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Research Article

Hermite-Hadamard Type Inequalities for GA-convex Functions on the Co-ordinates with Applications

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Abstract: In this paper, the concept of GA-convex functions on the co-ordinates is introduced. By using the concept of GA-convex functions on the co-ordinates, theHölder's integral inequality and a new identity established for twice differentiable functions, Hermite-Hadamard type inequalities for this class of functions are established. Finally, applications to special means of positive numbers are given.

Keywords and Phrases: Convex function, Hermite-Hadamard type inequality, co-ordinated GA-convex function, Hölder's integral integral inequality

(AMS SUBJECT CLASSFICATION: Primary: 26A15, 26A51; Secondary 52A30.**)**

1. INTRODUCTION

A function $\boldsymbol{f}: \boldsymbol{I} \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if

$$\boldsymbol{f}(\boldsymbol{\lambda x} + (\boldsymbol{1} - \boldsymbol{\lambda})\boldsymbol{y}) \leq \boldsymbol{\lambda f}(\boldsymbol{x}) + (\boldsymbol{1} - \boldsymbol{\lambda})\boldsymbol{f}(\boldsymbol{y})$$

for all $\boldsymbol{x}$, $\boldsymbol{y} \in \boldsymbol{I}$ and $\boldsymbol{\lambda} \in [\boldsymbol{0}, \boldsymbol{1}]$.

One of the most famous inequalities for convex functions is Hermite-Hadamard inequality. This double inequality is stated as follows:

Let $\boldsymbol{f}: \boldsymbol{I} \to \mathbb{R}$be a convex function on some nonempty interval $\boldsymbol{I}$of the set of real numbers $\mathbb{R}$. If $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{I}$with $\boldsymbol{a} < \boldsymbol{b}$. Then

$$\boldsymbol{f}\left(\frac{\boldsymbol{a}+\boldsymbol{b}}{\boldsymbol{2}}\right) \leq \frac{\boldsymbol{1}}{\boldsymbol{b}-\boldsymbol{a}} \int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{f}(\boldsymbol{x})\boldsymbol{dx} \leq \frac{\boldsymbol{f}(\boldsymbol{a}) + \boldsymbol{f}(\boldsymbol{b})}{\boldsymbol{2}}.$$

The Hermite-Hadamard inequality has received renewed attention in recent years and a number of papers have been written which provides noteworthy refinements, generalizations and new proofs of the Hermite-Hadamard inequality, see for instance [3, 5, 10, 20], and the references therein.

The classical convexity has been generalized in many ways and one of them is the so called GA-convexity, which is stated in the definition below:

Definition 1 *([14], [15]) A function* $f: I \subseteq \mathbb{R}_0 = [0, \infty) \to \mathbb{R}$ *is said to be GA-convex function on* I *if*

$$f\left(x^{\lambda} y^{1-\lambda}\right) \leq \lambda f(x) + (1-\lambda) f(y)$$

holds for all x, $y \in I$ and $\lambda \in [0,1]$, where $x^{\lambda} y^{1-\lambda}$ and $\lambda f(x) + (1-\lambda) f(y)$ are respectively the weighted geometric mean of two positive numbers x and y and the weighted arithmetic mean of $f(x)$ and $f(y)$.

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In what follows we will use the following notations of means:

For positive numbers $\alpha > 0$ and $\beta > 0$ with $\alpha \neq \beta$

$$A(\alpha,\beta) = \frac{\alpha+\beta}{2}, \qquad L(\alpha,\beta) = \frac{\beta-\alpha}{\ln\beta - \ln\alpha}$$

and

$$L_p(\alpha,\beta) = \begin{cases} \left[\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\beta-\alpha)}\right]^{\frac{1}{p}}, & p \neq -1,0 \\ L(\alpha,\beta), & p=-1 \\ \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right)^{\frac{1}{\beta-\alpha}}, & p=0 \end{cases}$$

are the arithmetic mean, the logarithmic mean and the generalized logarithmic mean of order $p \in \mathbb{R}$ respectively. For further information on means, we refer the readers to [4], [22], [23] and the references therein.

In a very recent paper, Zhang et al. in [26] established the following Hermite-Hadamard type integral inequalities for GA-convex function.

Theorem 1 *[26] Let* $f: I \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ *be a function differentiable function on* I° *and* $a,\ b \in I^\circ$ *with* $a < b$ *and* $f' \in L([a,b])$. *If* $|f'|^q$ *is GA-convex on* $[a,b]$ *for* $q \geq 1$, *we have the following inequality:*

$$\left|bf(b) - af(a) - \int_a^b f(x)dx\right| \leq \frac{[(b-a)A(a,b)]^{1-1/q}}{2^{1/q}}$$
$$\times\left\{[L(a^2,b^2)-a^2]|f'(a)|^q + [b^2 - L(a^2,b^2)]|f'(b)|^q\right\}^{1/q}. \qquad (1)$$

Theorem 2 *[26] Let* $f: I \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ *be a function differentiable function on* I° *and* $a,\ b \in I^\circ$ *with* $a < b$ *and* $f' \in L([a,b])$. *If* $|f'|^q$ *is GA-convex on* $[a,b]$ *for* $q > 1$, *we have the following inequality:*

$$\left|bf(b) - af(a) - \int_a^b f(x)dx\right|$$
$$\leq (\ln b - \ln a)\left[L\left(a^{\frac{2q}{(q-1)}}, b^{\frac{2q}{(q-1)}}\right)\right]^{1-\frac{1}{q}}\left[A\left(|f'(a)|^q |f'(b)|^q\right)\right]^{\frac{1}{q}}. \qquad (2)$$

Theorem 3 *[26] Let* $f: I \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ *be a function differentiable function on* I° *and* $a,\ b \in I^\circ$ *with* $a < b$ *and* $f' \in L([a,b])$. *If* $|f'|^q$ *is GA-convex on* $[a,b]$ *for* $q \geq 1$, *we have the following inequality:*

$$\left|bf(b) - af(a) - \int_a^b f(x)dx\right| \leq \frac{(\ln b - \ln a)^{1-1/q}}{(2q)^{1/q}}\left[L\left(a^{2q/(q-1)}, b^{2q/(q-1)}\right)\right]^{1-1/q}$$
$$\times\left\{[L(a^{2q},b^{2q})-a^{2q}]|f'(a)|^q + [b^{2q} - L(a^{2q},b^{2q})]|f'(b)|^q\right\}^{1/q}. \qquad (3)$$

Theorem 4 *[26] Let* $f: I \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ *be a function differentiable function on* I° *and* $a,\ b \in I^\circ$ *with* $a < b$ *and* $f' \in L([a,b])$. *If* $|f'|^q$ *is GA-convex on* $[a,b]$ *for* $q > 1$ *and* $2q > p > 0$. *Then*

$$\left|bf(b)-af(a)-\int_a^b f(x)dx\right| \leq \frac{(\ln b-\ln a)^{1-1/q}}{p^{1/q}}\left[L\left(a^{(2q-p)/(q-1)},b^{(2q-p)/(q-1)}\right)\right]^{1-1/q}$$

$$\times\left\{[L(a^p,b^p)-a^p]\left|f'(a)\right|^q+[b^p-L(a^p,b^p)]\left|f'(b)\right|^q\right\}^{1/q}. \tag{4}$$

For more recent results on the class of (α,m)-GA convex functions we refere the interested readers to [9] and the references therein.

