ON CERTAIN CLASS OF ANALYTIC FUNCTIONS

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Abstract: \(P_{\alpha}^{\alpha}[A, B]\) and \(Q_{\alpha}^{\alpha}[A, B]\) denote classes of functions analytic in the disc 
\(E = \{z : |z| < 1\}\) defined by a bounded radius rotation functions. In this paper we have obtained the distortion theorems, coefficients estimate, some radius problems, geometrical properties and studied convolution conditions.

Keywords: Analytic, starlike, convex, positive real part function, bounded radius rotation, convolution

Introduction

Let \(A\) denote the class of analytic functions \(f(z)\) in \(E = \{z : |z| < 1\}\), given by

\[f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)\]

and let \(S, S^*\) and \(C\) be classes of functions in \(A\), which are respectively univalent, starlike and convex in the unit disc \(E\).

Janowski [4] introduced the class \(P[A, B]\) as follows:

Definition 1

An analytic function in \(E\) given by the form \(P(z) = 1 + C_1 z + C_2 z^2 + \ldots\) belongs to \(P[A, B]\) if it satisfies the condition

\[p(z) = \frac{1 + Aw(z)}{1 + Bw(z)}, \quad -1 \leq B < A \leq 1,\]

where, \(w(0) = 0\) and \(|w(z)| \leq 1\). \(P[1, -1] = P\) (the class of analytic function with positive real part satisfying \(\text{Rep}(z) > 0\)).

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**Definition 2**

An analytic function in $E$ given by (1) belongs to $S^*[A,B], -1 \leq B < A \leq 1$, if and only if, \[ \frac{zf'(z)}{f(z)} \in P[A, B] \] and $S^*[1,-1] = S^*$. Also it is well known that an analytic function given by (1) belongs to $C[A,B]$ , if and only if, \[ \frac{(zf'(z))'}{f'(z)} \in P[A, B] \] and $C[1,-1]=C$.

**Definition 3**

A function $f \in A$ is close to convex denoted by $K[A,B,C,D]$ if $\exists$ a starlike function $g(z) \in S^*[C,D]$ such that $\frac{zf'(z)}{g(z)} \in P[A, B]$ and $K[1,-1,1,-1]=K$ (the well known close to convex class due to Kaplan).

**Definition 4**

Let $P_k(\alpha), k \geq 2$ and $0<\alpha \leq 1$, be the class of functions $p$ analytic in $E$ and have the representation

$$p(z) = \frac{1}{2} \int_{-\pi}^{\pi} \frac{1+(1-2\alpha)ze^{-it}}{1-ze^{-it}} d\mu(t),$$

where $\mu(t)$ is a function with bounded variation on $[-\pi,\pi]$ and satisfies the conditions

$$\int_{-\pi}^{\pi} d\mu(t) = 2, \quad \int_{-\pi}^{\pi} |d\mu(t)| \leq k.$$  

We note that $k \geq 2$ and $P_2(\alpha) = P[1-2\alpha,-1] = P(\alpha)$ are the class of analytic function with positive real part greater than $\alpha$. It can easily be seen [5] that $p \in P_k(\alpha)$, if and only if, there exist two analytic functions $p_1, p_2 \in P(\alpha)$ such that

$$p(z) = \frac{k+2}{4} p_1(z) - \frac{k-2}{4} p_2(z)$$

Let $R_k(\alpha)$ denote a subclass of $A$ of functions of bounded radius rotation of order $\alpha$. Then $f \in R_k(\alpha)$, if and only if,

$$\frac{zf'(z)}{f(z)} \in P_k(\alpha), \quad k \geq 2, \quad z \in E.$$  \hspace{1cm} (2)

It is clear that $R_2(\alpha) = S^*(\alpha)$.

Let $f$ be given by (1) and $g$ given by $g(z) = z + \sum_{n=2}^{\infty} a_n z^n \in A$. Then the convolution $f*g$ is
defined by \((f\star g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n\).

**Definition 5**

Let \(f \in A\). Then \(f\) belongs to \(P_k^\alpha[A, B]\) if it satisfies the condition

\[
\frac{f(z)}{g(z)} = \frac{1 + A w(z)}{1 + B w(z)},
\]

where \(g \in R_k(\alpha), -1 \leq B < A \leq 1, w(z)\) is regular, \(w(0) = 0\) and \(w(z)\leq 1\) and \(0 < \alpha \leq 1\).

**Definition 6**

Let \(Q_k^\alpha[A, B]\) denote the class of functions \(F(z) = z^{-1} + c_0 + c_1 z + c_2 z^2 + \ldots\), which are regular in \(0 < |z| < 1\) and satisfy the condition

\[
\frac{F(z)}{G(z)} = \left[\frac{1 + A w(z)}{1 + B w(z)}\right]^{-1},
\]

where \(-1 \leq B < A \leq 1, w(z)\) is regular in \(0 < |z| < 1\) and \(G(z) = z^{-1} + d_0 + d_1 z + d_2 z^2 + \ldots\), is of bounded radius rotation of order \(\alpha\), i.e.

\[
\frac{-z G'(z)}{G(z)} \in P_k(\alpha), \quad 0 < |z| < 1.
\]

**Distortion theorem for the class \(P_k^\alpha[A, B]\)**

**Theorem 1**

If \(f \in P_k^\alpha[A, B]\), then for \(|z| = r, 0 < r < 1\)

\[
1 - A r \frac{(1-r)^{(k-2)(1-\alpha)/2}}{1 - B r \frac{(1+r)^{(k+2)(1-\alpha)/2}}} \leq |f(z)| \leq \frac{1 + A r \frac{(1+r)^{(k-2)(1-\alpha)/2}}}{1 + B r \frac{(1+r)^{(k+2)(1-\alpha)/2}}} \ldots
\]

(3)

This result is sharp.

