_3$$

$$-\frac{9z_0^2}{160\left\{\lambda - (1 - 5\mu)^2\right\}}Cos\xi_4$$

$$-\frac{7z_0^2}{160\left\{\lambda - (1 - 3\mu)^2\right\}}Cos\xi_5$$

$$+\frac{3z_0^2}{20\left\{\lambda - (-1 + \mu)^2\right\}}Cos\xi_6 +$$

$$\frac{9z_0^2}{160}\left\{\frac{1}{\lambda - (1 + 5\mu)^2} + \frac{1}{\lambda - (1 - 5\mu)^2}\right\}$$

$$+\frac{7z_0^2}{160}\left\{\frac{1}{\lambda - (1 + 3\mu)^2} + \frac{1}{\lambda - (1 - 3\mu)^2}\right\}$$

$$\left.-\frac{3z_0^2}{20}\left\{\frac{1}{\lambda - (1 + \mu)^2} + \frac{1}{\lambda - (1 - \mu)^2}\right\}\right]. \tag{74}$$

where,

$$\xi_1 = (1 + 5\mu)\,t + 2e\,Sin\,t + 10\mu e Sin\,t - \frac{3\,(1 + 5\mu)}{10\mu}Sin\left(2\mu t + 4e\mu Sin\,t\right), \tag{75}$$

$$\xi_2 = (1 + 3\mu)\,t + 2e\,Sin\,t + 6\mu e Sin\,t - \frac{3\,(1 + 3\mu)}{10\mu}Sin\left(2\mu t + 4e\mu Sin\,t\right), \tag{76}$$

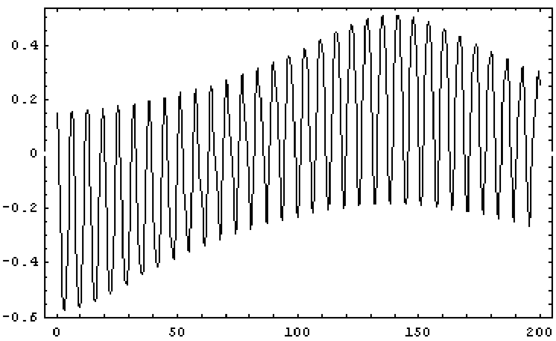
$$\xi_2 = (1 + 3\mu)\,t + 2e\,Sin\,t + 6\mu e Sin\,t - \frac{3\,(1 + 3\mu)}{10\mu}Sin\left(2\mu t + 4e\mu Sin\,t\right), \tag{77}$$

$$\xi_4 = (1 - 5\mu)\,t + 2e\,Sin\,t - 10\mu e Sin\,t - \frac{3\,(1 - 5\mu)}{10\mu}Sin\left(2\mu t + 4e\mu Sin\,t\right), \tag{78}$$

$$\xi_5 = (1 - 3\mu)\,t + 2e\,Sin\,t - 6\mu e Sin\,t - \frac{3\,(1 - 3\mu)}{10\mu}Sin\left(2\mu t + 4e\mu Sin\,t\right), \tag{79}$$

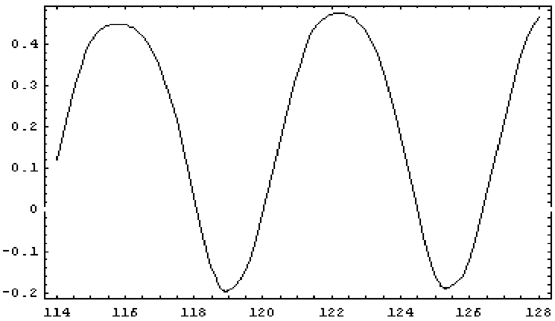
$$\xi_6 = (1 - \mu)\,t + 2e\,Sin\,t - 2\mu e Sin\,t - \frac{3\,(1 - \mu)}{10\mu}Sin\left(2\mu t + 4e\mu Sin\,t\right). \tag{80}$$

The solution (73) is plotted for $z_0 = 0.2$ and $e = 0.4$ in the figure 2(a) and 2(b).



$0 < t \le 200$

Fig. 2 (a). Sitnikov four body motion for e = 0.04, z = 0.2.



$114 < t \leq 120$

Fig. 2 (b). Sitnikov four body motion for e = 0.04, z = 0.2

4. CONCLUSIONS

We have studied an elliptic Sitnikov problem, when extended to four body problem by two methods. In the first method we have found out the analytical solution in (28) which is presented in the figure-1(a) and 1(b). These are drawn for a long time interval and short time interval respectively. To perform the second method we have used the solution (28) and by applying the Courant and Snyder transformation, we get an equation of Harmonic oscillator type (32). The equation of the form of duffing oscillator (45) is obtained by truncating the equation after the first non-linear term. We have used the Lindstedt-Poincare method to find the solution of this equation. The solution of zeroth order equation ($\varepsilon = 0$) is applied to the first order equation to obtain again a Hill's type of equation. Solution of this equation is obtained by the repetition of Courant and Snyder transformation. The final solution is obtained by implementation of inverse transformation to the dependent and independent variables (73). The graph of the solution so obtained, is plotted in the figure 2(a) and 2(b). These are plotted for the long and the short time of interval respectively.

As we are dealing with the elliptic case of the Sitnikov four body problem, the Equation (29) is unperturbed system, term $g_2(t)z^3$ appears as perturbation in the Equation (4). Hence there is the perturbation in the z-direction as well. The effect of this nonlinear term is to shift the frequency as a function of the amplitude and to distort the trajectory $z(t)$. We intend studying the other aspects in our subsequent work.

We observe that the nature of the graphs is oscillatory with amplitude and frequency depending on time.

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