We now recall some basic concepts about convex functions on the co-ordinates on rectangle from the plane.

Let $\Delta =: [a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$ be a bidimensional interval. A mapping $f:\Delta\to\mathbb{R}$ is said to be convex on Δ if the inequality

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)\leq\lambda f(x,y)+(1-\lambda)f(z,w)$$

holds for all $(x,y),(z,w)\in\Delta$ and $\lambda\in[0,1]$.

A modification for convex functions on Δ, known as co-ordinated convex functions, was introduced by Dragomir [6] as follows:

A function $f:\Delta\to\mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial mappings $f_y:[a,b]\to\mathbb{R}$, $f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}$, $f_x(v)=f(x,v)$ are convex where defined for all $x\in a,b]$, $y\in c,d]$.

Remark 1 *It is clear that if a function $f:\Delta\to\mathbb{R}$ is convex on the co-ordinates on Δ. Then*

$$f(tx+(1-t)z,sy+(1-s)w)$$
$$\leq tsf(x,y)+t(1-s)f(x,w)+s(1-t)f(z,y)+(1-t)(1-s)f(z,w),$$

holds for all $(t,s)\in[0,1]\times[0,1]$ and $x,z\in[a,b],y,w\in[c,d]$.

Clearly, every convex mapping $f:\Delta\to\mathbb{R}$ is convex on the co-ordinates but converse may not be true see for instance [6].

The following Hermite-Hadamard type inequalities for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$ were established in [6, Theorem 1, page778]:

Theorem 5 *[6] Suppose that $f:\Delta\to\mathbb{R}$ is co-ordinated convex on Δ, then*

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\leq\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]$$
$$\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx$$
$$\leq\frac{1}{4}\left[\frac{1}{b-a}\int_a^b[f(x,c)+f(x,d)]dx+\frac{1}{d-c}\int_c^d[f(a,y)+f(b,y)]dy\right]$$
$$\leq\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \tag{5}$$

The above inequalities are sharp.

Most recently, the concept of co-ordinated convexity has been generalized in a diverse manner. A number of papers have been written on Hermite-Hadamard type inequalities for the classes of co-ordinated s-convex functions, co-ordinated m-convex functions, (α,m)-convex functions, co-ordinated h-convex functions and co-ordinated quasi-convex functions see for example [1]-[3], [6]-[8], [11]-[19], [24] and [25] and the references therein.

Motivated by the above results for GA-convex functions, we first introduce the notion of GA-convex functions on the co-ordinates and establish Hermite-Hadamard type inequalities for this class of functions in Section 2. We will also present applications of our results to special means of positive real numbers in Section

2. MAIN RESULTS

In this section we first give the notion of GA-convex functions on the co-ordinates and then we prove inequalities of Hermite-Hadamard type for this class of functions.

Definition 2 *A function* $f:\Delta\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ *is GA-convex on* Δ *if*

$$f\left(x^{\lambda}z^{1-\lambda},y^{\lambda}w^{1-\lambda}\right)\leq\lambda f(x,y)+(1-\lambda)f(z,w)$$

holds for all $(x,y),(z,w)\in\Delta$ and $\lambda\in[0,1]$.

Definition 3 *A function* $f:\Delta\subseteq(0,\infty)\times 0,\infty)\to\mathbb{R}$ *is said to be GA-convex on the co-ordinates on* Δ *if the partial mappings* $f_y:[a,b]\subseteq(0,\infty)\to\mathbb{R}$, $f_y(u)=f(u,y)$ *and* $f_x:[c,d]\subseteq(0,\infty)\to\mathbb{R}$, $f_x(v)=f(x,v)$ *are GA-convex where defined for all* $x\in[a,b]$, $y\in[c,d]$.

Remark 2 *If a function* $f:\Delta\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ *is GA-convex on the co-ordinates on* Δ. *Then*

$$f(x^tz^{1-t},y^sw^{1-s})\leq tf(x,y^sw^{1-s})+(1-t)f(z,y^sw^{1-s})$$
$$\leq t[sf(x,y)+(1-s)f(x,w)]+(1-t)[sf(z,y)+(1-s)f(z,w)]$$
$$\leq tsf(x,y)+t(1-s)f(x,w)+s(1-t)f(z,y)+(1-t)(1-s)f(z,w)$$

holds for all $(t,s)\in[0,1]\times[0,1]$ and $x,z\in[a,b],y,w\in[c,d]$.

The following Lemma will be used to establish our main results:

Lemma 1 *Let* $f:\Delta\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ *be a twice differentiable mapping on* Δ° *and* $[a,b]\times[c,d]\subseteq\Delta^{\circ}$ *such that* $\frac{\partial^2 f}{\partial s\,\partial t}\in L([a,b]\times[c,d])$. *Then*

$$acf(a,c)-adf(a,d)-bcf(b,c)+bdf(b,d)-b\int_c^d f(b,y)dy$$
$$+a\int_c^d f(a,y)dy-d\int_a^b f(x,d)dx+c\int_a^b f(x,c)dx+\int_a^b\int_c^d f(x,y)dydx$$
$$=(\ln b-\ln a)(\ln d-\ln c)\int_0^1\int_0^1 b^{2t}a^{2(1-t)}d^{2s}c^{2(1-s)}\frac{\partial^2 f(b^ta^{1-t},d^sc^{1-s})}{\partial s\,\partial t}dsdt. \tag{6}$$

Proof. By the change of the variables $x=b^ta^{1-t}$, $y=d^sc^{1-s}$ and by integration by parts with respect to y and then with respect to x, we have

$$(\ln b-\ln a)(\ln d-\ln c)\int_0^1\int_0^1 b^{2t}a^{2(1-t)}d^{2s}c^{2(1-s)}\frac{\partial^2 f(b^ta^{1-t},d^sc^{1-s})}{\partial s\,\partial t}dsdt$$
$$=\int_a^b\int_c^d xy\frac{\partial^2 f(x,y)}{\partial y\,\partial x}dydx=\int_a^b x\left[d\frac{\partial f(x,d)}{\partial x}-c\frac{\partial f(x,c)}{\partial x}-\int_c^d\frac{\partial f(x,y)}{\partial x}dy\right]dx$$

$$= d\int_a^b x\frac{\partial f(x,d)}{\partial x}dx - c\int_a^b x\frac{\partial f(x,c)}{\partial x}dx - \int_c^d\left[\int_a^b x\frac{\partial f(x,y)}{\partial x}dx\right]dy$$

$$= bdf(b,d) - adf(a,d) - bcf(b,c) + acf(a,c) - d\int_a^b f(x,d)dx$$

$$+c\int_a^b f(x,c)dx - b\int_c^d f(b,y)dy + a\int_c^d f(a,y)dy + \int_a^b\int_c^d f(x,y)dydx. \tag{7}$$

Which is the desired identity. This completes the proof of the lemma.