**Proof**

Since \(f \in P_k^\alpha[A, B]\), we have
\[
\frac{f(z)}{g(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}, \quad -1 \leq B < A \leq 1,
\]

where \( g \in R_k(\alpha) \). By Schwarz's lemma, we have \(|w(z)| \leq |z|\).

If \( p(z) = \frac{1 + Aw(z)}{1 + Bw(z)} \), \(-1 \leq B < A \leq 1\), then it is well known [4] that \( p \in P[A,B] \) and satisfies

\[
\frac{1 - Ar}{1 - Br} \leq |p(z)| \leq \frac{1 + Ar}{1 + Br} \quad \ldots (4)
\]

Further if \( g(z) \) is a function of bounded radius rotation of order \( \alpha \), then by [7]

\[
\frac{(1 - r)^{(k-2)(1-\alpha)/2}}{(1 + r)^{(1-\alpha)/2}} \leq g(z) \leq \frac{(1 + r)^{(k-2)(1-\alpha)/2}}{(1 - r)^{(1-\alpha)/2}} \quad \ldots (5)
\]

equations (4),(5) together imply the inequality (3).

This result is sharp, if we take

\[
p(z) = \frac{1 + Az}{1 + Bz} \quad \text{and} \quad g(z) = \frac{(1 + \theta_1 z)^{(k-2)(1-\alpha)/2}}{(1 + \theta_2 z)^{(1+2)(1-\alpha)/2}}, \quad |\theta_1| = |\theta_2| = 1.
\]

**Remarks**

1. On taking \( k = 2 \), we have a result of Ganesan [2].
2. On taking \( k = 2, B = -\lambda \beta \) and \( A = \beta \) with \( w(z) \) replaced by \( -w(z) \), we get the result of Goel and Sohi [3].

**Coefficient estimates for the class \( P_k^\alpha[A,B] \)**

To find the coefficient estimates for the class \( P_k^\alpha[A,B] \), we need the following lemmas:

**Lemma 1** [4]

Let \( p \in P[A,B] \) and \( p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \). Then \( |c_n| \leq A - B \).

**Lemma 2**

If \( p \in P_k(\alpha), p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \), then \( |c_n| \leq k(1 - \alpha) \).
Proof

This can be easily seen using Lemma 1 and the relation

\[ p(z) = \frac{k+2}{4} p_1(z) - \frac{k-2}{4} p_2(z) \]

with \( A = 1 - 2\alpha \) and \( B = -1 \).

Using Lemma 2, we can prove Lemma 3

**Lemma 3**

Let \( g \in \mathbb{R}_k (\alpha) \), \( g(z) = z + b_2z^2 + b_3z^3 + \ldots \). Then

\[ |b_2| \leq k(1 - \alpha) \quad \text{and} \quad |b_3| \leq \frac{k(1 - \alpha)}{2}(k - k\alpha + 1) \]

**Proof**

Let \( g \in \mathbb{R}_k (\alpha) \). Then \( zg'(z) = P(z)g(z), P(z) \in \mathbb{R}_k (\alpha) \). If \( g(z) = z + b_2z^2 + \ldots \) and \( p(z) = 1 + c_1z + c_2z^2 + \ldots \), then

\[ z + 2b_2z^2 + 3b_3z^3 + \ldots = (z + b_2z^2 + b_3z^3 + \ldots)(1 + c_1z + c_2z^2 + \ldots) \]

Equating the coefficient of \( z^2 \) and \( z^3 \) on both sides and using Lemma 1 and Lemma 2, we have

\[ 2b_2 = c_1 + b_2 \quad |b_2| \leq |c_1| \leq k(1 - \alpha) \]

and

\[ 3b_3 = b_3 + c_1b_2 + c_2 \]

\[ |b_3| \leq \frac{b_3c_1 + c_2}{2} \leq \frac{k^2(1 - \alpha)^2 + k(1 - \alpha)}{2} = \frac{k(1 - \alpha)}{2}(k - k\alpha + 1) \]

**Theorem 2**

Let \( f \in \mathbb{P}_k^n [A, B] \), where \( f(z) = z + \sum_{n=2}^\infty a_n z^n \). Then

\[ |a_2| \leq (1 - \alpha)k + (A - B) \]

and

\[ |a_3| \leq (A - B) + k(1 - \alpha)(A - B) + \frac{k(1 - \alpha)}{2}(k - k\alpha + 1) \]
These bounds are sharp.

**Proof**

Since \( f \in P_k^\alpha[A,B] \), there exists a function \( g \in R_k(\alpha) \) such that 
\[
f(z) = g(z)p(z), \quad p \in P[A,B].
\]

If \( g(z) = z + \sum_{n=2}^\infty b_n z^n \)
and \( p(z) = 1 + c_1 z + c_2 z^2 + \ldots \),
then 
\[
z + a_2 z^2 + a_3 z^3 + \ldots = (z + b_2 z^2 + b_3 z^3 + \ldots)(1 + c_1 z + c_2 z^2 + \ldots)
\]

Equating the coefficient of \( z^2 \) and \( z^3 \) on both sides and using Lemma 1 and Lemma 3 we have 
\[
a_2 = b_2 + c_1
\]
\[
|a_2| = (1 - \alpha)k + (A - B)
\]
and 
\[
a_3 = c_2 + b_2 c_1 + b_3
\]
\[
|a_3| \leq (A - B) + k (1 - \alpha)(A - B) + \frac{k (1 - \alpha)}{2}(k - k^\alpha + 1).
\]

This result is sharp as can be seen by the function 
\[
f(z) = \frac{(1 - z)^{(k-2)(1-\alpha)/2}}{(1 + z)^{(k+2)(1-\alpha)/2}} \frac{1 + Az}{1 + Bz}.
\]

**Remarks**

i. If \( k = 2 \), this result agrees with the result of Ganesan [2] and when \( k = 2, A = \beta, B = -\lambda \beta \), these results correspond to the result of Goel and Sohi [3].

ii. If \( B = 0 \), we get 
\[
|a_n| \leq A(n - 1) + n, \quad n \geq 2
\]
with sharp bounds as discussed in [3] is also obtainable.