Theorem 6 *Let* $f:\Delta \subseteq (0,\infty)\times(0,\infty)\to\mathbb{R}$ *be a twice differentiable mapping on* Δ° *and* $[a,b]\times[c,d]\subseteq\Delta^\circ$ *such that* $\frac{\partial^2 f}{\partial s\,\partial t}\in L([a,b]\times[c,d])$. *If* $\left|\frac{\partial^2 f}{\partial s\,\partial t}\right|^q$ *for* $q\geq 1$, *is GA-convex on the co-ordinates on* $[a,b]\times[c,d]$, *we have*

$$\left|acf(a,c) - adf(a,d) - bcf(b,c) + bdf(b,d) - b\int_c^d f(b,y)dy\right.$$

$$\left.+a\int_c^d f(a,y)dy - d\int_a^b f(x,d)dx + c\int_a^b f(x,c)dx + \int_a^b\int_c^d f(x,y)dydx\right|$$

$$\leq \frac{[(b-a)(d-c)A(a,b)A(c,d)]^{1-1/q}}{2^{2/q}}\left\{\left|\frac{\partial^2 f(b,d)}{\partial s\,\partial t}\right|^q [b^2 - L(a^2,b^2)][d^2 - L(c^2,d^2)]\right.$$

$$+\left|\frac{\partial^2 f(b,c)}{\partial s\,\partial t}\right|^q [b^2 - L(a^2,b^2)][L(c^2,d^2) - c^2] + \left|\frac{\partial^2 f(a,d)}{\partial s\,\partial t}\right|^q [L(a^2,b^2) - a^2][d^2 - L(c^2,d^2)]$$

$$\left.+\left|\frac{\partial^2 f(a,c)}{\partial s\,\partial t}\right|^q [L(a^2,b^2) - a^2][L(c^2,d^2) - c^2]\right\}^{1/q}. \tag{8}$$

Proof. From Lemma 1, Hölder's inequality for double integrals and By the GA-convexity of $\left|\frac{\partial^2 f}{\partial s\,\partial t}\right|^q$ for $q\geq 1$ on the co-ordinates on $[a,b]\times[c,d]$, we have

$$\left|acf(a,c) - adf(a,d) - bcf(b,c) + bdf(b,d) - b\int_c^d f(b,y)dy\right.$$

$$\left.+a\int_c^d f(a,y)dy - d\int_a^b f(x,d)dx + c\int_a^b f(x,c)dx + \int_a^b\int_c^d f(x,y)dydx\right|$$

$$\leq a^2c^2(\ln b - \ln a)(\ln d - \ln c)\int_0^1\int_0^1\left(\frac{b}{a}\right)^{2t}\left(\frac{d}{c}\right)^{2s}\left|\frac{\partial^2 f(b^t a^{1-t}, d^s c^{1-s})}{\partial s\,\partial t}\right|dsdt$$

$$\leq a^2c^2(\ln b - \ln a)(\ln d - \ln c)\left[\int_0^1\int_0^1\left(\frac{b}{a}\right)^{2t}\left(\frac{d}{c}\right)^{2s}dsdt\right]^{1-1/q}$$

$$\times\left[\int_0^1\int_0^1\left(\frac{b}{a}\right)^{2t}\left(\frac{d}{c}\right)^{2s}\left|\frac{\partial^2 f(b^t a^{1-t}, d^s c^{1-s})}{\partial s\,\partial t}\right|dsdt\right]^{1/q} \leq a^2c^2(\ln b - \ln a)(\ln d - \ln c)$$

$$\times\left[\frac{(b^2-a^2)(d^2-c^2)}{4a^2c^2(\ln b - \ln a)(\ln d - \ln c)}\right]^{1-1/q}\left[\left|\frac{\partial^2 f(b,d)}{\partial s\,\partial t}\right|^q\int_0^1\int_0^1\left(\frac{b}{a}\right)^{2t}\left(\frac{d}{c}\right)^{2s}stdsdt\right.$$

$$+\left|\frac{\partial^2 f(b,c)}{\partial s\,\partial t}\right|^q \int_0^1\int_0^1 \left(\frac{b}{a}\right)^{2t}\left(\frac{d}{c}\right)^{2s} t(1-s)dsdt + \left|\frac{\partial^2 f(a,d)}{\partial s\,\partial t}\right|^q \int_0^1\int_0^1 \left(\frac{b}{a}\right)^{2t}\left(\frac{d}{c}\right)^{2s} (1-t)sdsdt$$

$$+\left|\frac{\partial^2 f(a,c)}{\partial s\,\partial t}\right|^q \int_0^1\int_0^1 \left(\frac{b}{a}\right)^{2t}\left(\frac{d}{c}\right)^{2s} (1-t)(1-s)dsdt\Bigg]^{1/q}. \tag{9}$$

Since

$$\int_0^1\int_0^1 \left(\frac{b}{a}\right)^{2t}\left(\frac{d}{c}\right)^{2s} stdsdt$$

$$=\left[\frac{2b^2(\ln b-\ln a)-b^2+a^2}{4a^2(\ln b-\ln a)^2}\right]\left[\frac{2d^2(\ln d-\ln c)-d^2+c^2}{4c^2(\ln d-\ln c)^2}\right]=\frac{[b^2-L(a^2,b^2)][d^2-L(c^2,d^2)]}{4a^2c^2(\ln b-\ln a)(\ln d-\ln c)}.$$

Similarly

$$\int_0^1\int_0^1 \left(\frac{b}{a}\right)^{2t}\left(\frac{d}{c}\right)^{2s} t(1-s)dsdt=\frac{[b^2-L(a^2,b^2)][L(c^2,d^2)-c^2]}{4a^2c^2(\ln b-\ln a)(\ln d-\ln c)}$$

$$\int_0^1\int_0^1 \left(\frac{b}{a}\right)^{2t}\left(\frac{d}{c}\right)^{2s} s(1-t)dsdt=\frac{[L(a^2,b^2)-a^2][d^2-L(c^2,d^2)]}{4a^2c^2(\ln b-\ln a)(\ln d-\ln c)},$$

and

$$\int_0^1\int_0^1 \left(\frac{b}{a}\right)^{2t}\left(\frac{d}{c}\right)^{2s} t(1-s)dsdt=\frac{[L(a^2,b^2)-a^2][L(c^2,d^2)-c^2]}{4a^2c^2(\ln b-\ln a)(\ln d-\ln c)}.$$

Using the above four equalities in (9) and simplifying, we get the required inequality (8).

Corollary 1 *Under the assumptions of Theorem 6, if $q=1$. Then*

$$\Bigg|acf(a,c)-adf(a,d)-bcf(b,c)+bdf(b,d)-b\int_c^d f(b,y)dy$$

$$+a\int_c^d f(a,y)dy-d\int_a^b f(x,d)dx+c\int_a^b f(x,c)dx+\int_a^b\int_c^d f(x,y)dydx\Bigg|$$

$$=\frac{1}{4}\Bigg\{\left|\frac{\partial^2 f(b,d)}{\partial s\,\partial t}\right|[b^2-L(a^2,b^2)][d^2-L(c^2,d^2)]$$

$$+\left|\frac{\partial^2 f(b,c)}{\partial s\,\partial t}\right|[b^2-L(a^2,b^2)][L(c^2,d^2)-c^2]+\left|\frac{\partial^2 f(a,d)}{\partial s\,\partial t}\right|\left[L\left(a^2,b^2\right)-a^2\right]\left[d^2-L\left(c^2,d^2\right)\right]$$

$$+\left|\frac{\partial^2 f(a,c)}{\partial s\,\partial t}\right|[L(a^2,b^2)-a^2][L(c^2,d^2)-c^2]\Bigg\}. \tag{10}$$

Theorem 7 *Let $f:\Delta\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ be a twice differentiable mapping on Δ° and $[a,b]\times[c,d]\subseteq\Delta^\circ$ such that $\frac{\partial^2 f}{\partial s\,\partial t}\in L([a,b]\times[c,d])$. If $\left|\frac{\partial^2 f}{\partial s\,\partial t}\right|^q$ for $q>1$, is GA-convex on the co-ordinates on $[a,b]\times[c,d]$, we have*

$$\Bigg|acf(a,c)-adf(a,d)-bcf(b,c)+bdf(b,d)-b\int_c^d f(b,y)dy$$
$$+a\int_c^d f(a,y)dy-d\int_a^b f(x,d)dx+c\int_a^b f(x,c)dx+\int_a^b\int_c^d f(x,y)dydx\Bigg|$$
$$\leq(\ln b-\ln a)(\ln d-\ln c)\left[L\left(a^{2q/(q-1)},b^{2q/(q-1)}\right)L\left(c^{2q/(q-1)},d^{2q/(q-1)}\right)\right]^{1-1/q}$$
$$\times\left[A\left(\left|\frac{\partial^2 f(b,d)}{\partial s\,\partial t}\right|^q,\left|\frac{\partial^2 f(b,c)}{\partial s\,\partial t}\right|^q,\left|\frac{\partial^2 f(a,d)}{\partial s\,\partial t}\right|^q,\left|\frac{\partial^2 f(a,c)}{\partial s\,\partial t}\right|^q\right)\right]^{1/q}. \qquad (11)$$

Proof. From Lemma 1, Hölder's inequality for double integrals and By the GA-convexity of $\left|\frac{\partial^2 f}{\partial s\,\partial t}\right|^q$ for $q>1$ on the co-ordinates on $[a,b]\times[c,d]$, we have

$$\Bigg|acf(a,c)-adf(a,d)-bcf(b,c)+bdf(b,d)-b\int_c^d f(b,y)dy$$
$$+a\int_c^d f(a,y)dy-d\int_a^b f(x,d)dx+c\int_a^b f(x,c)dx+\int_a^b\int_c^d f(x,y)dydx\Bigg|$$
$$\leq a^2c^2(\ln b-\ln a)(\ln d-\ln c)\int_0^1\int_0^1\left(\frac{b}{a}\right)^{2t}\left(\frac{d}{c}\right)^{2s}\left|\frac{\partial^2 f(b^t a^{1-t},d^s c^{1-s})}{\partial s\,\partial t}\right|dsdt$$
$$\leq a^2c^2(\ln b-\ln a)(\ln d-\ln c)\left[\int_0^1\int_0^1\left(\frac{b}{a}\right)^{2qt/(q-1)}\left(\frac{d}{c}\right)^{2qs/(q-1)}dsdt\right]^{1-1/q}$$
$$\times\left[\int_0^1\int_0^1\left|\frac{\partial^2 f(b^t a^{1-t},d^s c^{1-s})}{\partial s\,\partial t}\right|dsdt\right]^{1/q}\leq(\ln b-\ln a)(\ln d-\ln c)$$
$$\times\left[\frac{(q-1)^2\left(b^{2q/(q-1)}-a^{2q/(q-1)}\right)\left(d^{2q/(q-1)}-c^{2q/(q-1)}\right)}{4q^2(\ln b-\ln a)(\ln d-\ln c)}\right]^{1-1/q}$$
$$\times\left[\left|\frac{\partial^2 f(a,c)}{\partial s\,\partial t}\right|^q\int_0^1\int_0^1(1-t)(1-s)dsdt+\left|\frac{\partial^2 f(b,c)}{\partial s\,\partial t}\right|^q\int_0^1\int_0^1 t(1-s)dsdt\right.$$
$$\left.+\left|\frac{\partial^2 f(a,d)}{\partial s\,\partial t}\right|^q\int_0^1\int_0^1(1-t)sdsdt+\left|\frac{\partial^2 f(b,d)}{\partial s\,\partial t}\right|^q\int_0^1\int_0^1 stdsdt\right]^{1/q}. \qquad (12)$$

Since

$$\int_0^1\int_0^1 stdsdt=\int_0^1\int_0^1 t(1-s)dsdt=\int_0^1\int_0^1 s(1-t)dsdt=\int_0^1\int_0^1 t(1-s)dsdt=\frac{1}{4}.$$

Hence from (12), we get the required result.