**Argument of** \( \frac{f(z)}{z} \) **when** \( f \in P_k^\alpha[A,B] \)

To discuss the argument of the class  \( P_k^\alpha[A,B] \), we need the following Lemma:
**Lemma 4**

Let $f \in R_k(\alpha)$. Then

$$\left| \arg \frac{f(z)}{z} \right| \leq k(1-\alpha) \sin^{-1} r.$$  

**Proof**

It is well known that if $f \in R_k(\alpha)$, then there exist two functions $s_1, s_2 \in S^*(\alpha)$ such that

$$f(z) = \left( \frac{k+2}{4} \right)^{-\frac{k+2}{2}} \left( \frac{k-2}{4} \right)^{\frac{k-2}{4}}.$$  

Thus

$$\left| \arg \frac{f(z)}{z} \right| = \left| \frac{k+2}{4} \arg \frac{s_1(z)}{z} - \frac{k-2}{4} \arg \frac{s_2(z)}{z} \right| \leq \frac{k+2}{4} \left| \arg \frac{s_1(z)}{z} \right| + \frac{k-2}{4} \left| \arg \frac{s_2(z)}{z} \right|.$$  

It is known [8] that if $s \in S^*(\alpha)$, then

$$\left| \arg \frac{s(z)}{z} \right| \leq 2(1-\alpha) \sin^{-1} r.$$  

Hence

$$\left| \arg \frac{f(z)}{z} \right| \leq k(1-\alpha) \sin^{-1} r.$$  

Sharpness is satisfied for $f(z) = \frac{(1+\theta_1 z)^{(1-\alpha)(\frac{k+2}{2})}}{(1+\theta_2 z)^{(1-\alpha)(\frac{k-2}{2})}}$.

**Lemma 5** [4]

Let $p \in P[A, B]$. Then

$$\left| \arg \frac{p(z)}{z} \right| \leq \sin^{-1} \frac{(A-B)r}{1-ABr^2}.$$  

Using Lemma 4 and Lemma 5, we can prove
**Theorem 3**

Let \( f \in P^\alpha_k[A, B] \). Then

\[
\left| \arg \frac{f(z)}{z} \right| \leq k (1 - \alpha) \sin^{-1} r + \sin^{-1} \frac{(A - B)r}{1 - AB r^2}.
\]

**Proof**

Since \( f \in P^\alpha_k[A, B] \), therefore

\[
f(z) = g(z)p(z), \quad p(z) \in P[A, B] \quad \text{and} \quad g \in R^\alpha_k(\alpha). \]

By Lemma 4, we have

\[
\left| \arg \frac{g(z)}{z} \right| \leq k (1 - \alpha) \sin^{-1} r
\]

and by Lemma 5, we have

\[
\left| \arg p(z) \right| \leq \sin^{-1} \frac{(A - B)r}{1 - AB r^2}
\]

Using (6) and (7), we have the result.

Sharpness follows by taking

\[
\frac{f(z)}{g(z)} = \frac{1 + A \theta_1 z}{1 + B \theta_1 z}, \quad |\theta_1| = 1
\]

and

\[
g(z) = \frac{(1 + \theta_2 z)^{(1 - \alpha)k - 2}/2}{(1 + \theta_2 z)^{(1 - \alpha)k + 2}/2}, \quad |\theta_2| = 1.
\]

Then

\[
\left| \arg \frac{f(z)}{g(z)} \right| = \sin^{-1} \frac{(A - B)r}{1 - AB r^2}
\]

and

\[
\left| \arg \frac{g(z)}{z} \right| = \arg(1 + \theta_2 z)^{(1 - \alpha)k - 2}/2 + \arg(1 + \theta_2 z)^{(1 - \alpha)k + 2}/2
\]
Using Lemma 4, we have
\[
\arg \frac{g(z)}{z} = (1 - \alpha)k \sin^{-1} r
\] ...

(9)

Using (8) and (9), we have that
\[
\arg \frac{f(z)}{z} = (1 - \alpha)k \sin^{-1} r + \sin^{-1} \frac{(A - B)r}{1 - ABr^2}.
\]

**Remark**

For \( k = 2 \) again this result agrees with the result in [2], and when \( A = \beta > 0, B = -\lambda \beta \) and replacing \( w(z) \) by \( -w(z) \), we have the result of Goel and Sohi [3].

**Some radius problems for \( P_k^\alpha[A, B] \)**

**Lemma 6** [1]

Let \( p \in P[A, B] \). Then for \( z \in E \)
\[
\text{Re} \left\{ \alpha p(z) + \beta z p'(z) \right\} p(z) > \left\{ \frac{\alpha - [(A - B)\beta + 2\alpha A]r + \alpha A^2 r^2}{(1 - Ar)(1 - Br)} \right\} \quad \text{if} \quad R_1 < R_2,
\]
\[
\left\{ \frac{\beta A + B + \frac{2}{A - B}}{(A - B)(1 - r^2)} \left\{ (L_1 K_1)^{1/2} - \beta (1 - ABr^2) \right\} \right\} \quad \text{if} \quad R_2 < R_1.
\]

where
\[
R_1 = \left( \frac{L_1}{K_1} \right)^{1/2}, R_2 = \frac{1 - Ar}{1 - Br}, L_1 = (1 - A)(1 + Ar^2) \text{ and } K_1 = (1 - B)(1 + Br^2)
\]

This result is sharp.