Theorem 8 *Let* $f:\Delta\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ *be a twice differentiable mapping on* Δ° *and* $[a,b]\times[c,d]\subseteq\Delta^\circ$ *such that* $\frac{\partial^2 f}{\partial s\,\partial t}\in L([a,b]\times[c,d])$. *If* $\left|\frac{\partial^2 f}{\partial s\,\partial t}\right|^q$ *for* $q\geq1$, *is GA-convex on the co-ordinates on* $[a,b]\times[c,d]$, *we have*

$$\left|acf(a,c)-adf(a,d)-bcf(b,c)+bdf(b,d)-b\int_c^d f(b,y)dy\right.$$
$$\left.+a\int_c^d f(a,y)dy-d\int_a^b f(x,d)dx+c\int_a^b f(x,c)dx+\int_a^b\int_c^d f(x,y)dydx\right|$$
$$\leq\frac{[(\ln b-\ln a)(\ln d-\ln c)]^{1-1/q}}{(2q)^{2/q}}\left\{\left|\frac{\partial^2 f(b,d)}{\partial s\,\partial t}\right|^q[b^{2q}-L(a^{2q},b^{2q})][d^{2q}-L(c^{2q},d^{2q})]\right.$$
$$+\left|\frac{\partial^2 f(b,c)}{\partial s\,\partial t}\right|^q[b^{2q}-L(a^{2q},b^{2q})][L(c^{2q},d^{2q})-c^{2q}]$$
$$+\left|\frac{\partial^2 f(a,d)}{\partial s\,\partial t}\right|^q[L(a^{2q},b^{2q})-a^{2q}][d^{2q}-L(c^{2q},d^{2q})]$$
$$\left.+\left|\frac{\partial^2 f(a,c)}{\partial s\,\partial t}\right|^q[L(a^{2q},b^{2q})-a^{2q}][L(c^{2q},d^{2q})-c^{2q}]\right\}^{1/q}. \tag{13}$$

Proof. From Lemma 1, Hölder's inequality for double integrals and By the GA-convexity of $\left|\frac{\partial^2 f}{\partial s\,\partial t}\right|^q$ for $q\geq 1$ on the co-ordinates on $[a,b]\times[c,d]$, we have

$$\left|acf(a,c)-adf(a,d)-bcf(b,c)+bdf(b,d)-b\int_c^d f(b,y)dy\right.$$
$$\left.+a\int_c^d f(a,y)dy-d\int_a^b f(x,d)dx+c\int_a^b f(x,c)dx+\int_a^b\int_c^d f(x,y)dydx\right|$$
$$\leq a^2c^2(\ln b-\ln a)(\ln d-\ln c)\left[\int_0^1\int_0^1 1dsdt\right]^{1-1/q}$$
$$\times\left[\int_0^1\int_0^1\left(\frac{b}{a}\right)^{2t}\left(\frac{d}{c}\right)^{2s}\left|\frac{\partial^2 f(b^ta^{1-t},d^sc^{1-s})}{\partial s\,\partial t}\right|^q dsdt\right]^{1/q}$$
$$\leq a^2c^2(\ln b-\ln a)(\ln d-\ln c)\left[\left|\frac{\partial^2 f(b,d)}{\partial s\,\partial t}\right|^q\int_0^1\int_0^1\left(\frac{b}{a}\right)^{2qt}\left(\frac{d}{c}\right)^{2qs}stdsdt\right.$$
$$+\left|\frac{\partial^2 f(b,c)}{\partial s\,\partial t}\right|^q\int_0^1\int_0^1\left(\frac{b}{a}\right)^{2qt}\left(\frac{d}{c}\right)^{2qs}t(1-s)dsdt+\left|\frac{\partial^2 f(a,d)}{\partial s\,\partial t}\right|^q\int_0^1\int_0^1\left(\frac{b}{a}\right)^{2qt}\left(\frac{d}{c}\right)^{2qs}(1-t)sdsdt$$
$$\left.+\left|\frac{\partial^2 f(a,c)}{\partial s\,\partial t}\right|^q\int_0^1\int_0^1\left(\frac{b}{a}\right)^{2qt}\left(\frac{d}{c}\right)^{2qs}(1-t)(1-s)dsdt\right]^{1/q}. \tag{14}$$

Since

$$\int_0^1\int_0^1\left(\frac{b}{a}\right)^{2qt}\left(\frac{d}{c}\right)^{2qs}stdsdt=\frac{[b^{2q}-L(a^{2q},b^{2q})][d^{2q}-L(c^{2q},d^{2q})]}{4q^2a^{2q}c^{2q}(\ln b-\ln a)(\ln d-\ln c)}.$$

Similarly

$$\int_0^1\int_0^1\left(\frac{b}{a}\right)^{2qt}\left(\frac{d}{c}\right)^{2qs}t(1-s)dsdt=\frac{[b^{2q}-L(a^{2q},b^{2q})][L(c^{2q},d^{2q})-c^{2q}]}{4q^2a^{2q}c^{2q}(\ln b-\ln a)(\ln d-\ln c)}$$

$$\int_0^1\int_0^1\left(\frac{b}{a}\right)^{2qt}\left(\frac{d}{c}\right)^{2qs}s(1-t)dsdt=\frac{[L(a^{2q},b^{2q})-a^{2q}][d^{2q}-L(c^{2q},d^{2q})]}{4q^2a^{2q}c^{2q}(\ln b-\ln a)(\ln d-\ln c)},$$

and

$$\int_0^1 \int_0^1 \left(\frac{b}{a}\right)^{2qt} \left(\frac{d}{c}\right)^{2qs} t(1-s)dsdt = \frac{[L(a^{2q}, b^{2q}) - a^{2q}][L(c^{2q}, d^{2q}) - c^{2q}]}{4q^2 a^{2q} c^{2q} (\ln b - \ln a)(\ln d - \ln c)}.$$

Using the above four in (14) and simplifying, we get the required inequality (13).

Theorem 9 *Let $f: \Delta \subseteq (0,\infty) \times (0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on Δ° and $[a,b] \times [c,d] \subseteq \Delta^\circ$ such that $\frac{\partial^2 f}{\partial s\, \partial t} \in L([a,b] \times [c,d])$. If $\left|\frac{\partial^2 f}{\partial s\, \partial t}\right|^q$ is GA-convex on the co-ordinates on $[a,b] \times [c,d]$ for $q > 1$ and $q > p > 1$. Then*

$$\begin{aligned}
&\Bigg| acf(a,c) - adf(a,d) - bcf(b,c) + bdf(b,d) - b\int_c^d f(b,y)dy \\
&+a\int_c^d f(a,y)dy - d\int_a^b f(x,d)dx + c\int_a^b f(x,c)dx + \int_a^b \int_c^d f(x,y)dydx \Bigg| \\
&\leq \frac{[(\ln b - \ln a)(\ln d - \ln c)]^{1-1/q}}{p^{2/q}} \\
&\times \left[L\left(a^{(2q-p)/(q-1)}, b^{(2q-p)/(q-1)}\right) L\left(c^{(2q-p)/(q-1)}, d^{(2q-p)/(q-1)}\right)\right]^{1-1/q} \\
&\times \Bigg\{ \left|\frac{\partial^2 f(b,d)}{\partial s\, \partial t}\right|^q [b^p - L(a^p, b^p)][d^p - L(c^p, d^p)] + \left|\frac{\partial^2 f(b,c)}{\partial s\, \partial t}\right|^q [b^p - L(a^p, b^p)][L(c^p, d^p) - c^p] \\
&+ \left|\frac{\partial^2 f(a,d)}{\partial s\, \partial t}\right|^q [L(a^p, b^p) - a^p][d^p - L(c^p, d^p)] \\
&+ \left|\frac{\partial^2 f(a,c)}{\partial s\, \partial t}\right|^q [L(a^p, b^p) - a^p][L(c^p, d^p) - c^p] \Bigg\}^{1/q}.
\end{aligned} \tag{15}$$