**Lemma 7** [7]

Let \( g \in R_k(\alpha) \). Then
\[
\text{Re} \left\{ \frac{zg(z)}{g(z)} \right\} > \frac{1 - k(1 - \alpha)r + (1 - 2\alpha)r^2}{1 - r^2}.
\]

Further, since \( g \in R_k(\alpha) \) implies \( \frac{zg(z)}{g(z)} = f(z) \in P_k(\alpha) \), we have for all \( f \in P_k(\alpha) \)
\[ \text{Re } f(z) \geq \frac{1 - k(1 - \alpha)r + (1 - 2\alpha)r^2}{1 - r^2}. \]

**Theorem 4**

Let \( f \in P_\kappa^n[A, B] \). Then

\[
\text{Re } \frac{zf'(z)}{f(z)} \geq \begin{cases} M_1(r) & \text{for } R_1 \leq R_2, \\ M_2(r) & \text{for } R_2 \leq R_1 \end{cases},
\]

where

\[
M_1(r) = \frac{1 - k(1 - \alpha)r + (1 - 2\alpha)r^2}{1 - r^2} - \frac{(A - B)r}{(1 - Ar)(1 - Br)},
\]

\[
M_2(r) = \frac{1 - k(1 - \alpha)r + (1 - 2\alpha)r^2}{1 - r^2} + \frac{A + B}{A - B} + \frac{2}{(1 - r^2)(A - B)} \left[ (L_1K_1)^{1/2} - (1 - ABr^2) \right]
\]

and \( R_1, R_2, L_1 \text{ and } K_1 \) are defined in Lemma 6.

**Proof**

Since \( f \in P_\kappa^n[A, B] \), there exists a function \( g \in R_\kappa(\alpha) \) such that

\[
\frac{f(z)}{g(z)} = P(z) \in P[A, B]
\]

Using logarithmic differentiation, we obtain

\[
\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \frac{zp'(z)}{p(z)}
\]

and

\[
\text{Re } \frac{zf'(z)}{f(z)} \geq \min \text{Re } \frac{zg'(z)}{g(z)} + \min \text{Re } \frac{zp'(z)}{p(z)}.
\]

Using Lemma 6 with \( \alpha = 0, \beta = 1 \) and Lemma 7, we have the result.

Sharpness of the bounds follow if we choose \( g_i(z)(i = 1, 2) \), of bounded radius rotation of order \( \alpha \) such that
**Case 1:** If $R_1 \leq R_2$, we take $P_1(z) = \frac{1 + Az}{1 + Bz}$, and 
\[ \frac{zg'(z)}{g(z)} = \frac{1 + (1 - \alpha)z + (1 - 2\alpha)z^2}{1 - r^2}. \]

Then \[
\frac{zP_1'(z)}{P_1(z)} = \frac{(A - B)z}{(1 + Az)(1 + Bz)}. \]

Thus at $z = -r$, \[
\operatorname{Re} \frac{zP_1'(z)}{P_1(z)} = \frac{-(A - B)r}{(1 - Ar)(1 - Br)}. \]

**Case 2:** If $R_2 \leq R_1$, we take $p_2(z) = \frac{f_2(z)}{g_2(z)} = \frac{1 + Aw_1(z)}{1 + Bw_1(z)}$ and

\[
\frac{zg'(z)}{g(z)} = \frac{1 + k(1 - \alpha)w_1(z) + (1 + 2\alpha)w_1^2}{1 - w^2(z)} \quad \text{with} \quad w_1(z) = \frac{z(z - c_i z)}{(1 - c_i z)}, \quad \text{where} \quad c_i \text{ defined by the condition}
\]

\[
\operatorname{Re} \left[ \frac{1 + Aw_1(z)}{1 + Bw_1(z)} \right] = R_1 \text{ at } z = -r.
\]

Now \[
\frac{zp_2'(z)}{p_2(z)} = \frac{(A - B)zw_1'(z)}{(1 + Aw_1(z))(1 + Bw_1(z))}.
\]

In fact from the inequalities $R_2 \leq R_1 \leq c + p$, where $c = \frac{1 - ABr^2}{1 - B^2r^2}$, $p = \frac{(A - B)r}{1 - B^2r^2}$ and we have

\[
\frac{1 - Ar}{1 - Br} \leq \frac{1 + AT}{1 + BT} \leq \frac{1 + Ar}{1 + Br}, T = w_1(-r).
\]

Hence $|T| \leq r$ and $T^2 \leq r^2$ which yields

\[
\frac{r^2(r + c_i)^2}{(1 + rc_i)^2} \leq r^2. \quad \text{Thus} \quad |c_i| \leq 1
\]

Further $|zw_1'(z) - w_1(z)| = \frac{|z|^2 - |w(z)|^2}{1 - |z|^2}$, for $w_1(z) = \frac{z(z - c_i z)}{(1 - c_i z)}$, $|c_i| \leq 1$.

\[
w_1(-r) = T = \frac{1 - R_1}{BR_1 - A} = \frac{r(r - c_i)}{(1 + c_i^2)}.
\]

Hence $c_i = \frac{r^2 - T^2}{r(T - 1)}$ and $\frac{r^2 - T^2}{(1 - r^2)} = \frac{r^2(1 - q^2)}{(1 + qr)}$ and $\left[ zw_1'(z) - w_1(z) \right]_{z=t} = \frac{r^2 - T^2}{(1 - r^2)}$.
Now \[ \text{Re} \left[ \frac{zp'(z)}{p(z)} \right] = \frac{(A - B)}{(1 - AT)(1 - BT)} \left\{ T - \frac{r^2 - T^2}{1 - r^2} \right\} \]

Using \[ T = \frac{1 - R_i}{BR_i - A} \] with \[ R_i = \frac{(1 - A)(1 + Ar^2)}{(1 - B)(1 + Br^2)} \] (see [1]),

and simplifying, we have \[ \text{Re} \left[ \frac{zp'(z)}{p(z)} \right]_{z=-r} = \frac{A + B}{A - B} + \frac{2}{1 - r^2} \left\{ (L_1 K_1)^{1/2} - (1 - ABr^2) \right\}, \]

where \[ L_i = (1 - A)(1 + Ar^2), K_i = (1 - B)(1 + Br^2), \] (see[1]).