Proof. From Lemma 1, Hölder's inequality for double integrals and by the GA-convexity of $\left|\frac{\partial^2 f}{\partial s\, \partial t}\right|^q$ on the co-ordinates on $[a,b] \times [c,d]$ for $q > 1$ and $2q > p > 1$, we have

$$\begin{aligned}
&\Bigg| acf(a,c) - adf(a,d) - bcf(b,c) + bdf(b,d) - b\int_c^d f(b,y)dy \\
&+a\int_c^d f(a,y)dy - d\int_a^b f(x,d)dx + c\int_a^b f(x,c)dx + \int_a^b \int_c^d f(x,y)dydx \Bigg| \\
&\leq a^2 c^2 (\ln b - \ln a)(\ln d - \ln c) \left[\int_0^1 \int_0^1 \left(\frac{b}{a}\right)^{(2q-p)t/(q-1)} \left(\frac{d}{c}\right)^{(2q-p)s/(q-1)} dsdt\right]^{1-1/q} \\
&\times \left[\int_0^1 \int_0^1 \left(\frac{b}{a}\right)^{pt} \left(\frac{d}{c}\right)^{ps} \left|\frac{\partial^2 f(b^t a^{1-t}, d^s c^{1-s})}{\partial s\, \partial t}\right|^q dsdt\right]^{1/q} \leq a^2 c^2 (\ln b - \ln a)(\ln d - \ln c) \\
&\left[\frac{(q-1)^2 \left(b^{(2q-p)/(q-1)} - a^{(2q-p)/(q-1)}\right)\left(d^{(2q-p)/(q-1)} - c^{(2q-p)/(q-1)}\right)}{a^{(2q-p)/(q-1)} c^{(2q-p)/(q-1)} (2q-p)^2 (\ln b - \ln a)(\ln d - \ln c)}\right]^{1-1/q}
\end{aligned}$$

$$\times\left[\left|\frac{\partial^2 f(b,d)}{\partial s\,\partial t}\right|^q \int_0^1\int_0^1 \left(\frac{b}{a}\right)^{pt}\left(\frac{d}{c}\right)^{ps} stdsdt + \left|\frac{\partial^2 f(b,c)}{\partial s\,\partial t}\right|^q \int_0^1\int_0^1 \left(\frac{b}{a}\right)^{pt}\left(\frac{d}{c}\right)^{ps} t(1-s)dsdt\right.$$

$$+\left|\frac{\partial^2 f(a,d)}{\partial s\,\partial t}\right|^q \int_0^1\int_0^1 \left(\frac{b}{a}\right)^{pt}\left(\frac{d}{c}\right)^{ps} (1-t)sdsdt$$

$$\left.+\left|\frac{\partial^2 f(a,c)}{\partial s\,\partial t}\right|^q \int_0^1\int_0^1 \left(\frac{b}{a}\right)^{pt}\left(\frac{d}{c}\right)^{ps} (1-t)(1-s)dsdt\right]^{1/q}. \tag{16}$$

Since

$$\int_0^1\int_0^1 \left(\frac{b}{a}\right)^{pt}\left(\frac{d}{c}\right)^{ps} stdsdt = \frac{[b^p - L(a^p,b^p)][d^p - L(c^p,d^p)]}{p^2a^pc^p(\ln b - \ln a)(\ln d - \ln c)}.$$

Similarly

$$\int_0^1\int_0^1 \left(\frac{b}{a}\right)^{pt}\left(\frac{d}{c}\right)^{ps} t(1-s)dsdt = \frac{[b^p - L(a^p,b^p)][L(c^p,d^p) - c^p]}{p^2a^pc^p(\ln b - \ln a)(\ln d - \ln c)}$$

$$\int_0^1\int_0^1 \left(\frac{b}{a}\right)^{pt}\left(\frac{d}{c}\right)^{ps} s(1-t)dsdt = \frac{[L(a^p,b^p) - a^p][d^p - L(c^p,d^p)]}{p^2a^{2q}c^{2q}(\ln b - \ln a)(\ln d - \ln c)},$$

and

$$\int_0^1\int_0^1 \left(\frac{b}{a}\right)^{pt}\left(\frac{d}{c}\right)^{ps} t(1-s)dsdt = \frac{[L(a^p,b^p) - a^p][L(c^p,d^p) - c^p]}{p^2a^pc^p(\ln b - \ln a)(\ln d - \ln c)}.$$

Using the above four in (16) and simplifying, we get the required inequality (15).

Corollary 2 *Under the assumptions of Theorem 9, if* $p = q$, *we have the inequality*

$$\left|acf(a,c) - adf(a,d) - bcf(b,c) + bdf(b,d) - b\int_c^d f(b,y)dy\right.$$

$$\left.+a\int_c^d f(a,y)dy - d\int_a^b f(x,d)dx + c\int_a^b f(x,c)dx + \int_a^b\int_c^d f(x,y)dydx\right|$$

$$\leq \frac{[(\ln b - \ln a)(\ln d - \ln c)]^{1-1/q}}{q^{2/q}}\left[L\left(a^{q/(q-1)}, b^{q/(q-1)}\right)L\left(c^{q/(q-1)}, d^{q/(q-1)}\right)\right]^{1-1/q}$$

$$\times\left\{\left|\frac{\partial^2 f(b,d)}{\partial s\,\partial t}\right|^q [b^q - L(a^q,b^q)][d^q - L(c^q,d^q)] + \left|\frac{\partial^2 f(b,c)}{\partial s\,\partial t}\right|^q [b^q - L(a^q,b^q)][L(c^q,d^q) - c^q]\right.$$

$$+\left|\frac{\partial^2 f(a,d)}{\partial s\,\partial t}\right|^q [L(a^q,b^q) - a^q][d^q - L(c^q,d^q)]$$

$$\left.+\left|\frac{\partial^2 f(a,c)}{\partial s\,\partial t}\right|^q [L(a^q,b^q) - a^q][L(c^q,d^q) - c^q]\right\}^{1/q}. \tag{17}$$

3. APPLICATIONS TO SPECIAL MEAN

In this section we apply our results to establish inequalities for special means.

Theorem 10 *For* $b > a > 0,\ d > c > 0,\ s_1,\ s_2 > 0,\ q \geq 1$ *and* $s_1 q \neq 1,\ s_2 q \neq 1$, *we have*

$$[L_{s_1+1}(a,b)]^{s_1+1}[L_{s_2+1}(c,d)]^{s_2+1} \leq \frac{[(a+b)(c+d)]^{1-1/q}}{4}$$
$$\times \Big[\Big\{(s_1 q+2)[L_{s_1 q+1}(a,b)]^{s_1 q+1} - s_1 q L(a^2,b^2)[L_{s_1 q-1}(a,b)]^{s_1 q-1}\Big\}$$
$$\times \Big\{(s_2 q+2)[L_{s_2 q+1}(c,d)]^{s_2 q+1} - s_2 q L(c^2,d^2)[L_{s_2 q-1}(c,d)]^{s_2 q-1}\Big\}\Big]^{1/q}. \tag{18}$$

Proof. Let

$$f(x,y) = \frac{x^{s_1+1}y^{s_2+1}}{(s_1+1)(s_2+1)}, (x,y) \in (0,\infty)\times(0,\infty), s_1, s_2 > 0.$$

Then $\left|\frac{\partial^2 f(x,y)}{\partial y\,\partial x}\right|^q = x^{s_1 q}y^{s_2 q}$ is GA-convex on the co-ordinates on $(0,\infty)\times(0,\infty)$ and both sides of the inequality (8) in Theorem 6 become

$$\Big|acf(a,c) - adf(a,d) - bcf(b,c) + bdf(b,d) - b\int_c^d f(b,y)dy$$
$$+a\int_c^d f(a,y)dy - d\int_a^b f(x,d)dx + c\int_a^b f(x,c)dx + \int_a^b\int_c^d f(x,y)dydx\Big|$$
$$= (b-a)(d-c)\left[\frac{b^{s_1+2}-a^{s_1+2}}{(s_1+2)(b-a)}\right]\left[\frac{d^{s_2+2}-c^{s_2+2}}{(s_2+2)(d-c)}\right]$$
$$= (b-a)(d-c)[L_{s_1+1}(a,b)]^{s_1+1}[L_{s_2+1}(c,d)]^{s_2+1}.$$
$$\frac{[(b-a)(d-c)A(a,b)A(c,d)]^{1-1/q}}{2^{2/q}}\left\{\left|\frac{\partial^2 f(b,d)}{\partial s\,\partial t}\right|^q [b^2 - L(a^2,b^2)][d^2 - L(c^2,d^2)]\right.$$
$$+\left|\frac{\partial^2 f(b,c)}{\partial s\,\partial t}\right|^q [b^2 - L(a^2,b^2)][L(c^2,d^2) - c^2] + \left|\frac{\partial^2 f(a,d)}{\partial s\,\partial t}\right|^q [L(a^2,b^2) - a^2][d^2 - L(c^2,d^2)]$$
$$\left.+\left|\frac{\partial^2 f(a,c)}{\partial s\,\partial t}\right|^q [L(a^2,b^2) - a^2][L(c^2,d^2) - c^2]\right\}^{1/q} = \frac{(b-a)(d-c)[(a+b)(c+d)]^{1-1/q}}{4}$$
$$\times \Big[\Big\{(s_1 q+2)[L_{s_1 q+1}(a,b)]^{s_1 q+1} - s_1 q L(a^2,b^2)[L_{s_1 q-1}(a,b)]^{s_1 q-1}\Big\}$$
$$\times \Big\{(s_2 q+2)[L_{s_2 q+1}(c,d)]^{s_2 q+1} - s_2 q L(c^2,d^2)[L_{s_2 q-1}(c,d)]^{s_2 q-1}\Big\}\Big]^{1/q}.$$

A combination of the above two equalities gives us the desired inequality (18).

Corollary 3 *Under the assumptions of Theorem 10, if* $q = 1$ *and* $s_1,\ s_2 \neq 1$. *Then*

$$[L_{s_1+1}(a,b)]^{s_1+1}[L_{s_2+1}(c,d)]^{s_2+1} \leq \frac{1}{4}\Big[\Big\{(s_1+2)[L_{s_1+1}(a,b)]^{s_1+1} - s_1 L(a^2,b^2)[L_{s_1-1}(a,b)]^{s_1-1}\Big\}$$
$$\times \Big\{(s_2+2)[L_{s_2+1}(c,d)]^{s_2+1} - s_2 L(c^2,d^2)[L_{s_2-1}(c,d)]^{s_2-1}\Big\}\Big]. \tag{19}$$

Theorem 11 *Let* $b > a > 0,\ s_1, s_2 > 0$ *and* $q > 1$. *Then*

$$[L(a,b)][L(c,d)]\left[L_{s_1+1}(a,b)\right]^{s_1+1}\left[L_{s_2+1}(c,d)\right]^{s_2+1}$$
$$\leq \left[L\left(a^{2q/(q-1)},b^{2q/(q-1)}\right)L\left(c^{2q/(q-1)},d^{2q/(q-1)}\right)\right]^{1-1/q}$$
$$\times\left[A(a^{s_1q},b^{s_2q})A(c^{s_1q},d^{s_2q})\right]^{1/q}. \tag{20}$$

Proof. The proof follows from Theorem 7 and using the following GA- convex function on the co-ordinates on $(0,\infty)\times(0,\infty)$

$$f(x,y)=\frac{x^{s_1+1}y^{s_2+1}}{(s_1+1)(s_2+1)},(x,y)\in(0,\infty)\times(0,\infty),s_1,s_2>0.$$

The following interesting inequalities of means can be obtained using the results of Theorem 8 and Theorem 9 and the GA-convex function on the co-ordinates on $(0,\infty)\times(0,\infty)$ as defined in Theorem 10, however the details are left to the interested reader.

Theorem 12 *Let* $b>a>0$, $s_1,s_2>0$ *and* $s_1q,s_2q\neq 1$. *Then*

$$[L(a,b)L(c,d)]^{1-1/q}\left[L_{s_1+1}(a,b)\right]^{s_1+1}\left[L_{s_2+1}(c,d)\right]^{s_2+1}\leq\frac{1}{(2q)^{2/q}}$$
$$\times\left\{(s_1+2)q\left[L_{(s_1+2)q-1}(a,b)\right]^{(s_1+2)q-1}-s_1qL(a^{2q},b^{2q})\left[L_{s_1q-1}(a,b)\right]^{s_1q-1}\right\}^{1/q}$$
$$\times\left\{(s_2+2)q\left[L_{(s_2+2)q-1}(c,d)\right]^{(s_2+2)q-1}-s_2qL(c^{2q},d^{2q})\left[L_{s_2q-1}(c,d)\right]^{s_2q-1}\right\}^{1/q}. \tag{21}$$