Thus the equality in our theorem holds at \( z=-r \) for

\[ f_1(z) = \frac{1 + Az}{1 + Bz} g_1(z), \text{if} R_1 \leq R_2 \]

and for \[ f_2(z) = \frac{1 + Aw_1(z)}{1 + Bw_1(z)} g_2(z), \text{if} R_2 \leq R_1, \text{where} \ g_1(z), g_2(z) \in R_i(\alpha). \]

**Theorem 5**

If \( f \in P_k^\alpha[A, B] \), then \( f \) is starlike in

\[ |z| < \begin{cases} r_1 & \text{for } R_1 \leq R_2, \\ r_2 & \text{for } R_2 \leq R_1, \end{cases} \]

where \( R_1 \) and \( R_2 \) are defined as in Lemma 6 and \( r_1, r_2 \) are respectively the positive roots of the following two equations

\[ (1 - k (1 - \alpha) r + (1 - 2 \alpha) r^2)(1 - Ar)(1 - Br) - (A - B)r(1 - r^2) = 0 \]

\[ (1 - k (1 - \alpha) r + (1 - 2 \alpha) r^2)(A - B) + (1 - r^2)(A + B) + 2 \left[ (L_i K_i)^{1/2} - (1 - ABr^2) \right] = 0, \]

where \( K_i \) and \( L_i \) are defined in Lemma 6. This result is sharp.

**Proof**

It follows from Theorem 4 that if \( f \in P_k^\alpha[A, B] \), then \( \text{Re} \left[ \frac{zf'(z)}{f(z)} \right] \geq M_1(r) \), if \( R_1 \leq R_2 \) and
\[ \text{Re} \frac{zf''(z)}{f(z)} \geq M_2(r), \text{if } R_2 \leq R_1. \] Then

\[ \text{Re} \frac{zf''(z)}{f(z)} \geq \frac{(1-k(1-\alpha)r + (1-2\alpha)r^2)(1-Ar)(1-Br) - (A-B)r(1-r^2)}{(1-r^2)(1-Ar)(1-Br)} > 0, \text{forall } |z| < r. \]

If \( R_1 \leq R_2 \) and

\[ \text{Re} \frac{zf''(z)}{f(z)} \geq \frac{(1-k(1-\alpha)r + (1-2\alpha)r^2)(A-B) + (1-r^2)(A+B) + 2(L_{1}K_{1})z - (1-ABr^2)}{(1-r^2)(A-B)} > 0. \]

For all, if \( R_2 \leq R_1 \).

For special cases see [2] and [3].

**Lemma 8**

Let \( g_1(z) \) and \( g_2(z) \in R_k(\alpha) \). Then \( G(z) = (g_1(z))^\rho (g_2(z))^{1-(\rho + \gamma)} \) belongs to \( R_k(\alpha_1) \) where \( \alpha_1 = 1-(1-\alpha)(\rho + \gamma) \).

**Proof**

A logarithmic differentiation yields

\[ \frac{zG'(z)}{G(z)} = \rho \frac{zg'_1(z)}{g_1(z)} + \gamma \frac{zg'_2(z)}{g_2(z)} + (1-(\rho + \gamma)) \]

\[ = \rho K_1(z) + \gamma K_2(z) + (1-(\rho + \gamma)) \]

where \( K_1, K_2 \in P_k(\alpha) \). From the definition of \( P_k(\alpha) \), there exists \( h_i, i = 1,2,3,4 \in P(\alpha) \) such that

\[ \frac{zG'(z)}{G(z)} = \rho \left[ \frac{k+2}{4} h_1(z) - \frac{k-2}{4} h_2(z) \right] + \gamma \left[ \frac{k+2}{4} h_3(z) - \frac{k-2}{4} h_4(z) \right] + (1-(\rho + \gamma)) \]

It is well known that if \( h \in P(\alpha) \), then \( h(z) \) can be written as

\[ h(z) = (1-\alpha)p(z) + \alpha, \text{where } \text{Re } p(z) > 0 \]

and
\[
\frac{zG'(z)}{G(z)} = \rho \left[ \frac{k+2}{4} [(1-\alpha)p_1(z) + \alpha] - \rho \frac{k-2}{4} [(1-\alpha)p_2 + \alpha] + \gamma \frac{k+2}{4} [(1-\alpha)p_3 + \alpha] - \frac{k-2}{4} [(1-\alpha)p_4 + \alpha] + (1 - (\rho + \gamma)) \right].
\]

Since the class \( P \) is a convex set, then
\[
\frac{\rho p_1(z) + \gamma p_3(z)}{\rho + \gamma} = H_1(z) \quad \text{and} \quad \frac{\rho p_2(z) + \gamma p_4(z)}{\rho + \gamma} = H_2(z),
\]
where \( \text{Re} \ H_i(z) > 0, i = 1,2 \). Hence (10) can be written as
\[
\frac{zG'(z)}{G(z)} = \frac{k+2}{4} [(1-\alpha)(\rho + \gamma)H_1(z) + [1 - (1 - \alpha)(\rho + \gamma)]
- \frac{k-2}{4} [(1-\alpha)(\rho + \gamma)H_2(z) + [1 - (1 - \alpha)(\rho + \gamma)]
= \frac{k+2}{4} T_1(z) + \frac{k-2}{4} T_2(z), T_1, T_2 \in P(\alpha_1) \text{and} \alpha_1 = 1 - (1 - \alpha)(\rho + \gamma)
\]
This shows that \( G \in R_k(\alpha_1) \).