Theorem 13 *Let* $b>a>0$, $s_1,s_2>0$, $q>1,2q>p>0$ *and* $s_1q,s_2q\neq 1$. *Then*

$$[L(a,b)L(c,d)]^{1-1/q}\left[L_{s_1+1}(a,b)\right]^{s_1+1}\left[L_{s_2+1}(c,d)\right]^{s_2+1}$$
$$\leq\frac{1}{(p)^{2/q}}\left[L\left(a^{(2q-p)/(q-1)},b^{(2q-p)/(q-1)}\right)L\left(c^{(2q-p)/(q-1)},d^{(2q-p)/(q-1)}\right)\right]^{1-1/q}$$
$$\left\{(p+s_1q)\left[L_{p+s_1q-1}(a,b)\right]^{p+s_1q-1}-s_1qL(a^p,b^p)\left[L_{s_1q-1}(a,b)\right]^{s_1q-1}\right\}^{1/q}$$
$$\times\left\{(s_2+2)q\left[L_{(s_2+2)q-1}(c,d)\right]^{(s_2+2)q-1}-s_2qL(c^{2q},d^{2q})\left[L_{s_2q-1}(c,d)\right]^{s_2q-1}\right\}^{1/q}. \tag{22}$$

4. CONLUSIONS

In our paper, a new notion of GA-convex fucntions on the co-ordinates on the rectangle from the plane is introduced and some of the properties of this class of functions are discussed. A new integral inequality for twice differitiable mappings of two variables which aredefined on a rectangle from the plane is established. By using the notion of GA convexity of the mappings on the co-ordinates, Holder inequality and mathematical analysis, some new inequalities of Hermite-Hadamard type are established. Applications of our results to special means of positive real numbers are given as well. We believe that by using the notion GA conveity on the co-ordinates introduced in this article and some identitities for functions of two variables, many other intersting inequalities of Hermite-Hadamard type can be investigated. Moreover, some weighted generilazations can also be proved by using some appropriate choice of the weight function.

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6. REFERENCES

1. Alomari, M. & M. Darus. Co-ordinated s-convex function in the first sense with some Hadamard-type inequalities. *International Journal of Contemporary Mathematical Sciences* 32 (3):1557-1567 (2008).
2. Alomari, M. &M. Darus. The Hadamard's inequality for s-convex function of 2-variables on the co-ordinates. *International Journal of Mathematical Analysis* 2 (13): 629–638 (2008).
3. Alomari, M., M. Darus& S.S. Dragomir. Inequalities of Hermite-Hadamard's type for functions whose derivatives absolute values are quasi-convex. *Research Group of Mathematical Inequalities and Applications, Research Report Collection* 12 (suppl. 14) (2009).
4. Bullen, P.S. *Handbook of Means and Their Inequalities*. Kluwer Academic Publishers, Dordrecht (2003).
5. Dragomir, S.S. & R. P. Agarwal. Two inequalities for differentiable mappings and applications to special means of real numbers and to Trapezoidal formula. *Applied Mathematics Lett*ers 11(5): 91-95 (1998).
6. Dragomir, S. S. On Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane.*Taiwanese Journal of Mathematics* 4: 775-788 (2001).
7. Dragomir, S. S. & C.E.M. Pearce.*Selected Topics on Hermite-Hadamard Inequalities and Applications*. Research Group of Mathematical Inequalities and Applications, Monographs, Victoria University, 2000. Online: [http://www.staff.vu.edu.au/RGMIA/monographs/hermite_hadamard.html].
8. Hwang, D. –Y., K. L. Tseng & G.S. Yang. Some Hadamard's inequalities for co-ordinated convex functions in a rectangle from the plane. *Taiwanese Journal of Mathematics* 11: 63-73 (2007).
9. Ji, A.-P., T.-Y. Zhang & F. Qi. Integral inequalities of Hermite-Hadamard type for (α, m)-GA-convex functions.*Journal of Function Spaces and Applications* 2013: Article ID 823856, 8 pages (2013).
10. Kirmaci, U.S. Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula. *Applied Mathematics and Comput*ation 147: 137-146 (2004).
11. Latif, M.A. & M. Alomari. Hadamard-type inequalities for product two convex functions on the co-ordinates. *International Mathematical Forum* 4(47): 2327-2338 (2009).
12. Latif, M.A. & M. Alomari. On the Hadamard-type inequalities for h-convex functions on the co-ordinates. *International Journal of Mathematical Analysis* 3(33): 1645-1656 (2009).
13. Latif, M.A. & S.S. Dragomir. On Some New inequalities for differentiable co-ordinated convex functions.*Journal of Inequalities and Applications* 2012: Article 28 (2012).
14. Niculescu, C. P. Convexity according to the geometric mean. *Mathematical Inequalities and Appl*ications 3 (2) 155-167 (2000).
15. Niculescu, C.P. Convexity according to means. *Mathematical Inequalities and Appl*ications 6 (4) 571-579 (2003).
16. Özdemir, M.E., E. Set & M.Z. Sarikaya. New some Hadamard's type inequalities for coordinated m-convex and (α,m)-convex functions. *Research Group of Mathematical Inequalities and Applications, Research Report Collection* 13 (Supplement): Article 4 (2010).
17. Özdemir, M.E., H.Kavurmac, A. O. Akdemir & M. Avc. Inequalities for convex and s-convex functions on Δ=[a,b]×[c,d].*Journal of Inequalities and Applications* 2012: Article 20 (2012).
18. Özdemir, M.E., M. A. Latif & A. O.Akdemir, On some Hadamard-type inequalities for product of two s-convex functions on the co-ordinates.*Journal of Inequalities and Applications* 2012: Article 21 (2012).
19. Özdemir, M E., A. O. Akdemir & M. Tunc. On the Hadamard-type inequalities for co-ordinated convex functions.*arXiv*: 1203.4327v1[math.CA] (2012).
20. Pearce, C.M. E. & J. E. Pečarić. Inequalities for differentiable mappings with applications to special means and quadrature formula. *Applied Mathematics Letters* 13: 51-55 (2000).
21. Pečarić, J.E., F. Proschan& Y. L. Tong.*Convex Functions, Partial Ordering and Statistical Applications*, Academic Press, New York(1991).
22. Qi, F. Generalized abstracted mean values. *Journal of Inequalities in Pure Applied Mathematics* 1 (1): Article 4 (2000).
23. Qi, F. Generalized weighted mean values with two parameters. *Proceedings of the Royal Society A: Mathematical, Physical & Engineering Sciences* 454(1998): 2723–2732 (1998).
24. Sarikaya, M.Z., E. Set, M.E. Özdemir & S.S. Dragomir. New some Hadamard's type inequalities for co-ordinated convex functions.*arXiv*:1005.0700v1 [math.CA] (2010).
25. Sarikaya, M.Z. On the Hermite-Hadamard-type inequalities for co-ordinated convex function via fractional integrals.*Integral Transforms and Special Functions*25(2): 134-147 (2014).
26. Zhang, T.–Y., A.-P. Ji & F. Qi. Some inequalities of Hermite-Hadamard type for GA-Convex functions with applications to means. *Le Matematiche* 48(1): 229-239 (2013).