**Theorem 6**

Let \( f_1, f_2 \in P_k^\alpha[A,B] \). Then
\[
F(z) = (f_1(z))^\rho (f_2(z))^\gamma z^{1-(\rho + \gamma)}
\]
belongs to \( P_k^\alpha[A,B] \), where \( \alpha_1 = 1 - (1 - \alpha)(\rho + \gamma) \).

**Proof**

Let \( G(z) = (g_1(z))^\rho (g_2(z))^\gamma z^{1-(\rho + \gamma)} \). Then
\[
\frac{F(z)}{G(z)} = \left( \frac{f_1(z)}{g_1(z)} \right)^\rho \left( \frac{f_2(z)}{g_2(z)} \right)^\gamma
= (h_1(z))^{\rho} (h_2(z))^{\gamma}, \quad (\rho + \gamma) \leq 1,
\]
where \( h_1, h_2 \in P[A,B] \).
Hence $F \in P^\alpha_k[A, B], \alpha_1 = 1 - (1 - \alpha)(\rho + \gamma)$.

Some geometrical properties

In this part we shall investigate the behavior of $\arg f(z)$ at a point $w(\theta) = F(re^{i\theta})$ to the image $\Gamma_r$ of the circle $C_r = \{ z : |z| = r \}, 0 \leq r < 1$ and where $\theta$ is any number of the interval $(0, 2\pi)$ under the mapping by means of function $f$ from the class $P^\alpha_k[A, B]$. We have

**Theorem 7**

If $F \in P^\alpha_k[A, B]$ and $0 \leq r < 1$, then for $\theta_2 < \theta_1, \theta_1, \theta_2 \in [0, 2\pi]$

$$\arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1}) = \int_{\theta_1}^{\theta_2} \text{Re} \left[ \frac{re^{i\theta}f'(re^{i\theta})}{re^{i\theta}} \right] \geq -\pi + \{1 - (1 - \alpha)k + (1 - 2\alpha)(\theta_2 - \theta_1)\} + 2\pi C \cos \frac{A - B}{1 - AB}$$

where $-1 < B < A \leq 1$ and $0 < \alpha \leq 1$.

**Proof**

If $f \in P^\alpha_k[A, B]$, then $\frac{f(z)}{p(z)} = p(z)$, where $p \in P[A, B]$.

Thus

$$\text{Re} \frac{zf'(z)}{f(z)} = \text{Re} \frac{zg'(z)}{g(z)} + \text{Re} \frac{zp'(z)}{p(z)} \quad \ldots \quad (11)$$

Let $z = re^{i\theta}, 0 < r < 1, \theta \in [0, 2\pi]$. Integrating (11) with respect to $\theta$ in the interval $[\theta_1, \theta_2], \theta_1 < \theta_2$, we have

$$\int_{\theta_1}^{\theta_2} \text{Re} \frac{re^{i\theta}f'(re^{i\theta})}{f(re^{i\theta})} d\theta = \arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1})$$

$$= \int_{\theta_1}^{\theta_2} \text{Re} \frac{re^{i\theta}g'(re^{i\theta})}{g(re^{i\theta})} d\theta + \int_{\theta_1}^{\theta_2} \text{Re} \frac{re^{i\theta}p'(re^{i\theta})}{p(re^{i\theta})} d\theta$$

Since $f \in R_k(\alpha)$, it follows that

$$\min_{g \in R_k(\alpha)} \int_{\theta_1}^{\theta_2} \text{Re} \frac{re^{i\theta}g'(re^{i\theta})}{g(re^{i\theta})} d\theta \geq \frac{1 - k(1 - \alpha)r + (1 - 2\alpha)r^2}{1 - r^2}(\theta_2 - \theta_1), \quad \text{See [7]}. $$
Now in the second integral, we observe that
\[
\frac{\partial}{\partial \theta} \arg p(re^{i\theta}) = \frac{\partial}{\partial \theta} \text{Re} \{ -i \ln p(re^{i\theta}) \} = \text{Re} \frac{re^{i\theta} p'(re^{i\theta})}{p(re^{i\theta})}.
\]
Consequently
\[
\int_0^\theta \text{Re} \left[ \frac{re^{i\theta} p'(re^{i\theta})}{p(re^{i\theta})} \right] d\theta = \arg p(re^{i\theta_2}) - \arg p(re^{i\theta_1})
\]
and
\[
\max_{p \in \mathcal{P}[A,B]} \left| \int_0^\theta \text{Re} \frac{re^{i\theta} p'(re^{i\theta})}{p(re^{i\theta})} d\theta \right| \leq \max_{p \in \mathcal{P}[A,B]} \left| \arg p(re^{i\theta_2}) - \arg p(re^{i\theta_1}) \right|
\]
Using Lemma 5, we have
\[
\max_{p \in \mathcal{P}[A,B]} \arg p(re^{i\theta}) = \sin^{-1} \frac{(A-B)r}{1-ABr^2}
\]
\[
\leq 2 \sin^{-1} \frac{(A-B)r}{1-ABr} = \pi - 2 \cos^{-1} \frac{(A-B)r}{1-ABr}
\]
Hence
\[
\arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1}) \geq -\pi + 2 \cos^{-1} \frac{(A-B)r}{1-ABr} + \frac{1-k(1-\alpha)r + (1-2\alpha)r^2}{1-r^2} (\theta_2 - \theta_1).
\]
The value of the right side is depending on the value of \( r \) and it takes its smallest value at \( r = 1 \). Thereby we obtain the required result.

A convolution conditions for \( p^\alpha_k [A,B] \)

In 1973, Rushweyh and Sheil-Small [9] proved the polya-Schoenberg conjecture, namely, if \( f \) is convex or starlike or close to convex and \( \phi \) is convex then \( f * \phi \) belongs to the same class. In the following we shall prove the analogue of this conjecture for the class \( p^\alpha_k [A,B] \)and give some of its applications. We need the following lemma with simple modification.
Lemma 9 [6]

Let \( f \in R_\alpha (\alpha) \). Then \( G = f^* \phi \in R_\alpha (\alpha) \) where \( \phi \) is convex in \( E \).

Theorem 8

Let \( F \in P_k^{\alpha} [A,B] \) and \( \phi \) is convex. Then \( F^* \phi \in P_k^{\alpha} [A,B] \).

Proof:

Let \( F \in P_k^{\alpha} [A,B] \). Then \( F(z) = P(z) g(z) \), where \( g \) belongs to \( R_\alpha (\alpha) \) and \( P(z) \in P[A,B] \). It follows from the Lemma 9 that \( g^* \phi \in R_\alpha (\alpha) \). Then \( \frac{F^* \phi}{g^* \phi} \in P[A,B] \).

Remark

As an application of Theorem 8, we have the following

(1) The family \( P_k^{\alpha} [A,B] \) is invariant under the following operators.

\[
F_1(f) = \sum_{n=1}^{\infty} \frac{1}{n} z^n = -\log(1-z),
\]

\[
F_2(f) = \frac{2}{z} \int_{0}^{z} f(\xi) d\xi = (f^* \phi_1)(z)
\]

\[
F_3(f) = \int_{0}^{z} \frac{f(\xi) - f(x \xi)}{\xi - x} d\xi, \quad |x| \leq 1, \quad x \neq 1
\]

\[
F_4(f) = \frac{1 + c}{c} \int_{0}^{z} \xi^{c-1} f(\xi) d\xi, \quad \Re c > 0
\]

where \( F(f_i(z)) = (f^* \phi_i)(z) \) and \( \phi_i (i = 1, 2, 3, 4) \) are convex univalent functions which satisfy

\[
\phi_1(z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n = -\log(1-z),
\]

\[
\phi_2(z) = \sum_{n=1}^{\infty} \frac{2}{n+1} z^n = -2\left[z + \log(1-z)\right],
\]

\[
\phi_3(z) = \sum_{n=1}^{\infty} \frac{1-x^n}{n(1-x)} z^n = \frac{1}{1-x} \log \frac{1-xz}{1-z}, \quad |x| \leq 1, \quad x \neq 1,
\]

\[
\phi_4(z) = \sum_{n=1}^{\infty} \frac{1+c}{n+c} z^n, \quad \Re c > 0.
\]
Now let $D_\lambda F(z) = (1 - \lambda)F(z) + \lambda z F'(z) = (\psi_{\lambda} * F)(z)$...........................(12)

where $\lambda > 0$ and let $\psi_{\lambda}(z) = \frac{z[1 - (1 - \lambda)z]}{1 - z^2}$. Then $\psi_{\lambda}(z)$ is convex if

$$|z| = r_\lambda = \frac{1}{2\lambda + \sqrt{4\lambda^2 - 2\lambda + 1}}$$

...(13)

Thus, we have

(2) Let $F(z) \in P^n[A, B]$. Then $D_\lambda F(z) = \psi_{\lambda} * F$ belongs to the same class for $|z| < r_\lambda$,

where $r_\lambda$ is given by (13).

Now let $\mu(F) = zF'(z)$. This differential operator can be written as $\mu(F) = \phi * F$,

where

$$\phi(z) = \sum_{n=1}^{\infty} nz^n = -\frac{z}{1 - z^2}$$

...(14)

It can be easily verified that the radius of convexity of $\phi$ is given by $r_c(\phi) = 2 - \sqrt{3}$. This fact together with Theorem 8 yields

(3) If $f \in P^n[A, B]$ then $\phi * f \in P^n[A, B]$ where $\phi$ is given by (14) if $|z| = r_c < 2 - \sqrt{3}$.

**Radius of starlikeness for the class $Q^n_k[A, B]$**


The following lemma can be easily derived.

**Lemma 9**

Let $s_i, i = 1, 2$ be given by $s_1(z) = z^{-1} + c_1 + c_2 z + c_3 z^2 + ...$ and

$s_2(z) = z^{-1} + d_1 z + d_2 z^2 + ...$ , and let $s_i, i = 1, 2$ satisfy $-\frac{\text{Re} \frac{zs_1'(z)}{s_1(z)}}{s_1(z)} > \alpha$ . If

$G(z) = z^{-1} + b_1 z + b_2 z^2 + ...$ such that

$$\frac{G(z)}{(s_2(z))^4} = \frac{(s_1(z))^{k+2}}{(s_2(z))^{k-2}}$$

...(15)

then
\[- \frac{zG'(z)}{G(z)} \in P_k(\alpha) .\]

**Proof**

Differentiating (15) logarithmically yields

\[
\frac{zG'(z)}{G(z)} = \frac{k + 2}{4} \frac{zs_1'(z)}{s_1(z)} - \frac{k - 2}{4} \frac{zs_2'(z)}{s_2(z)} .
\]

This implies that

\[
- \frac{zG'(z)}{G(z)} = \frac{k + 2}{4} \left( - \frac{z}{s_1(z)} s_1'(z) \right) - \frac{k - 2}{4} \left( - \frac{z}{s_2(z)} s_2'(z) \right)
\]

or

\[
- \frac{zG'(z)}{G(z)} = \frac{k + 2}{4} p_1(z) - \frac{k - 2}{4} p_2(z) ,
\]

where \( \text{Re } p_i(z) > \alpha , i = 1, 2 \) and \( - \frac{zG'(z)}{G(z)} \in P_k(\alpha) .\)

**Theorem 9**

If \( F \in Q_k^n [A, B] , \) then for \( |z| = r < 1 \)

\[- \text{Re} \frac{zF'(z)}{F(z)} \geq \begin{cases} M_1(r) , & \text{for } R_1 \leq R_2 \\ M_2(r) , & \text{for } R_2 \leq R_1 \end{cases} ,
\]

where

\[
M_1(r) = \frac{1 - k(1 - \alpha)r + (1 - 2\alpha)r^2}{1 - r^2} - \frac{(A - B)r}{(1 - Ar)(1 - Br)} ,
\]

\[
M_2(r) = \frac{1 - k(1 - \alpha)r + (1 - 2\alpha)r^2}{1 - r^2} + \frac{A + B}{A - B} + \frac{2}{1 - r^2} \left( \frac{L_1 K_1}{(A - B)^2} - (1 - ABr^2) \right)
\]

and \( R_1, R_2, L_1, \text{and } K_1 \) are defined in Lemma 6 .

**Proof**

Since \( F \in Q_k^n [A, B] , \) therefore
\[ p(z) = \left( \frac{F(z)}{G(z)} \right)^{-1} = \frac{1+Aw(z)}{1+Bw(z)}, \text{ where } -1 \leq B < A \leq 1 \quad \ldots \quad (16) \]

\( w(z) \) is analytic in \( E \) and satisfies \( w(0) = 0, |w(z)| < 1 \),

Differentiating (16) logarithmically, we have

\[ \frac{zp'(z)}{p(z)} = -\frac{zF'(z)}{F(z)} + \frac{zG'(z)}{G(z)} \]

or

\[ -\frac{zF'(z)}{F(z)} = -\frac{zG'(z)}{G(z)} + \frac{zp'(z)}{p(z)}. \]

Using Lemma 6, we have

\[ -\operatorname{Re} \frac{zF'(z)}{F(z)} \geq -\operatorname{Re} \frac{zG'(z)}{G(z)} - \frac{(A-B)r}{(1-Ar)(1-Br)} \quad \text{if} \quad R_1 \leq R_2 \]

\[ \geq -\operatorname{Re} \frac{zG'(z)}{G(z)} + \frac{(A+B)}{(A-B)} + \frac{2\left[ (L,K_1)^{1/2} - (1-ABr^2) \right]}{(A-B)(1-r^2)} \]

and since \( G \) is of bounded radius rotation of order \( \alpha \), using Lemma 7 we have

\[ \operatorname{Re} \frac{zG'(z)}{G(z)} \geq 1 - (1-\alpha)kr + (1-2\alpha)r^2, \quad |z| < r \quad \ldots \quad (17) \]

Using (17), we have the required result. The bounds are sharp. This can be seen by choosing \( G_1(z) \) of bounded radius variation of order \( \alpha \) such that

\[ -\frac{zG'(z)}{G(z)} \geq \frac{1-(1-\alpha)kz + (1-2\alpha)z^2}{1-z^2} \quad \text{if} \quad R_1 \geq R_2, \]

\[ -\frac{zG'(z)}{G(z)} \geq \frac{1-(1-\alpha)kw_1(z) + (1-2\alpha)w_1^2(z)}{1-w_1^2(z)} \quad \text{if} \quad R_2 \geq R_1 \]

and take \( F_1(z) \) such that it satisfies

\[ p_1(z) = \left[ \frac{F_1(z)}{G_1(z)} \right]^{-1} = \frac{1+Az}{1+Bz}, \quad \text{if} \quad R_1 \leq R_2 \]
$$= \frac{1+Aw_1(z)}{1+Bw_1(z)} \text{ if } R_2 \leq R_1,$$

where $w_1(z) = \frac{z(1-c_1z)}{1-c_1z}$ with $|c_1| \leq 1$. Proceeding in the same way as in proving the sharpness of Theorem 4, we can prove that this result is sharp.

**Theorem 10**

If $F \in Q^\alpha_k[A, B]$, then $F$ is starlike for $|z| = r_i < 1, i = 1, 2$

i. $0 < |z| < r_1$ for $R_1 \leq R_2$

ii. $0 < |z| < r_2$ for $R_2 \leq R_1$

where $r_1$ and $r_2$ are the smallest positive roots of the following equations respectively

$$
\left[1-k(1-\alpha)r+(1-2\alpha)r^2\right](1-Ar)(1-Br)-(A-B)r(1-r^2) = 0
$$

$$
\left[1-k(1-\alpha)r+(1-2\alpha)r^2\right](A-B)+(1-r^2)(A+B)+2\left[(L_1K_1)^{1/2}-(1-ABr^2)\right] = 0
$$

**Proof**

Using Theorem 9, we have

$$\text{Re}\left(\frac{F'(z)}{F(z)}\right) \geq M(r)_1, \text{ when } R_1 \leq R_2 \text{ and } \text{Re}\left(\frac{F'(z)}{F(z)}\right) \geq M(r)_2, \text{ when } R_2 \geq R_1.$$ 

Hence

$$\text{Re}\left(\frac{F'(z)}{F(z)}\right) > 0 \text{ for } |z| < r_i, i = 1, 2,$$

and this gives a sufficient condition for any function $F$ to be starlike. Proceeding in the same way as in Theorem 5, we obtain the required result.

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**References**


